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Rings, modules, and algebras in stable homotopy theory, by A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, with an appendix by M. Cole, Mathematical Surveys and Monographs, no. 47, AMS, Providence, RI, 1997, xi + 249 pp., \$62.00, ISBN 0-8218-0638-6

Equivariant homotopy and cohomology theory, by J. P. May, with contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner, CBMS Regional Conference Series in Mathematics, no. 91, AMS, Providence, RI, 1996, xiii + 366 pp., \$49.00, ISBN 0-8128-0319-0

A concise course in algebraic topology, by J. P. May, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, IL, 1999, ix + 247 pp., \$18.00, ISBN 0-226-51183-9 (paper)

In the paper "Analysis Situs" that founded the enterprise of algebraic topology, Poincaré introduced the notion of cobordism. Two compact, closed, n-dimensional manifolds are **cobordant** if there is a compact manifold of dimension $n+1$ whose boundary is the disjoint union of the two manifolds. The collection of cobordism classes of compact n-manifolds is denoted by $\mathcal{N}_n$ and the union of the $\mathcal{N}_n$ over all dimensions by $\mathcal{N}_*$. This graded set enjoys some extra structure—it is an Abelian group with the addition induced by the disjoint union of n-manifolds. Furthermore, $\mathcal{N}_*$ is a graded ring with the multiplication induced by the Cartesian product of manifolds.

The determination of the structure of the ring $\mathcal{N}_*$ is a celebrated result of Thom [25]. To the orthogonal group $O(n)$ and its universal classifying bundle $EO(n) \to BO(n)$, one associates the Thom space $MO(n)$, which is constructed from the universal n-plane bundle over $BO(n)$ by collapsing the vectors outside the associated unit disk bundle to a point. The operation of adding a trivial line bundle to the universal n-plane bundle induces a mapping $f_n : \Sigma MO(n) \to MO(n+1)$, where ΣX is the suspension of a space X given by

$$\Sigma X = X \times I/(X \times \{0,1\} \cup \{x_0\} \times I).$$

Using transversality arguments, Thom analyzed the sequence of homotopy groups,

$$\cdots \to \pi_{q+n}(MO(n)) \to \pi_{q+n+1}(MO(n+1)) \to \pi_{q+n+2}(MO(n+2)) \to \cdots$$

where the homomorphisms are given by the composite

$$\pi_{q+n}(MO(n)) \xrightarrow{E} \pi_{q+n+1}(\Sigma MO(n)) \xrightarrow{f_{n*}} \pi_{q+n+1}(MO(n+1)).$$

Here E denotes the *Einhängung* or suspension homomorphism of Freudenthal that takes the homotopy class of a mapping $g : S^n \to Y$ to the homotopy class of the mapping $\Sigma g : \Sigma S^n = S^{n+1} \to \Sigma Y$. The long sequence stabilizes to a group dependent only on q that Thom proved to be isomorphic to $\mathcal{N}_q$. Thus the problem of classifying manifolds up to cobordism was reduced to the problem of computing

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certain limit groups of the sequence of spaces and mappings $\{MO(n), f_n\}$. Thom went further to compute the direct limits $\lim_n \pi_{q+n}(MO(n))$ using cohomological arguments.

Similar stabilizing sequences of homotopy groups are implied by the classical Freudenthal suspension theorem: the suspension homomorphism $E : \pi_q(X) \to \pi_{q+1}(\Sigma X)$ is an isomorphism when X is $(n-1)$-connected and $q \leq 2n-1$. For $X = S^n$, the sequence of groups $E : \pi_{n+j}(S^n) \to \pi_{n+j+1}(S^{n+1})$ stabilizes to a single isomorphism class of groups, π_j^S, the j^{th} stable homotopy group of spheres. Computation of these groups remains an important problem in homotopy theory.

Stable homotopy theory concerns phenomena like these that become independent of dimension after enough suspensions. Another stable phenomenon is the suspension isomorphism of cohomology groups with coefficients in a ring R; for $i \geq 1$, $H^i(X; R) \cong H^{i+1}(\Sigma X; R)$. In the case of coefficients in the finite field, $R = \mathbb{F}_p$, the cohomology ring is a module over the Steenrod algebra of stable cohomology operations, and this graded module is independent of dimension because the Steenrod operations commute with the suspension isomorphism.

The systems of spaces, $\{MO(1), MO(2), MO(3), \dots\}$ and $\{X, \Sigma X, \Sigma^2 X, \dots\}$, were generalized by Lima [15] to the notion of a **spectrum**, a collection of spaces, $\mathbf{X} = \{X_n\,;\, n \geq 0\}$, linked together by mappings, $f_n : \Sigma X_n \to X_{n+1}$ (now called a *prespectrum*). Equivalently, we could give the adjoints of the f_n, $f_n' : X_n \to \Omega X_{n+1}$, where $\Omega X = \mathrm{map}((S^1, 1), (X, x_0))$ denotes the space of based loops in X for a choice of basepoint x_0. Up to homotopy, we can choose a spectrum to have all its mappings f_n' be homeomorphisms. The homotopy groups of a spectrum, denoted $\pi_q(\mathbf{X})$, are given by the direct limit of the sequence of homomorphisms,

$$\pi_{q+n}(X_n) \xrightarrow{E} \pi_{q+n+1}(\Sigma X_n) \xrightarrow{f_{n*}} \pi_{q+n+1}(X_{n+1}).$$

Thom's computation of $\mathcal{N}_*$ generalizes to other cobordism theories by choosing the appropriate family of structure groups—$\{SO(n)\}$, for the cobordism theory of oriented smooth manifolds, $\{U(n)\}$ for smooth manifolds with a complex structure on their stable normal bundle. Each choice gives rise to a spectrum $\mathbf{MG}$, whose homotopy groups are isomorphic to the graded ring of cobordism classes of manifolds with structure group in the chosen family.

Other examples of spectra arise from the Brown representability theorem [5]. A generalized cohomology theory $h^*(\ )$, defined on a reasonable category of spaces and satisfying certain conditions, is representable in the sense that there is a system of spaces, $\mathbf{X} = \{X_n\}$ with $h^n(Z) \cong [Z, X_n]$, the set of homotopy classes of maps from Z to X_n. Excision leads to homotopy equivalences, $f_n' : X_n \to \Omega X_{n+1}$, for each n, and hence, a spectrum. By the time of Brown's paper, examples of generalized cohomology theories included topological K-theory, bordism and cobordism theories, and stable cohomotopy. The computation of the coefficients of a generalized cohomology theory, that is, $h^*(\text{a point})$, was made accessible with Adams's introduction of the Adams spectral sequence [1]. Milnor used the Adams spectral sequence in his computation of the cobordism groups associated with $\mathbf{MU}$ [20] in which he determined the structure of $H^*(\mathbf{MU}; \mathbb{F}_p)$ as a module over the Steenrod algebra, a key ingredient in later developments.

By the mid 1960's, Adams had pursued spectral sequence computations of the stable homotopy groups of spheres using K-theory, and Novikov [21] developed the analogous computations using complex cobordism, the generalized theory based on $\mathbf{MU}$. Brown and Peterson [6] introduced spectra BP, based on $\mathbf{MU}$, one for each prime, whose algebraic properties opened new opportunities for the computation of the stable homotopy groups of spheres via the Adams-Novikov spectral sequence. The possibility of the detection of infinite families of elements in these groups, organized in a periodic manner, was introduced by Adams [2] and developed by Toda [26] and Smith [24], who posed the existence of spectra, tailor-made to generalize the successful argument of Adams.

The construction of such spectra ought to be possible using the constructions available for spaces, such as the smash product, cofibre and fibre sequences, Postnikov towers and later, localization and completion. These computations made it clear that a homotopy category of spectra was needed. The objects of such a category are spectra and the morphisms homotopy classes of maps of spectra. The category should be equipped with everything one needs to set up the Adams spectral sequence and prove its convergence. Such a category was set up definitively by Boardman [4].

In the stable homotopy category of spectra, Thom's computation of $\mathcal{N}_*$ can be framed as a homotopy equivalence of spectra, $\mathbf{MO} \simeq \bigvee_i \Sigma^{n_i} K(\mathbb{Z}/2\mathbb{Z})$, where $K(\mathbb{Z}/2\mathbb{Z})$ is the spectrum representing mod 2 cohomology theory and the n_i are determined by Stiefel-Whitney classes. To see where the geometry leaves off and the topology takes over, we could ask if the product structure on $\mathcal{N}_*$ is realized at the level of spectra; that is, is there a mapping of spectra $\mathbf{MO} \wedge \mathbf{MO} \to \mathbf{MO}$ that induces the multiplication of cobordism classes in the isomorphism $\mathcal{N}_* \cong \pi_*(\mathbf{MO})$? The first difficulty that this question poses is how to define the smash product of spectra. At the level of the stable homotopy category of spectra, we require constructions to work up to homotopy. Hence we can use a lot of flabby constructions as long as we keep a strict account of them. Boardman shunted the accounting for the smash product over to linear algebra by introducing the notion of coordinate-free spectra. Let U denote a countably infinite dimensional real inner product space isomorphic to $\mathbb{R}^\infty = \bigcup_n \mathbb{R}^n$. If $V \subset U$ is a linear subspace of finite dimension, then let S^V denote the one-point compactification of V. If we write $\Sigma^V X$ for $X \wedge S^V$ and $\Omega^V X$ for $\mathrm{map}(S^V, X)$, then a spectrum will now be taken as a collection of spaces, EV, one for each finite dimensional linear subspace of U, together with structure maps $f_{V,W} : \Sigma^{W-V} EV \to EW$ whenever $V \subset W$ and $W - V = V^\perp \subset W$. One requires the usual transitivity relations when $Z \subset W \subset V$ and that the adjoints of the structure maps $\tilde{f}_{W,V} : EV \to \Omega^{W-V} EW$ be homeomorphisms. The canonical inner product structure on $\mathbb{R}^\infty$ gives a classical spectrum. In the coordinate-free formulation the definition of the smash product of two spectra $\mathbf{E} \wedge \mathbf{F}$ is obtained by associating $EV \wedge FV'$ to $V \oplus V' \subset U \oplus U$. The isomorphisms between countably infinite dimensional real inner product spaces legislate the identifications required and are organized under the rubric of an operad.

Another development of the 1960's that expanded the store of generalized theories and put new demands on the representing spectra was the axiomatization of equivariant homotopy theory, the algebraic topology of spaces on which a group acts. Notions like the equivariant K-theory of Atiyah and Segal [3] required a foundation that made clear the relations between equivariant and nonequivariant phenomena. In order to fix the representation theory of the transformation group in its expected place, a broader definition of spectrum was required. Boardman's coordinate-free spectra provided a rich enough structure after replacing $\mathbb{R}^\infty$ with a countably infinite dimensional representation space that contains every finite dimensional representation infinitely often. This point of view was fruitfully taken by tom Dieck in his study of equivariant bordism [10], and it was later considerably generalized by May and his coauthors [18].

At the end of the 1960's, Quillen [22] singled out complex cobordism as special among generalized cohomology theories by identifying the connection between complex-oriented theories and formal group laws. The subsequent algebraic framework has led to considerable progress toward a global understanding of stable homotopy theory. The detection of periodic infinite families of elements in the stable homotopy groups of spheres is clearer in this algebraic context. The algebraic structure of $BP_*(BP)$ led Ravenel [23] to conjecture many aspects of the global structure of the stable homotopy category of spectra that were proved in the 1980's by Hopkins and his coauthors [12]. This is the chromatic viewpoint that is central to stable homotopy theory.

Although the picture of stable homotopy theory available through the eyes of complex cobordism is well-structured, the underlying category of spectra and topological maps of spectra is complicated. At the heart of the complications stood the problem of defining the smash product of two spectra in such a way that it is associative at the level of spectra, not simply associative up to homotopy. Without such a notion, the promise of the algebraic picture in which ring spectra and module spectra over such a ring spectrum have a manageable homological algebra is fraught with difficulty. In fact, Lewis [13] showed that a reasonable set of axioms for a symmetric monoidal structure with respect to smash product on a category of spectra and maps between them was too much to ask.

The monograph of Elmendorf, Kriz, Mandell and May (EKMM) overcomes this major difficulty. The principal corollary of what May has termed a 'brave new world' [17] is the potential to develop a satisfying homological algebra of spectra. A ring spectrum $\mathbf{E}$ has a multiplication $\mathbf{E} \wedge \mathbf{E} \to \mathbf{E}$ that is already associative on the point-set level, not merely up to higher homotopies, as in earlier constructions. Associated to a commutative ring spectrum is a category of module spectra over it and a derived category obtained by inverting homotopy equivalences. This includes classical homological algebra as a special case by associating to a commutative algebraic ring R the Eilenberg-Mac Lane ring spectrum HR. The derived category of module spectra over HR agrees with the derived category of chain complexes over the ring. In the case of $\mathbf{MU}$-module spectra, the constructions of familiar spectra are made precise and extended. Furthermore, invariants of algebraic rings, such as Hochschild homology, can be constructed for ring spectra more naturally in this appropriate category of spectra.

The second book surveys the work of May, his coauthors, and students on equivariant homotopy theory, especially in the light of the advances made in EKMM. The collection of essays is the work of ten authors and grew out of a conference that took place in Alaska in 1993. The topics cover the foundations and advances in equivariant homotopy theory with some emphasis on relations between nonequivariant stable homotopy theory and its equivariant analogue. The organizing principles of homotopy limits and colimits, diagrams, and closed model categories are developed, and signal results like Miller's proof of the Sullivan conjecture [19] and Carlsson's proof of the Segal conjecture [8] are presented in context. The exposition and choice of topics by May and his collaborators are well crafted to bring the uninitiated up to speed in a subject that has a long technical past.

The third book under review is based on May's third-quarter first-year graduate course in algebraic topology at the University of Chicago. Algebraic topology is a tool in many areas of mathematics, and its influence will only increase. Like statistics, one might like a first course in algebraic topology that is tailored to one's own needs, be they geometric, differentiable, or homological. This concise course steers its way through the essential parts of the subject, with emphasis on the success of categorical formulations and methods. This may not appeal to every reader, but it makes for a clear and concise account, attractive in its directness.

In previous publications ([7], [9], [11], [14], [18]), May has presented large scale pictures of the state of certain important aspects of algebraic topology, especially as seen through the eyes of his well-developed programs. He has involved his students and collaborators in these projects and subsequently brought to the community the work of new researchers with fresh ideas. That his programs have been successful in generating such new ideas is clear in these publications and the many others published by his students. The concise course notes reviewed here give students the background for the literature that has become part of the foundations of algebraic topology. The 'brave new worlds' of stable homotopy theory and equivariant stable homotopy theory developed by May and his coworkers hold great promise for the future of algebraic topology, and these books offer the reader an entrance into this world.

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