BOOK REVIEWS

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Metric structures for Riemannian and non-Riemannian spaces, by M. Gromov, Birkhäuser, Boston, 1999, xix + 585 pp., $89.95, ISBN 0-8176-3898-9

The concept of *distance* is already present in everyday language, where it refers to two physical objects or even abstract ideas being mutually close or far apart. The most common (but by no means most general) mathematical incarnation of this idea is the notion of a *metric space* (X, d). Here X is an abstract set, and the distance $d(x, x')$ between arbitrary points x and x' in X is a nonnegative real number. The most important restriction on the so-called *distance function* $d : X \times X \to \mathbb{R}$ is the famous *triangle inequality*

$$d(x, x'') \leq d(x, x') + d(x', x'')$$

for all x, x' and x'' in X. In addition one also insists that it is *symmetric*, i.e., $d(x, x') = d(x', x)$ for all x and x' in X, and that it satisfies the *separation axiom*, $d(x, x') = 0$ if and only if $x = x'$.

Metric spaces of all kinds permeate the book under review. The introduction and role of various notions of distances between even general metric spaces is at the heart of the book. In this review we will describe some of these ideas and topics related to them.

The ordinary Euclidean space $\mathbb{R}^n$ with its *pythagorean* distance between points $x = (x_1, \ldots, x_n)$ and $x' = (x'_1, \ldots, x'_n)$ given by

$$d(x, x') = \sqrt{(x_1 - x'_1)^2 + \ldots + (x_n - x'_n)^2}$$

is the archetypical example of a metric space of basic importance to both *geometry* and *analysis*. Subsets with the induced distance function provide a variety of other interesting examples including many discrete or even finite sets. For sufficiently nice subsets $X \subset \mathbb{R}^n$, where any two points can be joined by a *rectifiable curve*, i.e., a path of finite *length*, there is another natural metric, where the distance between x and x' in X is the infimum of lengths of curves joining x and x' inside X. A metric space (X, d) with this property is called a *length* or *inner metric* space. In such spaces, the *geodesics* (locally shortest curves) play a significant role in the *geometry* of the space. If X is a smooth submanifold of $\mathbb{R}^n$, its induced length metric is *Riemannian*. Although this hides much of the beauty and richness of *Riemannian geometry* emerging from its metric tensor g (an inner product in each

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tangent space), any Riemannian manifold M can be defined in this way according to the famous embedding theorem of J. Nash.

By means of a metric $d : X \times X \to \mathbb{R}$ it is possible to express notions of *convergence, size* and *shape*. Typical examples of size are the *diameter*, $\operatorname{diam}(X) = \max d(x, x')$, and *volume*, $\operatorname{vol}(X)$ (Riemannian or α-dimensional Hausdorff measure), of a space X. Other, often more complicated, *metric invariants*, are used to describe local or global shape. The emperor among all these is *curvature* in all of its guises. The idea behind curvature is to express, infinitesimally, local or global deviation from flatness as exhibited in *euclidean geometry*. The mathematical notion originated in the study of smooth surfaces. It was Gauss who discovered that the apparently extrinsic notion of curvature of a surface $M^2 \subset \mathbb{R}^3$, measuring how it bends (in terms of principal curvatures), is indeed intrinsic and can be detected by the *angle sum* of geodesic triangles on the surface. If $L(r)$ is the length of the boundary of a small ball of radius r around a point p on the surface, the (Gauss)curvature K_p at p can be expressed in terms of the Taylor expansion for $L(r)$ by

$$L(r) = 2\pi r - \frac{2\pi}{6} K_p r^3 + O(r^4).$$

Here the first term, $2\pi r$ is exactly the formula in the flat euclidean plane $\mathbb{R}^2$. In general, for a Riemannian manifold, Riemann introduced the *curvature tensor* R_p in terms of the Taylor expansion of the metric tensor g at the point p in M. Algebraic manipulations (notably taking traces) with the curvature tensor lead to other curvature invariants, the most important ones being *sectional curvature, Ricci curvature, and scalar curvature.*

The sectional curvature assigns to any two-dimensional subspace P of the tangent space at a point $p \in M$ a number $\sec(P)$. For surfaces this is the Gauss curvature, and in general complete information about sectional curvature is equivalent to complete information about the curvature tensor. It controls the local expansion/contraction behavior of geodesics emanating from a point compared with that of euclidean geometry. A lower bound on the sectional curvature, e.g. $\sec \geq 0$, is equivalent to global comparison of geodesic triangles; i.e., triangles in M are "fatter" (have larger angles) than euclidean triangles with the same side lengths. This so-called Toponogov *comparison theorem* is the key to most global results for manifolds with a lower (sectional) curvature bound.

The Ricci curvature assigns to any one dimensional subspace L of the tangent space at a point $p \in M$ a number $\operatorname{Ric}(L)$ (the sum of sectional curvatures of two planes spanned by L and an orthonormal basis of its complement). Its most direct geometric significance is related to volume control. In particular, if $\operatorname{Ric} \geq 0$ and $\operatorname{vol} B_0(r) = c_n r^n$ denotes the volume of an r-ball in euclidean n-space, then the relative volume function

$$F(r) = \operatorname{vol} B(p, r) / \operatorname{vol} B_0(r) \tag{1}$$

is nonincreasing and $F(r) \to 1$ as $r \to 0$. This was first proved by Bishop for small r (depending on p) and globally by Gromov in the original French version [35] of the book under review. It was also Gromov who pointed out the real significance of this *relative volume comparison estimate* in many different contexts. More generally, this estimate holds for arbitrary *sectors*, i.e., sets consisting of minimal geodesics emanating from a fixed point p. In this generality it is equivalent to a lower bound for Ric. In sharp contrast to sectional curvature, any manifold of dimension at least 3 admits a metric of negative Ricci curvature. This striking result was first proved by Gao and Yau [29] in dimension three, and then by Lohkamp [49] in all dimensions.

The scalar curvature assigns to any point $p \in M$ a number (the sum of Ricci curvatures of lines spanned by an orthonormal basis at p). It controls volume of balls only infinitesimally, i.e., enters in the Taylor expansion for $\operatorname{vol} B(p, r)$. Although it has little metric significance (almost all "geometry" has been "washed out"), this weakest curvature measure is still restricted by the topology of the manifold in general (many manifolds do not admit a metric with positive scalar curvature). The large and beautiful body of work devoted to understanding relations between topology and scalar curvature involves a mixture of analytic and topological methods (see e.g. [69], [63] and [48]).

As indicated above (local) bounds on sectional curvature are expressible in purely metric terms via bounds on fatness/slimness of small geodesic triangles. This approach to curvature in length spaces with sufficiently many geodesics was pioneered and developed by A.D. Alexandrov and his school. It is amazing that up to a small loss of regularity of the metric tensor (g is only $C^{1,\alpha}$ in general), complete Riemannian manifolds (with locally convex boundary) can be characterized as finite dimensional complete inner metric spaces with locally bounded curvature according to Nicolaev [52] and Plaut [59].

Geodesic spaces, i.e., metric spaces in which any two points are joined by a minimal curve (e.g. any locally compact complete inner metric space), in which arbitrary geodesic triangles are slimmer than euclidean ones are commonly referred to as CAT(0) spaces. These possibly quite singular spaces have played a significant role in recent years, for example in the context of rigidity problems and geometric group theory [6], [30], and even in billiard problems [7]. When the space is the graph of a group, a similar idea yields the notion *Gromov hyperbolic groups*. This concept has been intensely investigated for algebraic, geometric and topological reasons (see e.g. [25]).

A purely metric approach to curvature, which can be used in general even for finite metric spaces, goes back to A. Wald. According to Berestovskii [4], we say that $curvX \geq k$ if any four tuple of points $x = (x_0, x_1, x_2, x_3) \in X^4$ can be isometrically embedded in the simply connected 3-manifold $S^3_{k(x)}$ with constant curvature $k(x) \geq k$. This is equivalent to the statement

$$\angle_{1,2}(k) + \angle_{2,3}(k) + \angle_{3,1}(k) \leq 2\pi \tag{2}$$

where $\angle_{i,j}(k)$, the so-called *comparison angle*, is the angle at $x_0(k)$ in the geodesic triangle in S^2_k with vertices $(x_0(k), x_i(k), x_j(k))$ the isometric image of (x_0, x_i, x_j). It was also Berestovskii who observed that for Riemannian manifolds M this condition is equivalent to $\sec M \geq k$. The class of finite dimensional complete inner metric spaces with a lower curvature bound, so-called *Alexandrov spaces* (curved from below), have an astoundingly rich structure developed primarily by Perelmann [53] and in [8] (cf. also [60] for a dimension independent approach). Locally such spaces are conelike and include all orbit spaces of Riemannian manifolds by proper isometric actions. This structure is obtained by extending the *critical point theory* for (non-smooth) distance functions in Riemannian geometry, originating in [44] (cf. [11] and [38]), to Alexandrov spaces.

Metric aspects of Riemannian geometry and related topics (including the ones alluded to above) have witnessed a tremendous evolution over the last few decades. Much of this is in one way or another tied to concepts for "closeness" between different Riemannian manifolds or even general metric spaces. For manifolds this development can be traced back on the one hand to Shikata's work on the differentiable sphere theorem [68], where he introduced a notion of (Lipschiz) distance between differentiable structures on a smooth manifold, and on the other hand to Cheeger's work on *finiteness problems* [10] and the general approach to *pinching theorems* [9] (such theorems assert that a manifold has the same type as one of a suitable collection of model spaces if some of its geometric invariants are similar to those of the model spaces). In his thesis, the idea that abstract manifolds can converge to each other is also present. The fact that the class of closed n-manifolds M with arbitrary fixed bounds

$$|\sec M| \leq C, \quad \operatorname{diam} M \leq D \quad \text{and} \quad \operatorname{vol} M \geq v > 0 \tag{3}$$

contains at most finitely many diffeomorphism types is a consequence of the interpretation that this class is precompact in a certain topology where sufficiently close manifolds are diffeomorphic.

The idea of measuring the distance between subspaces of a given metric space goes back to Hausdorff. If (X, d) is a metric space and $A, B \subset X$ are compact subsets, the Hausdorff distance between A and B is given by

$$d_H^X(A, B) = \inf\{\epsilon | D_\epsilon(A) \supset B, D_\epsilon(B) \supset A\}$$

where $D_\epsilon(A) = \{x \in X | d(x, A) \leq \epsilon\}$ is the ϵ-neighborhood of A in X. This idea was extensively studied in the Russian and Polish schools led by Urysohn and Borsuk.

The dramatic phase transition came with Gromov's far-reaching idea to extend the Hausdorff distance to arbitrary (compact) metric spaces. This distance is now called the *Gromov-Hausdorff* distance and is denoted by d_{GH}. If A and B are two abstract compact metric spaces, $d_{GH}(A, B) \leq \epsilon$ if A and B admit isometric embeddings into a metric space X and $d_H^X(A, B) \leq \epsilon$. The actual distance is then the infimum of all such distances for all X and all isometric embeddings. It turns out that it suffices to take $X = A \coprod B$, the disjoint union of A and B, and consider all metrics on $X = A \coprod B$ extending the ones on A and on B. Thus

$$d_{GH}(A, B) = inf_{X = A \coprod B} d_H^X(A, B).$$

A simple but illustrative example is to take $A = pt$ and $B = \{x_0, x_1, x_2\}$ with all distances equal to 1. Then $d_{GH}(A, B) = \frac{1}{2}$. The Gromov-Hausdorff distance is indeed a distance function on the collection of isometry classes of compact metric spaces. For non-compact but locally compact spaces there is a natural notion of *pointed convergence* based on convergence of balls with a fixed center. With this notion the tangent space T_pM of a Riemannian manifold at a point p is the pointed Gromov-Hausdorff limit of the scaled manifolds $(\lambda M, p)$ with the scale λ blowing up to infinity. By their very definition, these notions of distance/topology are very *coarse*. For example, by definition of compactness it is clear that any such space can be approximated arbitrarily well by finite metric spaces; i.e., the collection of finite metric spaces is Gromov-Hausdorff dense in the space of all compact metric spaces. This coarseness is both a strength and a weakness: Most anything converges, but limit spaces are in general not of much use. In fact, what is probably most shocking about this metric is how powerful it actually is. The first spectacular application of this idea was Gromov's solution of the Milnor conjecture for groups of *polynomial growth* (the number of words of length at most ℓ in a fixed finite set of generators of the group grows at most polynomially in ℓ). Based on the apparent naive idea that the integers $\mathbb{Z}$ when viewed as a metric space converge to the real numbers $\mathbb{R}$ when the metric on $\mathbb{Z}$ is scaled to zero, Gromov [34] proved that

> Any group of polynomial growth is a finite extension of a lattice in a nilpotent Lie group.

In contrast to this impressive result, the following very useful so-called *Gromov compactness criterion* is quite easy to prove :

> A space $\mathcal{C}$ of compact metric spaces is Gromov-Hausdorff precompact if and only if for every $\epsilon > 0$, any $X \in \mathcal{C}$ can be covered by the same number of ϵ-balls.

It then follows directly from the relative volume estimate (1) that the class of closed Riemannian n-manifolds M with bounds

$$\operatorname{Ric} M \geq C \quad \text{and} \quad \operatorname{diam} M \leq D \tag{4}$$

is relatively compact in the Gromov-Hausdorff topology. Hence for any ϵ there are finitely many manifolds $M_1, \ldots, M_{k(\epsilon)}$ from this class such that any other manifold with these properties is at most ϵ away from one of these finitely many. This led to the natural question whether there might be any topological finiteness properties for this class. Many examples, e.g., with positive Ricci curvature, have been constructed (cf. [67] and [55]) showing for example that there is not even a bound on the *Betti numbers* except for $b_1(M^n) = 0$ (Myers theorem). When the Ricci curvature is nonnegative, $b_1(M^n) \leq n$ (Bochner). The problem is that spaces in the Gromov-Hausdorff closure can be very complicated and are not in general obviously related to the manifolds close to them. Nonetheless, Gromov-Hausdorff convergence techniques have played a central role in the most recent far-reaching progress due to Cheeger and Colding in understanding manifolds with a lower bound on Ricci curvature (see e.g. [19]). The main breakthrough came with Colding's $L^2 -$ *average version* of Toponogov's triangle comparison theorem for *thin triangles* [17]. Prior to that Abresch and Gromoll [1] had obtained an estimate for the *excess* (failure of triangle inequality from equality) of thin triangles. This delicate estimate can be viewed as a weakened finite quantitive version of the Cheeger-Gromoll *splitting theorem* [15], asserting that a *line* (geodesic which is minimal between any two of its points) splits off isometrically in a complete manifold of nonnegative Ricci curvature. These and other new techniques allow one to transfer the splitting theorem and Cheng's *maximal diameter/volume theorem* [16] to limit spaces [12], and yields among other things corresponding almost rigidity results for manifolds. It follows in particular that an n-manifold M with $\operatorname{Ric} M \geq \operatorname{Ric} S_1^n$ and $\operatorname{vol} M \geq \operatorname{vol} S_1^n$-$\epsilon$ is Gromov-Hausdorff close to the unit sphere S_1^n and diffeomorphic to it [17], [13]. Also a compact manifold with almost nonnegative Ricci curvature and $b_1(M) = n$ is diffeomorphic to the n-torus $T^n = S^1 \times \ldots \times S^1$ [18], [13].

One might expect that the topology induced by the Gromov-Hausdorff metric is stronger when restricted to smaller classes. Indeed, if we replace the lower bound for Ricci curvature with one for sectional curvature, one gains much more control on the limit objects since the distance comparison expressed in (2) is preserved in the limit. In particular the Gromov-Hausdorff limit, $X = \lim M_i$, of a sequence of Riemannian

n-manifolds M_i with $\sec M_i \geq k$ is an Alexandrov space with $\operatorname{curv} X \geq k$. Even for this class of manifolds, though, one still knows very little in general when *collapse* occurs, i.e., when $\dim X < \dim M_i$ (see [71] and [27] though). By an ingenious use of critical point theory for distance functions (and no use of convergence) Gromov [33] was able to prove his fabulous *Betti number finiteness theorem*: For any n, k and D there is an a priori bound $C = C(n, k, D)$ for the number of generators for the homology $H_*(M)$ of any n-manifold M with $\sec M \geq k$ and $\operatorname{diam} M \leq D$. When $k = 0$, D is obviously irrelevant due to the fact that this class is scale invariant.

For the smaller class where $\sec M \geq k$, $\operatorname{diam} M \leq D$ and in addition vol $M \geq v > 0$, the Gromov-Hausdorff convergence technique is strong enough to yield *finiteness of topological types* [42], [54], and hence via smoothing theory also of *diffeomorphism types* in all dimensions except possibly in dimension four. Again only critical point theory is needed for finiteness of homotopy types [40]. The homeomorphism result in all dimensions (including 3 left out in [42]) follows from Perelmann's amazing *stability theorem* for Alexandrov spaces [54]:

> If X is a compact n-dimensional Alexandrov space with $\operatorname{curv} X \geq k$, then any other compact n-dimensional Alexandrov space Y with $\operatorname{curv} Y \geq k$ Gromov-Hausdorff close to X is homeomorphic to it.

Since a lower volume bound prevents collapse, the Gromov-Hausdorff closure of the class of all closed Riemannian n-manifolds M with $\sec M \geq k$, $\operatorname{diam} M \leq D$ and $\operatorname{vol} M > v$ is a compact subset of the class of all n-dimensional Alexandrov spaces X with $\operatorname{curv} X \geq k$ and $\operatorname{diam} X \leq d$. The finiteness result is then an immediate consequence of the stability theorem. The stability theorem is also instrumental in achieving *recognition* type results for manifolds with almost extremal metric invariants of various types, as explained in [39] and first illustrated in [41] (these results are like pinching theorems except one does not know the model spaces ahead of time, and in general the model spaces that emerge are singular spaces).

It should be pointed out that Alexandrov geometry with lower curvature bounds is useful to Riemannian geometry not only via convergence techniques as described above. The reason is that there are other natural *operations* which are closed within Alexandrov geometry but not in Riemannian geometry. These include taking *quotients* by proper isometric group actions and taking *joins* among positively curved spaces (see e.g. [46], [43], [61], [62] and [45]).

A big difference between upper and lower curvature bounds is that there is no global triangle comparison for upper curvature bounds. This explains why upper curvature bounds are not in general preserved under the process of taking Gromov-Hausdorff limits. If this comparison holds in balls of a fixed size, however, then the upper bound carries over to the limit. This is crucial in the study of spaces with *nonpositive* curvature, since their universal covers, so-called Hadamard spaces, have this property even globally. This is the basis for the importance of convergence techniques in negative and nonpositive curvature.

For manifolds with bounded (sectional) curvature $|\sec M| \leq C$, one of the key points in Cheeger's proof of his finiteness theorem is that with an upper bound on the diameter $\operatorname{diam} M \leq D$, a lower volume bound is equivalent to a lower bound on the so-called *injectivity radius* (largest r such that all geodesics of length at most r are minimal). For this class of manifolds, Gromov-Hausdorff convergence is very strong; as a matter of fact it is equivalent to $C^{1,\alpha}$ convergence of metric tensors (see Anderson [2] for an extension to bounded Ricci curvature). This also provides the natural link to Nikolaev's work mentioned earlier.

In the context of bounded curvature there is a well developed theory for collapse due to Fukaya, Gromov and Cheeger. This is anchored in Gromov's milestone theorem for *almost flat manifolds*, i.e., manifolds with bounded diameter and (arbitrary) small curvature bounds [31]: Any such manifold is up to a finite cover a quotient of a nilpotent Lie group by a discrete subgroup. For the ultimate result see Ruh [64]. Although the proof of this result is not based on convergence, the idea behind it probably was. Note that almost flatness for M can be expressed as well by saying that M can collapse to a point with bounded curvature. In general, the presence of nilpotent groups is imminent when collapse occurs with bounded curvature. In vague terms such collapse yields a decomposition of the manifold into submanifolds, a singular foliation, whose leaves in local covers are orbits by actions of nilpotent groups. Moreover, the collapse takes place along these (infra)nilmanifolds (see [14]). This structure and additional convergence techniques have recently been used to obtain the following remarkable analogue of Cheeger's finiteness result for two-connected manifolds with bounded curvature and diameter, but no restrictions on volume [58] (cf. also [22]):

> The class of simply connected closed Riemannian n-manifolds M with finite $\pi_2(M)$, $|\sec M| \leq C$ and $\operatorname{diam} M \leq D$ contains at most finitely many diffeomorphism types.

When combined with Gromov's Betti number theorem, one arrives at the following amazing result [58]:

> For each n, C, and D, there exist a finite number of manifolds $M_1, \dots ,$ $M_{k(n,C,D)}$, such that any simply connected n-manifold, M with $|\sec M|$ $\leq C$ and $\operatorname{diam} M \leq D$ is diffeomorphic to a torus quotient of one of the M_i's.

The convergence ideas described above are well suited to describe and analyze *asymptotic properties/quasi-isometry types* of noncompact spaces. This enters significantly into *rigidity* aspects of nonpositively curved spaces (e.g. [20], [23], [24], [28], [47], [65], [66] and [70]) and via covering space theory into the geometry and large scale invariants for infinite groups [36].

Other geometries of interest in their own right as well, such as *Tits geometry* [3] and *Carnot-Caretheodory (or sub Riemannian) geometry* [37], also arise naturally in this context.

Much of the astounding development around Riemannian geometry described or alluded to above owes much to Gromov's inspirational deep insights and visions (cf. also [32]). The original French version of the book under review, *Structures Métriques pour les Variétés Riemanniennes*, written by J. Lafontaine and P. Pansu, arose from a course by Gromov at the University of Paris VII during the third semester of 1979. The purpose of that book was to describe some of the connections between the curvature of a Riemannian manifold M and some of its global properties reflected not only in its topology but also in relation to other metric invariants of the manifold and mappings between spaces. The influence of this "little green book" can hardly be overestimated. Its 150 pages were packed with striking new concepts and ideas. In particular, it was this book that spread the idea of convergence of Riemannian manifolds to a larger audience. Except for various survey articles (e.g. [26] and [56]), the only other text that treats this topic is the book by

Petersen [57]. In addition, new light was shed on classical topics such as *quasiconformal* maps, *isoperimetric-* and *isosystolic* inequalities.

Despite the fact that the current "translation", *Metric Structures for Riemannian and Non-Riemannian Spaces*, has quadrupled in size, most of the development described in this review is not treated in the book, at most hinted at. This illustrates not only the reviewer's personal taste and perspective on the place and influence of the original book, but also how much this general area stretching somewhere between the fields of topology and global Riemannian geometry has expanded during the last two decades.

In addition to natural elaborations and extensions of topics treated in the original version, the main additions in the new book are concerned with *relations* between *geometry* and *probability*, in particular pertaining to convergence theory. This development was stimulated as well by the Levy-Milmann *concentration phenomenon* [50], [51], encompassing the *law of large numbers* for metric spaces with measures and dimension going to infinity. This topic occupies a whole new chapter in the book and most of it has not been published elsewhere. As in the previous book, this addition contains a wealth of new ideas and concepts, including various notions of convergence of metric spaces with measure, and associated invariants such as *observable diameter* (see [5] for more details). One of the four appendices in the book is a reproduction of Gromov's important unpublished manuscript *Paul Levy's Isoperimetric Inequality*. The others are by P. Pansu, *"Quasiconvex" Domains*; by M. Katz, *Systolically Free Manifolds*; and by S. Semmes, *Metric Spaces and Mappings Seen at Many Scales*. The latter one is the most voluminous of them all. Here Stephen Semmes makes a delightful presentation of an analyst's view of metric spaces, including several key ideas of real analysis made inviting to geometrically inclined readers.

This is an unconventional book. It is too advanced and not detailed enough to be a *textbook*, and too broad and not sufficiently comprehensive (in providing proofs) to be a *research monograph*. In many ways it has the spirit of *lecture notes*. Although it is unlike others in the series, Birkhäuser's Progress in Mathematics is obviously an appropriate home for this book. Its treasure house of ideas and concepts is presented in a relaxed and rather unpolished style. This style, although pleasant in some ways, can also lead to frustration. Quite frequently (in particular in the new chapter about convergence of metric spaces with measures) the reader is urged to check or investigate things carefully for himself/herself: "(perhaps by looking through the literature)", p. 156; or since: "(I have not gone through the details of this exercise myself)", p. 157; or directly as in: "We suggest that the reader fill in the details by setting in order the qualifiers and chasing all the ϵ's and δ's (this will go smoothly unless we missed something in our II and/or III)", p. 208. At times it appears as if Gromov is aware that the reader might suffer: "We now plunge into the muddy waters of mixed algebraic and metric geometry, and we invite the courageous reader to swim along through a dozen pages until the finish in (G''')", p. 165. Just the number of ideas and different notions, and more or less complicated definitions of distances in this chapter, is overwhelming. I have no doubt that with sufficient persistence, the frustration will pass and one will be ready to reap the benefits (unfortunately I did not reach this point, but I urge others to try). If the original French book is any indication, we should all look forward to the influence this book will have over the next decades.

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