BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 38, Number 4, Pages 467–473
S 0273-0979(01)00911-9
Article electronically published on June 12, 2001

The geometry of schemes, by David Eisenbud and Joe Harris, Springer-Verlag, New York, 2000, x + 294 pp., \$69.95, ISBN 0-387-98638-3

The book is an elementary introduction to the theory of schemes. A scheme is a geometric object generalizing the notion of an algebraic variety. It is defined by gluing together the spectra of commutative rings.

1. Schemes: A history

The notion of spectrum of a commutative ring has an old and complicated history, as often happens with fundamental notions in mathematics. It is based on an idea familiar to all mathematicians: evaluating a function f at a point x is the same as evaluating x at the function f. More precisely, if F^X is the F-algebra of functions on a set X with values in a field F, any element $x \in X$ defines a homomorphism $F^X \to F$ by assigning to a function f its value at x. The kernel of this homomorphism is a maximal ideal $\mathfrak{m}$ in F^X, suggesting that one can recover X as the set of maximal ideals in F^X. For example, this is true when $F = \mathbb{F}_2$ is the field of two elements. The subsets of X form a Boolean lattice L and the commutative ring $\mathbb{F}_2^X$ is the commutative ring $A(L)$ associated to this lattice (with the product defined by $a \cdot b = \inf(a, b)$ and the sum defined by $a+b = \sup(\inf(a, Cb), \inf(b, Ca))$. A fundamental theorem of Marshall Stone [St1] says that any Boolean lattice L is isomorphic to the lattice of subsets of $A(L)$ assigning to each element a of the lattice the subset of maximal ideals containing a. An example of a Boolean lattice is the ordered set of orthogonal projectors in a Hilbert space, and the Spectral Theorem of Hilbert establishes a bijection between this Boolean lattice and the lattice of measurable subsets of $\mathbb{R}$. To show that any Boolean lattice is isomorphic to the lattice of subsets of a topological space which are open and closed at the same time, Stone introduced a topology in the set of maximal ideals of $A(L)$. A closed set in this topology is the set of maximal ideals containing a given ideal of $A(L)$ [St2].

In another direction, in 1941 Israel Gelfand [Gel] showed that any normed commutative algebra A over $\mathbb{C}$ admits a homomorphism to the ring of continuous functions on the set of its maximal ideals equipped with a certain structure of compact Hausdorff space. The kernel of this homomorphism is the *radical* of A, the intersection of all maximal ideals of A. The image of an element $a \in A$ is a function whose value at a maximal ideal $\mathfrak{m}$ is a unique complex number λ such that $a - \lambda \in \mathfrak{m}$, and, in particular, is not invertible. The set of possible values of a is called the *spectrum*

2000 *Mathematics Subject Classification.* Primary 14-01, 14A15.

of a. An example of a normed algebra is the set of operators on a Hilbert space which commute with a given bounded self-adjoint operator T. The set of maximal ideals corresponds to the spectrum of the operator T.

In 1945 Nathan Jacobson observed that the topology defined by Stone can be used to define a topology on the set of primitive two-sided ideals in an arbitrary ring [Jac]. A primitive ideal in a commutative ring is a maximal ideal. About the same time, Oscar Zariski introduced a topology on the set of valuations of a field of algebraic functions [Zar]. When restricted to the set of valuations defined by maximal ideals in the coordinate ring of an affine model of the field, it coincides with the topology defined by Jacobson. The Zariski topology is almost never Hausdorff, but quasi-compact. In spite of this fact, Andrè Weil [We2] and Jean-Pierre Serre [Ser] were able to show that many constructions of algebraic topology can be applied to algebraic varieties defined over an arbitrary field equipped with the Zariski topology.

When A is a finitely generated commutative algebra over an algebraically closed field F, a choice of its generators $t_1, \dots, t_n$ defines a surjective homomorphism of the polynomial algebra $F[T] = F[T_1, \dots, T_n]$ to A whose kernel I is an ideal of polynomials defining an affine algebraic set X in F^n. A point $x = (x_1, \dots, x_n) \in X$; i.e. a solution of the equations $P(T_1, \dots, T_n) = 0$, $P \in I$, defines a surjective homomorphism of $A \to F$ by assigning to each generator t_i the coordinate x_i. Its kernel is a maximal ideal $\mathfrak{m}_x$. The Nullstellensatz of Hilbert implies that this establishes a bijective correspondence between points of X and the set $\mathrm{Specm}(A)$ of maximal ideals of A. The Jacobson-Zariski topology on $\mathrm{Specm}(A)$ defines a topology on X which is called the Zariski topology. The pre-image of an element $a \in A$ in $F[T]$ can be considered as a polynomial (= regular) function on X. It is equal to zero if and only if a belongs to the radical of A. A more intrinsic way to view $a \in A$ as a function on $\mathrm{Specm}(A)$ is to consider $a(x)$ as the residue class of a modulo a maximal ideal $\mathfrak{m}$ representing $x \in \mathrm{Specm}(A)$. A homomorphism $f : A \to B$ of finitely generated F-algebras defines, by taking the inverse image, a map $\mathrm{Specm}(B) \to \mathrm{Specm}(A)$. This corresponds to a regular map of algebraic sets. Thus we see that there is a natural bijective correspondence between isomorphism classes of affine algebraic sets and reduced (i.e. with zero radical) finitely generated F-algebras.

As was observed by Serre [Ser], the notion of localization of a commutative ring introduces a natural sheaf $\mathcal{O}_X$ of rings on $X = \mathrm{Specm}(A)$. Its stalk at a maximal ideal $\mathfrak{m}$ is the localization of A with respect to the multiplicative set $A \setminus \mathfrak{m}$. A section of $\mathcal{O}_X$ over an open subset U can be interpreted as an element P/Q of the total ring of fractions of A such that Q does not vanish at any point $x \in U$.

The pair $(X, \mathcal{O}_X)$ gives an example of a ringed topological space. This notion was introduced earlier by Henri Cartan to define a geometric space by gluing local models of ringed spaces (like, for example, open subsets of $\mathbb{C}^n$ equipped with the sheaf of holomorphic functions). Serre showed that a similar construction can be applied to the abstract algebraic varieties introduced by Weil [We1].

The ideas of categories and functors which became popular around 1955 showed the inadequacy of the notion of the maximal spectrum $\mathrm{Specm}(A)$ of a commutative ring A. The correspondence $A \to \mathrm{Specm}(A)$ is not a functor from the category of commutative rings to the category of topological spaces since a homomorphism of rings does not define in general a natural map of the maximal spectra. However, if one enlarges the space $\mathrm{Specm}(A)$ by considering the set $\mathrm{Spec}(A)$ of all prime ideals instead of just maximal ideals and by defining a topology on Spec(A) in a similar fashion, the functoriality becomes obvious. A homomorphism $f : A \to B$ of rings defines a continuous map ${}^a f : \mathrm{Spec}(B) \to \mathrm{Spec}(A), \mathfrak{p} \to f^{-1}(\mathfrak{p})$. Although the pre-image of a maximal ideal is not always maximal, the pre-image of a prime ideal is always prime. When A is a finitely generated algebra over a field, a prime ideal corresponds to an irreducible closed subset of the corresponding affine algebraic variety X.

Around 1957 Pierre Cartier observed that a ringed space locally isomorphic to a ringed space of the form Spec(A) should be considered as a generalization of an algebraic variety. Following this suggestion Alexandre Grothendieck began to develop the foundations of algebraic geometry based on this generalized notion of algebraic variety called a *scheme*. The original plan of this grand and ambitious project under the modest title "Eléments de Géometrie Algébrique" (EGA) included 13 chapters in which a large part of algebraic geometry was supposed to be rewritten in the language of schemes. Grothendieck had also foreseen that the new ideas would be instrumental for the proof of the Weil conjectures for the zeta function of an algebraic variety over a finite field; he planned to devote the last chapter of his work to this proof. Although only four chapters filling around 2000 pages had been completed with the cooperation of Jean Dieudonné, Grothendieck's Seminaires de Géometrie Algébrique (SGA) at IHES in the sixties and early seventies give a glimpse of the material that was supposed to appear in the subsequent chapters. The influence which EGA made upon the development of algebraic geometry, commutative algebra and number theory is hard to overestimate. Even Grothendieck's foresight about the proof of the Weil conjectures turned out to be right, although it was not he but his former student Pierre Deligne who finished the proof. Needless to say it would be impossible without the techniques developed by Grothendieck. Although the impact of Grothendieck's opus on the development of algebraic geometry is tremendous, it is even more spectacular with respect to number theory. It is hard to believe that the proof of the Mordell conjecture by Gerd Faltings and the solution of the Fermat problem by Andrew Wiles would be possible if EGA did not exist.

2. Schemes versus algebraic varieties

One of the first immediate achievements of Grothendieck was making commutative algebra a part of algebraic geometry, establishing its goal as the study of the local structure of schemes. The immediate result was that many global constructions of algebraic geometry (e.g. resolution of singularities, or cohomology theory) became powerful tools in the study of commutative rings. The theory of schemes also unites Arithmetic and Geometry, as was contemplated by Leopold Kronecker in the last century. The spectrum of the ring of integers in an algebraic number field and a compact Riemann surface are just different examples of a regular scheme of dimension 1. A higher-dimensional generalization of the former scheme is an *arithmetic scheme*, a scheme obtained by gluing together the spectra of finitely generated algebras over the ring of integers. The study of such schemes now belongs to the subject of Arithmetic Geometry. The solution of the Fermat Problem is one of its most striking recent achievements.

The theory of schemes did not just add new objects of study in algebraic geometry. It brought some new insights and new techniques which were instrumental in solving many important problems in classical geometry. For example, admitting nilpotent elements in the structure sheaf $\mathcal{O}_X$ of a scheme X gives a very natural geometric interpretation of multiplicities of intersections of algebraic varieties, degenerations of algebraic varieties to "multiple varieties", infinitesimal neighborhoods of subvarieties and so on. Many pathologies in the theory of moduli of algebraic varieties (where the number of moduli computed from the local deformation theory turned out to be different from the actual number of parameters) were adequately explained by David Mumford in the late sixties by using the theory of schemes. The theory of schemes made it possible to approach some classical constructions in algebraic geometry (e.g. the Picard varieties or the Chow varieties) from a unified and simple point of view based on representability of functors in the category of schemes. It paved the way to the introduction of *algebraic spaces* by Michael Artin [Art] and Boris Moishezon [Moi] which generalize schemes. An algebraic space is also a special kind of *stack*. This new generalization of a scheme introduced by Deligne and Mumford in [DM] is very fashionable these days in connection with some new developments in enumerative geometry influenced by ideas coming from quantum field theory.

The construction of blowing up an arbitrary ideal in a commutative ring, or an arbitrary closed subscheme in a scheme, was made possible and easily dealt with only in the framework of schemes. This was used in an essential way by Heisuke Hironaka in his fundamental work on resolution of singularities of algebraic varieties over a field of characteristic zero [Hir].

The notion of extension of scalars for commutative algebras over an arbitrary ring, after globalizing, leads one to consider any morphism of schemes $f : X \to S$ as a family of schemes parameterized by S. For any point $s \in S$, the fibre X_s of f over s is a scheme defined over the residue field of the point s. The study of local and global properties of morphisms of schemes occupies the largest part of EGA. It has completely revolutionized the study of families of algebraic varieties based on the archaic techniques of valuation theory.

3. The book

According to the authors, the goal of the book is "to share the secret geometry of schemes with the average mathematician and many a beginner in algebraic geometry." The geometric nature of schemes is self-evident if one remembers the previous purely algebraic approach to the foundations of algebraic geometry developed in the works of van der Waerden, Chow, Weil, and Zariski. However, very often, especially in the case of a future number theorist, one's first acquaintance with algebraic geometry begins with studying Hartshorne's *Algebraic Geometry* [Har], an excellent digest of EGA. Its first chapter, devoted to classical algebraic geometry, occupies only 54 pages, which is hardly enough to build a good geometric intuition to fully appreciate the geometric nature of schemes. A student with a year's experience in algebraic geometry (e.g. from studying Shafarevich's book [Sha]) will have less trouble understanding geometry after reading Hartshorne's book. Also, Hartshorne's book was written with the goal of replacing EGA as a reference book in the theory of schemes. It contains a lot of serious technical material, like cohomology theory with its applications as well as the basic properties of morphisms of schemes. Constraints of space in Hartshorne's book left a gap which the present book attempts to fill. It provides an intuition of schemes, leaving till later the technical core of the theory.

In many aspects the book calls to mind the *Red book of varieties and schemes* by D. Mumford [Mu2]. I cite from the introduction to the latter book: "These notes attempted to show something that was controversial at that time [mid 60's]; that schemes really were the most natural language for algebraic geometry and that you did not need to sacrifice geometric intuition when you spoke 'scheme'. I think this thesis is now [1988] widely accepted within the community of algebraic geometry...." It seems that the goal of the book is to make the thesis accepted beyond this community. Mumford also suggested another approach to learning the geometry of schemes. One learns the language of schemes while studying at the same time a useful concrete geometric theory written in this language (for example, the theory of curves on algebraic surfaces [Mu1]). This approach served well many geometers of my generation, and probably the authors of the book. Taking into account that the tremendous progress in algebraic geometry in the past two decades makes the study of algebraic geometry even more time consuming, I believe that this approach still has some merit.

The book under review makes the life of a beginner much easier and more pleasant. There is no need to wander in the vast land of algebraic geometry before one gets to appreciate the geometric nature of schemes. Instead one is led to it straight along a pleasant and easy road paved by some of the best experts in the field. It is a masterpiece in choosing interesting topics and instructive examples which are presented in a very clear way. If one can afford the time to read it, this will be a source of tremendous joy and will constitute good preparation for a more serious study of the theory of schemes based on Hartshorne's book or EGA.

It is hard to place this book among the enormous number of already existing modern books in algebraic geometry. This is not a surprise since it is an original and unusual book. It is not a new textbook in algebraic geometry, of which there are already too many. It would be hard to use the book to teach a course on schemes because it does not go very deeply into the theory. For example, it does not contain cohomology theory. It also skips the proofs of some of the fundamental properties of schemes, such as, for example, the valuative criterion of properness or criteria of ampleness of invertible sheaves. However, it will certainly help a student to digest [Har] (which helps one to digest EGA). It will help an instructor to make the course more lively and motivating by borrowing some of the illuminating examples from the book. In this respect it is comparable to the book on classical algebraic geometry of the second author [Harr], which is also great fun to read but difficult to teach from. I would not recommend this book to a non-specialist who wants to get acquainted with some techniques of schemes without going into the details. Two articles of V. Danilov [Da1], [Da2] in *Encyclopaedia of Mathematical Sciences* will serve these needs much better.

Finally a few words about the contents of the book. First of all it represents a major revision and extension of the authors' earlier book [EH]. New chapters were added and the old ones have been improved and extended. This led to the increase in the size of the book by a factor of almost two. Two new chapters are Chapter 4, "Classical constructions", and Chapter 5, "Local constructions". The first one deals with many examples illustrating the role of nilpotents in the classical geometry of algebraic varieties. Although some of the examples are of the sort which can be easily left to the reader or included as exercises in any standard course on schemes, others are rather serious and very informative (for example, the examples concerning the limit behavior of the Fano scheme of lines on a hypersurface, blow-ups of nonreduced subschemes or forms). The second one treats the degenerations of dual plane curves and discriminants from the point of view of the definition of the image of a morphism of schemes. Some of this material could be useful even for an expert. Chapter 1, "Basic definitions", and Chapter 2, "Examples", are left almost without change. Chapter 3, "Projective schemes", now contains a lot of new material (Grassmannians, Bézout's Theorem, tangent cones). Chapter 6, "Schemes and functors", has not been changed much. Here one can find one of the most important applications of the theory of schemes, which is a construction of different parameter spaces in algebraic geometry. Although the goals of the book do not permit one to go into depth in this theory, one gets its flavor very easily. Every chapter of the book contains an extensive set of problems; their number has increased from the earlier book by more than double. Some of these contain new examples; others fill in technical details.

In summary, I find the book useful for a future researcher in algebraic geometry, number theory or commutative algebra. I hope one will share my enjoyment in reading it.

References

[Art] Artin M., *The implicit function theorem in algebraic geometry*, "Algebraic Geometry. Bombay 1968", Oxford Univ. Press (1969), 13-34. MR **41:**6847

[Da1] Danilov V., *Algebraic varieties and schemes*, in "Algebraic Geometry, I", Encyclopaedia Math. Sci., 23, Springer, Berlin (1994), 167-297. MR **95f:**14001

[Da2] Danilov V., *Cohomology of algebraic varieties*, in "Algebraic Geometry, II", Encyclopaedia Math. Sci., 35, Springer, Berlin (1996), 1-125. CMP 96:13

[DM] Deligne P., Mumford D., *The irreducibility of the space of curves of given genus*, Publ. Math. Inst. Hautes Études Sci. **36** (1969), 75-109. MR **41:**6850

[EH] Eisenbud D., Harris J., *Schemes: The Language of Modern Algebraic Geometry*, Wadsworth and Brooks/Cole, Pacific Grove, California, 1992. MR **93k:**14001

[Gel] Gelfand I., *Ideale und primäre Ideale in normierten Ringen*, Mat. Sbornik, N.S. **9(51)** (1941), 41-48. MR **3:**52a

[EGA1] Grothendieck A., Dieudonné J., *Élements de géométrie algébrique, I: Le langage des schémas*, Publ. Math. Inst. Hautes Études Sci. **4** (1960), 1-228. MR **29:**1207

[EGA2] Grothendieck A., Dieudonné J., *Élements de géométrie algébrique, II: Étude globale élementaire de quelques classes de morphismes*, Publ. Math. Inst. Hautes Études Sci. **8** (1961), 1-222. MR **29:**1208

[EGA3$_1$] Grothendieck A., Dieudonné J., *Élements de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents (premiére partie)*, Publ. Math. Inst. Hautes Études Sci. **11** (1961), 1-167. MR **29:**1209

[EGA3$_2$] Grothendieck A., Dieudonné J., *Élements de géométrie algébrique, III: Étude cohomologique des faisceaux cohérents (seconde partie)*, Publ. Math. Inst. Hautes Études Sci. **17** (1963), 1-91. MR **29:**1210

[EGA4$_1$] Grothendieck A., Dieudonné J., *Élements de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas (premiére partie)*, Publ. Math. Inst. Hautes Études Sci. **20** (1964), 1-259. MR **30:**3885

[EGA4$_2$] Grothendieck A., Dieudonné J., *Élements de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas (seconde partie)*, Publ. Math. Inst. Hautes Études Sci. **24** (1965), 1-231. MR **33:**7330

[EGA4$_3$] Grothendieck A., Dieudonné J., *Élements de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas (troisième partie)*, Publ. Math. Inst. Hautes Études Sci. **28** (1966), 1-255. MR **36:**178

[EGA4_4] Grothendieck A., Dieudonné J., *Élements de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas (quatrième partie)*, Publ. Math. Inst. Hautes Études Sci. **32** (1967), 1-361. MR **39:**220

[SGA1] Grothendieck A. et al., *Revêtements étales et Groupe Fondamental*, Lecture Notes in Math. 224, Springer-Verlag, Heidelberg, 1971. MR **50:**7129

[SGA2] Grothendieck A. et al., *Cohomologie Locale des Faisceaux Cohérents et Théorème de Lefschetz Locale et Globaux*, North-Holland, Amsterdam, 1968. MR **57:**16294

[SGA3] Grothendieck A. et al., *Schémas en Groupes, I, II, III*, Lecture Notes in Math. 151-153, Springer-Verlag, Heidelberg, 1970. MR **43:**223a; MR **43:**223b; MR **43:**223c

[SGA4] Grothendieck A. et al., *Théorie des Topos et Cohomologie Étale des Schémas*, Lecture Notes in Math. 269, 270, 305, Springer-Verlag, Heidelberg, 1972-73. MR **50:**7130; MR **50:**7131; MR **50:**7132

[SGA4$\frac{1}{2}$] Deligne P. et al., *Cohomologie Étale*, Lecture Notes in Math. 569, Springer-Verlag, Heidelberg, 1977. MR **57:**3132

[SGA5] Grothendieck A. et al., *Cohomologie l-adique et fonctions L*, Lecture Notes in Math. 589, Springer-Verlag, Heidelberg, 1977. MR **58:**10907

[SGA6] Grothendieck A. et al., *Théorie des intersections et Théorème de Riemann-Roch*, Lecture Notes in Math. 225, Springer-Verlag, Heidelberg, 1971. MR **50:**7133

[SGA7I] Grothendieck A. et al., *Groupes de Monodromie en Géometrie Algébrique, Part I*, Lecture Notes in Math. 288, Springer-Verlag, Heidelberg, 1972. MR **50:**7134

[SGA7II] Deligne P. , Katz N., *Groupes de Monodromie en Géometrie Algébrique, Part II*, Lecture Notes in Math. 340, Springer-Verlag, Heidelberg, 1973. MR **50:**7135

[Harr] Harris J., *Algebraic Geometry: A first course*, Springer, New-York, 1992. MR **93j:**14001

[Har] Hartshorne R., *Algebraic Geometry*, Springer-Verlag, New York, 1977. MR **57:**3116

[Hir] Hironaka H., *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.* Ann. of Math. (2) 79 (1964), 109–203; Ann. Math. **4** (1960), 1-228. MR **33:**7333

[Jac] Jacobson N., *A topology for the set of primitive ideals in an arbitrary ring*, Proc. Nat. Acad. Sci. USA **31** (1945), 333-338. MR **7:**110f

[Moi] Moishezon B., *An algebraic analogue of compact complex spaces with a sufficiently large field of meromorphic functions. I, II, III*, Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 174-238, 323-367, 506-548.

[Mu1] Mumford D., *Lectures on Curves on an Algebraic Surface*, Ann. Math. Studies, v. 59, Princeton Univ. Press, 1966. MR **35:**187

[Mu2] Mumford D., *The red book of varieties and schemes*, Lecture Notes in Math. 1358, Springer-Verlag, Berlin, 1988. MR **89k:**14001

[Ser] Serre J.-P., *Faisceaux algébriques cohérents*, Ann. Math. **61** (1955), 197-278. MR **16:**953c

[Sha] Shafarevich I., *Basic Algebraic Geometry. I. Varieties in projective space (second edition)*, Springer-Verlag, New York, 1994. MR **95m:**14001

[St1] Stone M., *The theory of representations for Boolean algebras*, Trans. Math. Math. Soc. **40** (1936), 37–111. CMP 95:18

[St2] Stone M., *Applications of the theory of Boolean rings to general topology*, Trans. Math. Math. Soc. **41** (1937), 375–481. CMP 95:18

[We1] Weil A., *Foundations of Algebraic Geometry*, A.M.S., New York, 1946. MR **9:**303c

[We2] Weil A., *Fibre-spaces in Algebraic Geometry (notes by A. Wallace)*, Chicago Univ., 1952.

[Zar] Zariski O., *The compactness of the Riemann manifold of an abstract field of algebraic functions*, Bull. Amer. Math. Soc. **50** (1944), 683-691. MR **6:**186b

Igor Dolgachev
University of Michigan
E-mail address: idolga@math.ISA.umich.edu