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Classical invariant theory, by Peter Olver, London Mathematical Society Student Texts, vol. 44, Cambridge Univ. Press, New York, 1999, xxi+280 pp., $21.95, ISBN 0-521-55821-2

Classical invariant theory was a hot topic in the 19$^{\text{th}}$ century and in the beginning of the 20$^{\text{th}}$ century. The book under review is an attempt to revive this beautiful subject. In our new era of computers and new interest in computational aspects, it is certainly worthwhile to recall the constructive methods of the 19$^{\text{th}}$ century invariant theorists.

In classical invariant theory one studies polynomials and their intrinsic properties. The book mostly deals with polynomials in one variable, or rather, homogeneous polynomials in two variables which are called *binary forms*.

A first example is treated in Chapter 1. Let $Q(x) = ax^2+bx+c$ be the quadratic polynomial[1] (throughout the book, coefficients are always either real or complex). As we learned in high school, an important characteristic of a quadratic polynomial is the discriminant $\Delta = b^2 - 4ac$.[2] The discriminant tells us for example whether there are 0, 1 or 2 real solutions, and it is invariant under translation $x \mapsto x + \alpha$. We can view Q as a *binary form of degree 2* by writing $Q(x, y) = y^2 Q(\frac{x}{y}) = ax^2 + bxy + cy^2$. We now have more symmetries available, namely the projective transformations

$$(x, y) \mapsto (\alpha x + \beta y, \gamma x + \delta y)$$

with nonzero determinant $\alpha\delta - \beta\gamma \neq 0$. Let us make a change of coordinates

$$\begin{aligned} \overline{x} &= \alpha x + \beta y \\ \overline{y} &= \gamma x + \delta y \end{aligned} \tag{1}$$

and let $\overline{Q}$ be the polynomial in two variables such that

$$\overline{Q}(\overline{x}, \overline{y}) = Q(x, y). \tag{2}$$

If $\overline{\Delta}$ is the discriminant of $\overline{Q}$, then $\Delta = (\alpha\delta - \beta\gamma)^2\overline{\Delta}$. This shows that the discriminant is invariant under unimodular transformations and semi-invariant under linear transformations (i.e., it only changes up to a scalar).

Let

$$Q(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$$

be a binary form[3] of degree n. A polynomial

$$I(\mathbf{a}) = I(a_0, a_1, \ldots, a_n)$$

in the coefficients of Q is called an *invariant of weight k*, if $I(\mathbf{a}) = (\alpha\delta - \beta\gamma)^k I(\overline{\mathbf{a}})$, where $\overline{a}_0, \ldots, \overline{a}_n$ are the coefficients of $\overline{Q}$ defined by (2) and (1).

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[1] In the book, the standard form $ax^2 + 2bx + c$ is used.

[2] The discriminant is defined with opposite sign in the book, conforming to a more general definition later on.

[3] The book follows the tradition of putting binomials in front of the coefficients.

For binary forms Q of arbitrary degree d one can also define the discriminant $\Delta[Q]$ using resultants. The discriminant is an invariant of weight $d(d-1)$. The discriminant vanishes if and only if Q has multiple zeroes. For $d=2$ and $d=3$, the quadric and the cubic, the discriminant is essentially the only invariant; i.e., any other invariant can be expressed as a polynomial in the discrimiant.

Besides invariants, also the more general *covariants* are considered. A covariant is a polynomial

$$J(a_0, a_1, \ldots, a_n, x, y),$$

homogeneous in the variables $\{x, y\}$, such that $J(\mathbf{a}, x, y) = (\alpha\delta - \beta\gamma)^k J(\overline{\mathbf{a}}, \overline{x}, \overline{y})$. In other words, a covariant is a binary form whose coefficients are polynomials in the coefficients of Q which is invariant under unimodular transformations. One of the most important covariants is the Hessian defined by

$$\Big(\frac{\partial^2}{\partial x^2}Q\Big)\Big(\frac{\partial^2}{\partial y^2}Q\Big) - \Big(\frac{\partial}{\partial x}\frac{\partial}{\partial y}Q\Big)^2.$$

The Hessian vanishes exactly when Q is the d^{th} power of a linear function.

A classical problem in invariant theory is to understand all the invariants and covariants for binary forms of a given degree. One would like to have a finite set $J_1, J_2, \ldots, J_r$ of covariants such that every covariant is a polynomial in those covariants. Such a set is called a *Hilbert basis*. It is not at all clear that such a finite Hilbert basis always exists. However, in 1868, Paul Gordan wrote a paper entitled "Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Funktion mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist". In that paper Gordan does exactly what he promises in the title, namely prove that for binary forms there is always a finite Hilbert basis.

At that time, it seemed very difficult to extend Gordan's result to homogeneous polynomials in more than 2 variables. It was in 1890 when Hilbert proved his general Basis Theorem, showing that for a very large class of examples, a finite Hilbert basis will exist. The methods of Hilbert were abstract, not constructive and completely new. For example, Hilbert proved his famous Nullstellensatz and Syzygy Theorem as lemmas for his results in Invariant Theory. Thus, his invariant theory papers did not only have a big impact on Invariant Theory, but also on other areas of mathematics like Commutative Algebra and Algebraic Geometry. The result of Hilbert almost put an end to the computational classical invariant theory approach. Olver describes it as follows in the introduction:

"Hilbert's paper did not immediately kill the subject, but rather acted as a progressive illness, beginning with an initial shock, and slowly consuming the computational body of the theory from within, so that by the early 1920's the subject was clearly moribund. Abstraction ruled: the disciples of Emmy Noether, a student of Gordan, led the fight against the discredited computational empire, perhaps as a reaction to Noether's original, onerous thesis topic that involved computing the invariants for a quartic form in three variables."

It is clear that the book under review needs only a small amount of prerequisites for the reader. So far the first two chapters have been described, and they only use elementary algebraic manipulation of polynomials. A knowledge of Group Theory or Representation Theory is not a requirement. In fact, in Chapters 3 and 4, groups, representation theory and some elementary invariant theory are introduced.

Covariants have a very rich structure (a point well-proven by this book). We can multiply and compose covariants, but there is yet another construction of covariants. Given two covariants $P(x,y)$ and $Q(x,y)$ respectively we can define so-called *transvectants*. Transvectants can be defined by directly appealing to the representation theory of SL_2 and the Clebsch-Gordan formula for tensor products of irreducible representations. In this book, the author uses differential operators to define transvectants. Consider the differential operator

$$\Omega_{\alpha,\beta} = \frac{\partial}{\partial x_\alpha}\frac{\partial}{\partial y_\beta} - \frac{\partial}{\partial y_\alpha}\frac{\partial}{\partial x_\beta}$$

where $x_\alpha, x_\beta, y_\alpha, y_\beta$ are variables. We define the k^{th} transvectant $(P,Q)^{(k)}$ of P and Q by

$$(P,Q)^{(k)} = \Omega^k_{\alpha,\beta} P(x_\alpha, y_\alpha) Q(x_\beta, y_\beta)\Big|_{\substack{x_\alpha=x_\beta=x \\ y_\alpha=y_\beta=y}} .$$

For example the Hessian is the second transvection $H = \frac{1}{2}(Q,Q)^{(2)}$ up to a scalar. If Q is a binary form, then starting with Q one can obtain new covariants by taking transvectants of covariants already constructed. In this way, one can obtain all covariants.

The construction of transvectants can be generalized to *partial transvectants*. For example, if P, Q, R are three covariants, and k, l and m are integers, then we can obtain a new covariant

$$\Omega^k_{\alpha,\beta}\Omega^l_{\beta,\gamma}\Omega^m_{\alpha,\gamma} P(x_\alpha, y_\alpha) Q(x_\beta, y_\beta) R(x_\gamma, y_\gamma)\Big|_{\substack{x_\alpha=x_\beta=x_\gamma=x \\ y_\alpha=y_\beta=y_\gamma=y}} .$$

It is clear how to generalize this to partial transvectants of arbitrarily many covariants. Besides the Omega operators, one also considers the operators $\sigma_\alpha = x_\alpha \frac{\partial}{\partial x_\alpha} + y_\alpha \frac{\partial}{\partial y_\alpha}$ which acts on $Q(x_\alpha, y_\alpha)$ as multiplication with the degree.

Let Q be a binary form. Covariants of Q, which were constructed by partial transvections and the scaling operators σ_α, can be written in the so-called *symbolic form*. For example, for the transvectant $(Q,Q)^{(k)}$ we could simply write $[\alpha\ \beta]^k$. For the covariant

$$\Omega^k_{\alpha,\beta}\Omega^l_{\beta,\gamma}\Omega^m_{\alpha,\gamma}\sigma^a_\alpha\sigma^b_\beta\sigma^c_\gamma Q(x_\alpha, y_\alpha) Q(x_\beta, y_\beta) Q(x_\gamma, y_\gamma)\Big|_{\substack{x_\alpha=x_\beta=x_\gamma=x \\ y_\alpha=y_\beta=y_\gamma=y}}$$

we would write $[\alpha\ \beta]^k[\beta\ \gamma]^l[\alpha\ \gamma]^m(\alpha\ \mathbf{x})^a(\beta\ \mathbf{x})^b(\gamma\ \mathbf{x})^c$. All covariants constructed in this way can be written as polynomials in *brackets* of the form $[\alpha\ \beta]$ and $(\alpha\ \mathbf{x})$.

The variables $\alpha, \beta, \gamma, \ldots$ are symbolic, and they may be permuted. For example, $[\alpha\ \beta]^2[\alpha\ \gamma]$ is the same covariant as $[\beta\ \alpha]^2[\beta\ \gamma]$ because one can be obtained from the other by interchanging α and β. Besides this, there are also relations between the brackets. In fact, all relations are generated by the ones below:

$$[\alpha\ \beta] = -[\beta\ \alpha],$$

$$[\alpha\ \beta][\gamma\ \delta] + [\gamma\ \alpha][\beta\ \delta] + [\beta\ \gamma][\alpha\ \delta] = 0,$$

$$[\alpha\ \beta](\gamma\ \mathbf{x}) + [\beta\ \gamma](\alpha\ \mathbf{x}) + [\gamma\ \alpha](\beta\ \mathbf{x}) = 0.$$

In the traditional symbolic method, all variables $\alpha, \beta, \ldots$ must appear exactly d times where d is the degree of Q.

The symbolic form $[\alpha\ \beta]$ represents a trivial covariant, because $[\alpha\ \beta] = [\beta\ \alpha] = -[\alpha\ \beta]$.

To visualize algebraic manipulations with symbolic forms, Clifford developed a graphical theory. For a symbolic form we can write a directed graph as follows. For each symbol $\alpha, \beta, \dots$ we have a vertex. For each $[\alpha\ \beta]$ we will draw an arrow from α to β. Up to a scalar, the Hessian is represented by

$$\circ \rightrightarrows \circ\,.$$

The pictures obtained in this way look a bit like molecules. Sometimes this visualization is called the chemical method, and one can use notions like *atoms* (for vertices), *molecules* (for graphs), *valence* and *chemical reaction* for some algebraic manipulations such as transvections. Sylvester got carried away with this. Olver writes, "Sylvester unveiled his 'algebro-chemical theory', the aim of which was to apply the methods of classical invariant theory to the then rapidly developing science of molecular chemistry. Sylvester's theory was never taken very seriously by chemists, and not developed any further by mathematicians, and soon succumbed to a perhaps well-deserved death."

Using the chemical method, transvections can be visualized. At the end of Chapter 7, Gordan's algorithm for finding a finite basis for the ring of covariants is explained. However, for the termination of the algorithm, Olver refers to Hilbert's basis theorem (which is proven in Chapter 9), or to Grace and Young (see [3]).

In Chapter 8, Lie Groups and moving frames are discussed. In this section, the author moves to the field of differential geometry. The Cartan theory of moving frames and differential invariants are discussed. A highlight in this section is the signature curve. Given a curve $\mathcal{C}$ in $\mathbb{R}^2$, we can view the set of all

$$S_{\mathcal{C}} = \{(\kappa(\mathbf{x}), \kappa_s(\mathbf{x}) \mid \mathbf{x} \in \mathbb{R}^2\}$$

where $\kappa(\mathbf{x})$ is the curvature and κ_s is the derivative of κ with respect to the arclength. The set $S_{\mathcal{C}}$ is again a curve in $\mathbb{R}^2$. The importance of the signature curve is the following. A curve $\mathcal{C}$ can be transformed to another curve $\mathcal{C}'$ by a Euclidean orientation-preserving transformation if and only if their signature curves $S_{\mathcal{C}} = S_{\mathcal{C}'}$ are the same. This result can be applied to the computer vision problem of object recognition (see [1]). A similar construction gives the definition of the signature curve S_Q of a binary form Q. The method of signature curves gives in this case a method for deciding whether one binary form can be transformed to another binary form by a transformation of the form (1).

In Chapters 9 and 10 generalizations of the theory of binary forms in more general settings are considered. First, Lie algebras are introduced. Their actions on forms are studied, and this is used in the proof of Hilbert's basis theorem. In Chapter 10, the author discusses the generalization of the symbolic method to forms in more than 2 variables.

The book under review is an elementary exposition of the classical theory of binary forms. The book is well-suited for advanced undergraduate students and graduate students who are trying to learn the subject. The level of abstraction is kept as low as possible. The book is self-contained and includes introductions to related subjects such as Lie groups, Lie algebras and Representation Theory. The book has a reasonable number of exercises, spread throughout the book.

The author does not hide his background in differential geometry. Throughout the book, the base field is the real or complex numbers. A large portion of Chapter 8 is dedicated to problems in differential geometry. Many of the constructions use differential operators. Olver also introduces a Fourier-like transform of differential polynomials, which can be used to compare the approach of the book with the more algebraic classical techniques.

As a modern in-depth study of binary forms, this book is one of a kind. The book uses a rather ad-hoc approach to binary forms. This keeps the abstraction as low as possible. If one is more interested in invariant theory and representation theory for arbitrary classical groups, centered around Weyl's "The Classical Groups" (see [7]), then [4] is probably a better reference.

The book is written in a pleasant style. The author cares a great deal about the subject. The many historical comments make it a quite enjoyable read.

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