BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 2, Pages 235–243
S 0273-0979(00)00898-3
Article electronically published on December 27, 2000

Inverse Galois theory, by G. Malle and B. H. Matzat, Springer-Verlag, Berlin, Heidelberg, New York, 1999, xv + 436 pp., \$59.95, ISBN 3-540-62890-8

1. History of the Inverse Galois Problem. From the work of Galois it emerged that an algebraic equation $f(x) = 0$, say over the rationals, is solvable by radicals if and only if the associated Galois group G_f is a solvable group. As a consequence, the general equation of degree $n \geq 5$ cannot be solved by radicals because the group S_n is not solvable.

This idea of encoding algebraic-arithmetic information in terms of group theory was the beginning of Galois theory as well as group theory. By now it has become one of the guiding principles of algebra. What remains unsatisfactory is the fact that it is very hard to compute the Galois group of a given polynomial: Even today's powerful computer algebra systems can only handle polynomials up to degree about 15. Therefore, the full correspondence between equations of degree n and subgroups of S_n can only be worked out for very small values of n. Since it is probably impossible to get a full understanding of this correspondence for general n, one is naturally led to the following more reasonable question: Do at least all subgroups of S_n occur in this correspondence; i.e., does every subgroup of S_n correspond to some equation of degree n? The most important case is that of irreducible equations, which correspond to the transitive subgroups of S_n.

This question is one formulation of the Inverse Problem of Galois Theory. It is often just called the **Inverse Galois Problem**. Hilbert was the first to study this problem. His irreducibility theorem shows that it suffices to realize groups as Galois groups over the function field $\mathbb{Q}(x)$. This allows us to use methods from Riemann surface theory and algebraic geometry. Hilbert applied his method to obtain Galois realizations of the symmetric and alternating groups. The next milestone was Shafarevich's realization of all solvable groups over $\mathbb{Q}$ (in the 1950's). His approach is purely number-theoretic and does not extend to non-solvable groups.

The classification of finite simple groups, completed around 1980, gave a new direction to the work on the Inverse Galois Problem. It now seemed natural to concentrate first on the simple groups, and get the composite groups later by some kind of inductive procedure. It is not yet clear how this inductive procedure – or embedding problem, in technical terms – would work in general. There are quite a few results in this direction, which are described in Chapter IV of the book. However, they are far from complete, just as the results on the simple groups. The Inverse Galois Problem is still wide open.

The results on simple groups and, more generally, almost simple groups (i.e., groups between a simple group and its automorphism group) are the main focus of the book and are contained in Chapters I-III.

2. The Regular Inverse Galois Problem. The most complete result over $\mathbb{Q}$ is Shafarevich's realization of all solvable groups. Most other results over $\mathbb{Q}$ use Hilbert's geometric method that produces Galois extensions of $\mathbb{Q}(x)$ with the additional property that they contain no proper algebraic extension of $\mathbb{Q}$. Such

2000 *Mathematics Subject Classification.* Primary 12F12, 12F10; Secondary 20C33, 20F36, 20G40, 11R32, 11R37.

extensions are called **regular** extensions of $\mathbb{Q}(x)$ (more precisely, $\mathbb{Q}$-regular extensions, but the common terminology is just to speak of regular extensions). We say a group G has a **regular realization** over $\mathbb{Q}$ if it occurs as the Galois group of a regular extension of $\mathbb{Q}(x)$. Same with $\mathbb{Q}$ replaced by any field k. Regular realizations have particularly nice properties that make them easier to work with than ordinary realizations. E.g., if G has a regular realization over k, then also over k' for every overfield k' of k. Furthermore, regular realizations admit Hilbert's trick that sometimes allows us to realize a subgroup H of a group G with known realization: Namely, if $G = Gal(L/k(x))$ and the fixed field of H is of the form $k(y)$, then also H has a regular realization over k. (Hilbert used this with H the alternating subgroup of the symmetric group G.)

A field k is called hilbertian if for every irreducible polynomial $f(x, y)$ in two variables over k there are infinitely many $b \in k$ such that the specialized polynomial $f(b, y)$ is irreducible. If k is hilbertian, then each regular realization of G over k specializes to an ordinary realization (actually, to infinitely many linearly disjoint realizations of G over k, if $G \neq 1$). Every finitely generated extension of a hilbertian field is hilbertian. But also many infinite algebraic extensions are again hilbertian, e.g., every (finite or infinite) abelian extension. See [Har] for some recent results on that. Hilbert's Irreducibility Theorem says that $\mathbb{Q}$ is hilbertian; hence so is every finite number field, and also the maximal cyclotomic field $\mathbb{Q}_{\rm ab}$ generated by all roots of unity (in $\mathbb{C}$). In particular, if G has a regular realization over $\mathbb{Q}$, then G occurs as a Galois group over every finite number field. The **Regular Inverse Galois Problem** asks whether this holds for all finite groups G.

Riemann's Existence Theorem implies that all finite groups G have a regular realization over $\mathbb{C}$, even over $\bar{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$). The descent from $\bar{\mathbb{Q}}$ to $\mathbb{Q}$ is the main difficulty, and can only be achieved under various special conditions. The difficulty is explained by the theory of Hurwitz spaces [FV1] showing that the regular realizations of G over $\mathbb{C}$ are parametrized by certain varieties (Hurwitz spaces), and those realizations that descend to regular realizations over $\mathbb{Q}$ correspond to rational points on these varieties. Certain aspects of that theory, phrased in Matzat's complicated field-theoretic language, are in Chapter III of the book.

The special case of rigidity, when the Hurwitz spaces become trivial, can be dealt with independently. This is contained in Chapter I. It yields a purely group-theoretic (sufficient) criterion for a group to have a regular realization over $\mathbb{Q}$ (or $\mathbb{Q}_{\rm ab}$). This is generally attributed to have been found independently by Belyi, Matzat and Thompson in the early 1980's. But it should be remarked that it is contained implicitly in earlier work of Fried ([Fr] 1977, p. 71, Example 4).

Chapter II is purely group-theoretic, applying the rigidity criteria to the almost simple groups.

The final chapter, Chapter V, surveys a method developed mainly by Harbater that uses weak analogues of Riemann's Existence Theorem over complete valued fields K other than $\mathbb{C}$. Among other things, Harbater proved that every finite group has a regular realization over such K. This applies in particular for K a p-adic field.

3. Riemann's Existence Theorem and rigidity. Let k be a subfield of $\mathbb{C}$. Geometrically, regular realizations of G over k correspond to Galois covers of the Riemann sphere with group G that together with all their automorphisms are defined over k. Invariants of such a cover are the Galois group G, the branch points

$p_1, ..., p_r$ and the associated conjugacy classes $C_1, ..., C_r$ of G (of distinguished inertia group generators over the p_i). By Riemann's Existence Theorem, the covers with given invariants are parametrized by classes of generating systems $(g_1, ..., g_r)$ of G with $g_i \in C_i$ and $g_1 \cdots g_r = 1$. Thus if there is only one such generating system, up to conjugation, then the associated cover is uniquely determined by the above invariants. This property of the generating system $(g_1, .., g_r)$ is called **rigidity**. If it holds and, additionally, G has trivial center, then the minimal field of definition of the cover can easily be read off from the invariants. This is the simple idea of rigidity, which can actually be developed using only Riemann's Existence Theorem and basic (finite) Galois theory; see [V1]. In the Malle/Matzat book, Chapter I is devoted to the rigidity method. The approach is based on the freeness of the algebraic fundamental group of a punctured $\mathbb{P}^1$.

In order to apply the rigidity criterion to a group G, one has to construct rigid generators $g_1, ..., g_r$ of G. At first sight, rigidity looks like a very strong condition that could hardly be satisfied in non-solvable groups. So it was surprising that Belyi [Be], Malle [Malle], Matzat [Mat1], Thompson [Th1] and others in the 1980's were able to find lots of rigid triples in almost simple groups. (Especially spectacular: Thompson's realization of the monster, the largest sporadic simple group). All these results were in the case $r = 3$ (triples), as one would expect because the rigidity condition seems the harder to satisfy the longer the tuple gets.

Thus it was even more surprising, at least for me, that now such generating systems of various lengths, growing with the rank of the group, can be constructed in abundance. Moreover, the longer tuples have analogues that are not quite rigid anymore, but satisfy a weaker condition (braid-abelian) which still can be used for Galois realizations. This has been emerging in the 90's from work of Thompson and Völklein [Th2], [V2], [ThV1]. In 1998, it got a totally unexpected boost from a connection to Katz's theory [Ka] of rigid local systems, a connection found by M. Dettweiler (University of Florida, 1998). None of this is mentioned in the book, which only treats the case $r = 3$.

4. Rigid triples in almost simple groups. Chapter II is devoted to the construction of rigid triples in almost simple groups. These constructions are scattered in the literature, and it is good to have a unified treatment here. It is easy to find such triples in S_n, and for the sporadic groups they can be found from the character tables. So the main emphasis is on the Lie type groups. For the classical Lie type groups, there is a natural class of triples to work with: the Belyi triples, defined by the property that one of the elements has an eigenspace of codimension 1 (in its natural matrix representation). Over $\mathbb{Q}_{ab}$, they suffice to realize all classical simple groups (Belyi 1980). Chapter II presents a streamlined version of that proof.

For the exceptional Lie type groups, there is no natural class of triples. Here, as for the sporadic groups, one uses a well-known formula for the number of triples (g_1, g_2, g_3) of elements of a finite group G with the following properties: $g_1 g_2 g_3 = 1$, and each g_i lies in a given conjugacy class C_i of G. This formula involves the values of all irreducible (complex) characters of G on the classes C_i. Using deep results from the character theory of Lie type groups (especially, Deligne-Lusztig theory), Malle was able to find rigid triples in most cases. This gets us close to a big result: Regular realization of all simple groups over $\mathbb{Q}_{ab}$. Only a few types of exceptional groups in characteristic 2 are missing.

For Galois realizations over $\mathbb{Q}$, the picture is much less complete. Here Belyi triples only yield realizations of classical groups defined over prime fields $\mathbf{F}_p$ (essentially), and one is even far from getting all of those. Work of Malle, Matzat and their students covers such groups under various conditions on p and the Lie type, e.g., $\mathrm{PSL}_n(p)$ for n odd, $\gcd(n, p-1) = 1$, $p > 3$ and $p \not\equiv -1 \pmod{12}$.

To include classical groups over any finite field, one needs to use generating systems whose length grows with the rank of the group. See the following section.

5. Linear rigidity and the connection to differential equations. A surprising connection between Inverse Galois Theory and linear differential equations surfaced in 1998, initiated by M. Dettweiler. A regular linear differential equation on the sphere (and its associated local system) is called **physically rigid** if it is globally determined by its local behavior around each of the singularities. This terminology was used by N. Katz [Ka], who gave a recursive construction method for all irreducible physically rigid local systems. His motivation goes back to Riemann's treatment of the hypergeometric equation, which was based on the fact that all local systems of rank 2 with 3 singularities are physically rigid. Local systems of rank n with r singularities can be parametrized by r-tuples $(g_1, ..., g_r)$ of complex $n \times n$-matrices with $g_1 \cdots g_r = 1$, taken up to simultaneous conjugation. Physical rigidity of local systems corresponds to a condition on these matrices that is very close to the rigidity condition of Galois theory. This condition on matrices $g_1, ..., g_r$ is now called **linear rigidity**; see [SV]. It requires the tuple to be uniquely determined, up to conjugation, by the condition $g_1 \cdots g_r = 1$ and by the Jordan normal forms of the g_i. (The difference to the previous rigidity condition is only that there is no condition on what group the g_i's generate).

In Galois theory, for a long time the only known class of linearly rigid tuples had been the **Belyi triples**, introduced by Belyi around 1980 and used to realize all classical groups over $\mathbb{Q}_{\mathrm{ab}}$. For Galois realizations over $\mathbb{Q}$ of Lie type groups defined over non-prime fields, one needs to use longer tuples. The first examples were found by Thompson and Völklein [Th2], [V2] in the 90's. They are now called **Thompson tuples**: A class of linearly rigid tuples in GL_n of length $n+1$. (They are actually the longest possible linearly rigid tuples in GL_n.) They gave the first examples of rigid generating systems of (almost) simple groups of length > 3. Glueing a Thompson tuple and its dual yields tuples in symplectic groups with a new property, called "braid-abelian"; see [ThV1], [ThV2]: The pure braid group (see below) induces an abelian permutation group on classes of such tuples. (The class of a rigid tuple is fixed by the pure braid group, while on general tuples it "usually" induces a full symmetric or alternating group.)

Katz's construction method now yields linearly rigid and braid-abelian tuples in abundance, allowing for vast improvements of the previous results on Galois realizations of classical groups over $\mathbb{Q}$ (see [DR1], [DR2]). This gets us close to a proof of Thompson's Conjecture: For any fixed finite field $\mathbf{F}_q$, realize all but finitely many groups $G(\mathbf{F}_q)$ over the rationals, where G is a simple algebraic group of adjoint type defined over $\mathbf{F}_q$.

Katz's construction was simplified by Dettweiler/Reiter [DR2] and Völklein [V3], and generalized to arbitrary ground fields. The application to Galois theory is in the case of a finite ground field (whereas in the case of differential equations, the ground field is the complex field). Völklein [V4] showed that for general tuples $(g_1, ..., g_r)$ (over a finite field), Katz's operation has an arithmetic significance for the associated covers of the Riemann sphere: Namely, it implies a close relationship between the fields of definition of a cover and its transform under the operation.

6. Hurwitz spaces and the braid group. As indicated above, Galois covers of the sphere with fixed branch points and fixed group G are finite in number and can be parametrized by certain classes of generating systems of G. This parametrization is not canonical, however, which gives rise to the monodromy action of the braid group (see below). If we let the branch points vary, we obtain a moduli space (for covers with fixed group G and a fixed number r of branch points) which has a finite-to-one map ψ to the space $\mathcal{U}_r$ of r-subsets of the sphere. (ψ maps a point of the moduli space to the branch point set of the corresponding cover.) The fundamental group of $\mathcal{U}_r$ is the Hurwitz braid group on r strings. In the natural topology on these spaces, ψ is an unramified covering. Thus the moduli space is determined topologically by the monodromy action of the braid group on the fiber of ψ (over the chosen base point). This action is the classical Hurwitz action of the braid group on generating systems of G. It describes how a cover of the sphere changes under a deformation of the branch points, and was already used by Hurwitz [Hur]. Its significance for Galois theory was first noticed by M. Fried [Fr]. Here is the explicit formula of how the standard generators $Q_1, ..., Q_{r-1}$ of the braid group act: Q_i maps $(g_1, ..., g_r)$ to the tuple that agrees with $(g_1, ..., g_r)$ in all but the i-th and $(i+1)$-th entry, and these entries are g_{i+1} and $g_{i+1}^{-1} g_i g_{i+1}$, respectively.

The above moduli space carries a natural structure as algebraic variety defined over $\mathbb{Q}$. Its main property is that rational points correspond to covers defined over $\mathbb{Q}$. This theory is due to Fried and Völklein [FV1]. A rather elementary exposition of it is given in [V1]. The subspaces of this moduli space parametrizing covers with a specific branching behavior are called **Hurwitz spaces**. A parallel approach in field-theoretic language was worked out by Matzat [Mat3], using the Galois correspondence between (connected) unramified coverings of $\mathcal{U}_r$ and subgroups of the braid group. This way he expressed everything in terms of the profinite Hurwitz braid group and its action on the maximal unramified extension of the function field of $\mathcal{U}_r$. This approach is reproduced in Chapter III. Since the geometric origin of these constructions is not explained, any reader who is not aware of it might be lost. The notation becomes very cumbersome.

Let me explain the geometry behind Matzat's application to the Mathieu group M_{24}. Besides M_{23} (which remains open), this is the only sporadic simple group that has not been realized over $\mathbb{Q}$ by rigidity. Consider the Hurwitz space parametrizing Galois covers of the sphere with group M_{24}, four branch points and ramification of type $(2A, 2A, 2A, 12B)$. Its quotient by the group of fractional linear transformations is a curve of genus zero, covering the quotient of $\mathcal{U}_4$ by the same group. From the branching of this covering one sees that the genus zero curve has rational points, hence is a split $\mathbb{P}^1$. From this one can deduce that also the original Hurwitz space has rational points, which realizes M_{24} over $\mathbb{Q}$.

7. Embedding problems. Chapter IV deals with embedding problems: The question of how to embed a given Galois realization over K of a quotient $\bar{G}$ of the given group G into a Galois realization of G over K. By induction, it suffices to solve embedding problems with characteristic simple kernel; i.e., where the kernel N of $G \to \bar{G}$ is a direct product of copies of a simple group S. The dichotomy between the abelian and non-abelian case (for S) is quite sharp. The abelian case is again divided into the split case (i.e., G splits over N) and the non-split case. Split abelian embedding problems are always solvable if the ground field is Hilbertian. This is proved in Chapter IV-2 using the standard wreath product construction. Non-split abelian embedding problems lead to questions of an arithmetic flavor: Their solvability depends on the vanishing of certain cohomological invariants, e.g., Serre's trace obstruction (see Chapter IV-6). The latter was originally used for Mestre's realization over $\mathbb{Q}$ of the double covers of the alternating groups; this result is proved in Chapter IV-5 by a different criterion due to the reviewer.

For embedding problems with non-abelian characteristic simple kernel, the solvability seems to depend on the group-theoretic properties of the simple factor S of this kernel. It is known for several classes of simple groups S that such an embedding problem is always solvable (independent of the properties of the given extension of K with group $\bar{G}$) if K is Hilbertian. This was proved by Matzat using his notion of GAR-realization [Mat2] (see Chapter IV-3): a regular realization of S that extends to a realization of $\mathrm{Aut}(S)$ with a particular rationality property. If every non-abelian simple group could be shown to have a GAR-realization over $\mathbb{Q}_{\mathrm{ab}}$, then this would solve the Inverse Galois Problem over $\mathbb{Q}_{\mathrm{ab}}$. Moreover, the absolute Galois group of $\mathbb{Q}_{\mathrm{ab}}$ would then be known to be isomorphic to the free profinite group $\hat{\mathcal{F}}_\omega$ of countable rank (Shafarevich Conjecture). So far, GAR-realizations over $\mathbb{Q}_{\mathrm{ab}}$ are known for roughly the same simple groups for which we have regular realizations over $\mathbb{Q}$.

Shafarevich's Conjecture implies the following exact sequence for the absolute Galois group $G_{\mathbb{Q}}$ of the rationals:

$$1 \;\rightarrow\; \hat{\mathcal{F}}_\omega \;\rightarrow\; G_{\mathbb{Q}} \;\rightarrow\; H \;\rightarrow\; 1$$

with H the group of units of the profinite completion of the ring of integers. Existence of such a sequence with H the direct product of all finite symmetric groups was proved in [FV2], using Hurwitz spaces.

The rest of Chapter IV is devoted to solving abelian embedding problems over global fields. This part is number-theoretic, based on class field theory. It leads up to the Scholz-Reichard theorem on nilpotent groups, a precursor of Shafarevich's realization of all solvable groups over $\mathbb{Q}$. So far, there is no corresponding result for regular realizations: It is not known whether all solvable (or at least nilpotent) groups have a regular realization over $\mathbb{Q}$ (or $\mathbb{Q}_{\mathrm{ab}}$).

8. Patching over complete valued fields. Chapter V uses Harbater's patching method to derive weak analogues of Riemann's Existence Theorem over fields that are complete with respect to a non-Archimedian absolute value. Now rigid analytic geometry replaces the use of Riemann surfaces and the complex topology. Section 1 gives a very short survey of rigid geometry. Section 2 presents Harbater's solution of the Regular Inverse Galois Problem over complete fields, and the proof of Harbater and Pop (independently) of the geometric case of Shafarevich's Conjecture: the absolute Galois group of $\bar{\mathbf{F}}_p(x)$ is free profinite of countable rank. Some of this generalizes to so-called "large" (or "ample") fields K, defined by the property that every curve defined over K that has at least one simple K-rational point has infinitely many such points. An example of such a field is the field of all totally real algebraic numbers (same for totally p-adic). This is contained in sections 3 and 4. (Reviewer's remark: These results can be proved without the machinery of rigid geometry; see [V1], Ch. 10 and for newer results, [HJ]. This is not mentioned in the book.)

The final section, section 5, gives some discussion of the proof of Abhyankar's Conjecture, due to Harbater [Ha] and Raynaud [Ra]. This result determines all finite groups that occur as the Galois group of an unramified covering of a given affine curve over an algebraically closed field of positive characteristic.

9. Explicit polynomials and the genus zero problem. Although many groups are known to have (regular) Galois realizations over $\mathbb{Q}$, it is very difficult to write down explicit polynomials that generate the corresponding field extensions. So far, this has been done for only finitely many almost simple groups other than A_n and S_n. These results rely heavily on computer calculation and are all in the genus zero case (see below). The appendix of the book lists polynomials realizing all transitive permutation groups up to degree 10 over $\mathbb{Q}$. Most of these results are due to the Malle/Matzat group.

In several papers, see e.g. [A], Abhyankar has found infinite series of polynomials in positive characteristic with various classical groups as Galois groups.

An extension of $\mathbb{C}(x)$ can be described by a polynomial $f(x, y) \in \mathbb{C}[x, y]$ in two variables. The first case to consider is when the extension has genus zero, i.e., $f(x, y) = p(y) - xq(y)$ with $p(y), q(y) \in \mathbb{C}(y)$. Then the Galois group of f is the monodromy group of the covering $y \mapsto p(y)/q(y)$ of the sphere by itself. Guralnick and Thompson [GT] conjectured that only finitely many non-abelian simple groups, other than alternating groups, can occur as a composition factor of such a monodromy group. Actually, the conjecture was more general, not only for genus zero covers but for covers of any fixed genus. Via Riemann's existence theorem, this conjecture translates into a purely group-theoretic problem on primitive permutation groups. It gave rise to a big project in which several group-theoretists took part. The proof has finally been completed by Frohardt and Magaard [FM].

10. The Malle/Matzat book and other texts on the subject. The main value of the book is that of a reference volume that collects a huge number of results in the area. Unfortunately, most of the book was written before 1995 and the newer developments (after '95) are largely ignored. This concerns in particular the material on braid-abelian generators, linear rigidity and Thompson's Conjecture (see above), which is not mentioned at all in the book, although it essentially changes the situation on Galois realizations of almost simple groups over $\mathbb{Q}$ (the book's main topic). Most contributions of the Malle/Matzat group are in Chapter II, which contains numerous results on rigid triples in almost simple groups and associated Galois realizations: realizations over $\mathbb{Q}_{\text{ab}}$, and realizations over $\mathbb{Q}$ of Lie type groups defined over prime fields.

The authors' policy was to quote all results that are available in books (except the other books in the area), and supply proofs only for material taken from journal articles. Many proofs are only sketched. In general, the exposition is very compressed and highly technical, involving more upper and lower indices on various levels than I have ever seen. Some of the terminology is non-standard, e.g., the expression "geometric realization" replaces "regular realization", which is used by all people in the area except the Malle/Matzat group. In particular, it is used in the other books on the subject, which are the following.

Serre's *Topics in Galois Theory* [Se] is a true topics volume, trying to stimulate further research rather than describe completed theories. Most of its material is subsumed in the much longer Malle/Matzat book. For anyone with some knowledge of algebraic geometry, Serre's book would be much easier to read. Fried/Jarden's *Field Arithmetic* [FJ] has material related to the Inverse Galois Problem, like hilbertian fields, specialization of regular realizations and realizations of abelian groups. Finally, the reviewer's *Groups as Galois Groups* [V1] gives a rather elementary introduction to the subject, requiring much less machinery than the other texts and giving full proofs of foundational material like Hilbert's Irreducibility Theorem and Riemann's Existence Theorem. It is nicely supplemented by the Malle/Matzat book, which contains many detailed results pertaining to the themes introduced in [V1].

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