BOOK REVIEWS

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A course in model theory: An introduction to contemporary mathematical logic, by Bruno Poizat (translated by Moses Klein), Springer, New York, 2000, 464 pp., $59.95, ISBN 0-387-98655-3

The book is a translation into English of Poizat's "white album", which was originally written in French and published by Poizat himself in 1985. The first half of the book (Chapters 1 to 10) is on classical notions and is an excellent basis for a first course in model theory, or even in mathematical logic. The second half of the book (Chapters 11 to 20) is on "stability theory": a collection of notions, techniques, and results coming out of and related to Shelah's work on classification theory. This can also serve as the basis for a first course in stability theory, although it bears the marks of the time it was written (the late 1970's and early 1980's).

Having recently written a survey article on model theory for the *Notices of the AMS* [5], I would prefer to refer the reader to that article for my perspective on the subject rather than simply repeating its content here. Briefly model theory concerns itself with structures, that is, sets X equipped with a distinguished family of relations (subsets of various Cartesian powers $X \times X \times .. \times X$), as well as relations *between* structures. What distinguishes the subject from say universal algebra is on the one hand the particular kinds of relations between structures which are considered: not only isomorphisms and embeddings, but also elementary equivalence and elementary embeddings. On the other hand when studying a given structure, we are not only interested in members of the given distinguished family of sets, but also in the sets that can be obtained from them by finite Boolean combinations and projections (that is by the operations of first order logic).

Poizat is quite forthright in his views regarding logic, especially in the introduction. He views model theory as the "least logical" part of mathematical logic, but at the same time that part of logic whose foundations cannot be ignored by logicians. His point of view is essentially "naturalist": as other mathematicians, model-theorists simply study the world of mathematics with their special tools, but without a priori foundational prejudices. In particular the notions of truth, implication, proof, and even effectiveness are not given any special status. This, it should be noted, is quite contrary to some traditional views regarding logic, in which the business of logic is or should be some kind of setting of the rules of the game, determining which kinds of reasoning are legitimate, etc. First order definability and elementary equivalence are taken by Poizat as the primitive notions of

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the subject and are expounded to begin with even without the notion of first order formulas. Notions such as "provability", "axiomatization", "decidability" make their appearance in the book only as aspects of definability in the structure $(\mathbf{N}, +, \cdot)$ of natural numbers. Gödel's completeness theorem (the result that a nice system of rules of proof captures the notion of material consequence) is seen as secondary to the compactness theorem. The reader is faced with other "prejudices" such as a rather dismissive attitude towards the mixing of notions from recursion theory and model theory. Poizat also systematically critiques the "axiomatic method" (including the Bourbaki project), which he sees as being undermined by Gödel's second incompleteness theorem. Poizat, through his research and writings over the past 25 years, has played an enormous role in the internal development of model theory and the identification of its proper scope and subject matter. In particular he has contributed crucially to the deepening of "geometric" rather than set-theoretic or "effective" sensibilities in the subject. The proper recognition of Poizat's decisive role here is long overdue.

Let me now get on to the material in the book. The first half is a general presentation and development of the key notions of first order logic and model theory. First order formulas, the syntactic raw material of logic, only appear in the second chapter. In both the first two chapters, the structures considered have the form of a set X equipped with a single m-place relation $R \subseteq X^m$ (for fixed m). Let me write such a structure as an ordered pair (X, R). One of the characteristic notions of logic and model theory is that of the *elementary equivalence* of two such structures (X_1, R_1) and (X_2, R_2). This is *weaker* than isomorphism. Poizat follows Roland Fraïssé's "back-and-forth" treatment. A 0-isomorphism between (X_1, R_1) and (X_2, R_2) is simply a bijection f between (possibly empty) subsets Y_1 and Y_2 of X_1 and X_2 respectively, such that $(x, y) \in R_1$ if and only if $(f(x), f(y)) \in R_2$ for all $x, y \in Y_1$. Working by induction, for p any natural number, a $p+1$-isomorphism is a bijection f between subsets of X_1 and X_2 such that for any $a \in X_1$, f extends to a p-isomorphism g with a in $dom(g)$, and dually for any $b \in X_2$ f extends to a p-isomorphism h with b in $ran(h)$. Ehrenfeucht subsequently presented this notion (p-isomorphisms) in the form of player II having a winning strategy in a certain two-person game of length p. In any case (X_1, R_1) is *elementarily equivalent* to (X_2, R_2) if the empty bijection is a p-isomorphism for all natural numbers p.

As an example with $m = 3$, any group G can be considered as a structure by taking X to be the underlying set of G and R the graph of the group operation. Let me note in passing that Tarski posed the problem whether any two free groups F_n and F_m (for $n, m \geq 2$) are elementarily equivalent.

The Fraïssé formalism is remarkably flexible as a way of introducing many of the basic notions of model theory. For example, if $(a_1, .., a_n)$, respectively $(b_1, .., b_n)$, are n-tuples of elements of X_1, respectively X_2, then these tuples are said to have the same *type* if the map which takes a_i to b_i for $i = 1, .., n$ is a p-isomorphism for all p. If $X_1 \subseteq X_2$, then (X_1, R_1) is said to be an *elementary substructure* of (X_2, R_2) (and (X_2, R_2) an *elementary extension* of (X_1, R_1)) if the identity map on X_1 is a p-isomorphism for all p.

Chapter 2 brings into play the language associated to a structure (X, R). This is simply an m-place "relation symbol" r (standing for R), as well as a supply of variables $x, y, ...$ and the logical connectives $\neg$ ("not"), $\wedge$ ("and"), $\vee$ ("or"), $\exists$ ("there exists"), $\forall$ ("for all"). First order formulas of this language are finite sequences of such symbols which are syntactically correct. A sentence in the language is a formula with no unquantified variables. Such a sentence σ say can be naturally construed as being true or false in a structure of the form (X, R). (If it is true in (X_1, R_1), (X_1, R_1) is said to be a *model* of σ, explaining the term "model theory".) The connection with the back-and-forth notions is that (X_1, R) is elementarily equivalent to (X_2, R_2) just if every sentence true in (X_1, R_1) is true in (X_2, R_2). Likewise, $(a_1, .., a_n) \in X_1^n$ and $(b_1 .., b_n) \in X_2^n$ have the same type if for each formula $\phi(x_1, .., x_n)$ of the language with free variables $x_1, .., x_n$, ϕ is true of $(a_1, .., a_n)$ in (X_1, R_1) just if ϕ is true of $(b_1, .., b_n)$ in (X_2, R_2)..

All of this can be done with several distinguished relations in place of the single relation R. A ring for example can be considered as a structure (X, R, S) where X is its underlying set, and R and S are the graphs of addition and multiplication. It is possibly more natural to consider the operations themselves rather than their graphs and the formalism can be suitably adapted. Again to a structure in the more general sense (a set equipped with a family of distinguished functions and relations) we can associate a suitable first order language, and structures with the same language can be compared, in terms of isomorphism, elementary equivalence, etc. Important syntactic objects of logic/model theory are *theories*. A theory is just a collection of first order sentences in a given language. Theorems such as compactness and Löwenheim-Skolem are proved in Chapter 4. The compactness theorem states that a theory T has a model if each of its finite subsets has a model. The compactness theorem is clearly expressed by Poizat in terms of the compactness of the space of all theories under a certain topology, the Stone space topology. A treatment via ultraproducts also appears. The upward Löwenheim-Skolem theorem is a consequence of compactness and says that a structure M whose underlying set is infinite has elementary extensions of arbitrarily large cardinality.

Given any specific theory T, there are a number of basic questions a logician or model-theorist will ask. One of them is whether T is complete, or equivalently whether all models of T are elementarily equivalent. Another is whether the formulas in the language have a simple description modulo T. The latter question is related to "definable sets": If M is a structure with underlying set X and ϕ is a formula in the associated language with n free variables, then ϕ *defines* a subset of the Cartesian power X^n, that is the set of all n-tuples for which the formula ϕ is true in M. This is exactly the way in which a polynomial equation in n indeterminates will carve out a subset of F^n for any field F, its solution set in F. The difference in the general case is that the formula ϕ has quantifiers, while a polynomial equation is a quantifier-free formula (in the language of rings). A theory T in a language L is said to have quantifier-elimination if all formulas are equivalent modulo T to quantifier-free formulas, equivalently if any definable set in a model T can be defined by a quantifier-free formula. This is of course very language-sensitive. Another question one can ask about T is whether or not it is *decidable*. The latter question used to be considered central to model theory. Partly under Poizat's influence decidability is in a sense no longer seen as a notion belonging to model theory proper.

In Chapter 5, Poizat approaches these questions of completeness and quantifier-elimination by a variant of the back-and-forth criteria for elementary equivalence described earlier. The method requires the notion of an ω-saturated structure. Such a structure M is "large" in a technical sense: if $\Phi(x)$ is a possibly infinite collection of formulas with free variable x and with a fixed finite number of parameters from M such that any finite subset of $\Phi(x)$ has a common solution in M, then $\Phi(x)$ has a common solution in M. Any structure has an ω-saturated elementary extension.

The result is then that T has quantifier-elimination if and only if for any two ω-saturated models M and N of T the set $I(M,N)$ of nonempty partial isomorphisms (that is 0-isomorphisms) between M and N with finite domain has the back-and-forth property: if $f \in I(M,N)$ and $a \in M$, then there is $g \in I(M,N)$ extending f and with $a \in dom(g)$ (and dually). If in addition $I(M,N)$ is nonempty for any such M, N, T is complete. In most situations this is the easiest method to simply *prove* quantifier-elimination and should be taught in all basic model theory courses. On the other hand explicit quantifier-elimination, namely finding a quantifier-free formula equivalent to a given formula, usually requires a little more work.

In Chapter 6 this method is applied to a number of examples, mostly of an algebraic nature. The power of model-theoretic methods is often exemplified in the deduction of higher dimensional features of a situation from assumptions about behaviour in dimension 1. For example the definition of an algebraically closed field is essentially just the Nullstellensatz property for polynomials in 1 variable. The back-and-forth method, plus some elementary algebra, quickly yields quantifier-elimination for ACF, the theory of algebraically closed fields, as well as completeness after fixing the characteristic. One immediately deduces the full Nullstellensatz: If K is algebraically closed, then any finite system of polynomial equations and inequations over K with a common solution in some field extension of K already has a solution in K. A similar phenomenon occurs with differentially closed fields. A differential field is a field K equipped with a derivation ∂. Such an object is considered as a structure by taking $+$, $\cdot$ and ∂ as the distinguished functions. Lenore Blum's axioms for differentially closed fields (of characteristic zero) essentially posit the existence of solutions of differential systems in one differential indeterminate. Again, with the help of some elementary differential algebra, completeness and quantifier-elimination are deduced for the theory DCF_0 of differentially closed fields of characteristic 0. A differential Nullstellensatz follows. Differential rings and fields have recently provided a meeting point for algebraic and diophantine geometry, and model theory. This review is not the place to go into these connections, but Poizat's work in this area, some of which is described below, set the stage for many of the later deep developments.

Chapter 7 is entitled "Arithmetic". What logicians have traditionally dubbed arithmetic (and what is the content of this chapter) is *not* applications of logic to number theory, but rather the complex of ideas around the undecidability and incompleteness theorems coming out of the work of Gödel. The basic model-theoretic object here is the structure consisting of the set $\mathbf{N}$ of natural numbers equipped with addition and multiplication, as well as its theory, $Th(\mathbf{N})$. From coding techniques originating with Gödel, definability in the structure $\mathbf{N}$ is rich and complicated. The study of definability in $\mathbf{N}$, at the appropriate level of generality, is called *recursion theory*, or nowadays *computability theory*. This subject is *not* usually considered part of number theory. (See the introduction to [4] for a discussion of number theory interpreted narrowly and broadly.) I strongly recommend Chapter 7 for a concise and clear "naturalist" account of Gödel's incompleteness theorems and Tarski's undefinability (of truth) theorem, as well as various reflections on mathematics, logic, and the axiomatic method.

Infinite cardinals traditionally play a big role in basic model theory, especially in issues around "saturated" models. So after a brief introduction to ordinals and cardinals in Chapter 8, the exposition of basic model theory is completed with Chapters 9 and 10 on saturated models and prime models. Roughly speaking saturated models are "big" and prime models "small". Still somewhat roughly speaking, one would like to think of a saturated model of a theory T as a "universal" model, one in which all other models are elementarily embeddable, but this does not make sense for cardinality reasons (T will typically have models of arbitrarily large cardinality). So fixing an infinite cardinal κ one just considers models of T of cardinality at most κ and looks for a universal model of cardinality κ. Saturation involves a bit more than universality, which I do not want to get into now, but in any case any two saturated models of the same cardinality will be isomorphic. The *existence* of saturated models is in general sensitive to set-theoretic assumptions. Prime models are dual to saturated models; a prime model of a theory is one which is elementarily embeddable *in* every model of the theory. It is not completely clear why these notions have been so important in model theory. It is probably because of the issue of "categoricity". One would like to produce certain *canonical* models of a theory, models that are determined uniquely up to isomorphism, and prime and saturated models fit the bill.

Chapter 11 begins the second part of the book. The subject is "stability theory", the study of models of stable theories and definability in these models. This is more or less synonymous with "pure model theory", although the past 20 years or so have seen a broadening of the scope of the methods beyond stable theories. It is worthwhile repeating a little the origins of the subject, although the story has been told many times before. It is natural for a logician to ask whether or not a given set T of (first order) axioms captures the *isomorphism type* of an object (model), namely whether T has a unique model, up to isomorphism. The Löwenheim-Skolem theorems tell us that this is impossible, for cardinality reasons, as long as some model of T is infinite. For κ an infinite cardinal, let us call T κ-categorical if T has at most one model of T of cardinality κ, up to isomorphism. So the best one can hope for is that T is κ-categorical for all κ. This is still a very restrictive condition on T. Moreover (assuming the language of T to be countable), the case $\kappa = \aleph_0$, the first infinite cardinal, is a bit special. The theory of dense linear orders is $\aleph_0$-categorical, but has 2^κ models of cardinality κ for any *uncountable* κ. In the 1960's Michael Morley proved that T is κ-categorical for *some* uncountable κ iff it is κ-categorical for *all* uncountable κ. In the late 1960's Saharon Shelah set himself the task of generalizing Morley's result by determining what can be the possible functions $I_T(-)$ for T a countable theory: where for uncountable κ, $I_T(\kappa)$ is the number of models of T of cardinality κ, up to isomorphism. By the mid 1970's a manuscript of Shelah's book *Classification theory* [7] was in circulation, and many people were grappling with its complexities. Most successful was the Parisian school, led by Daniel Lascar and Bruno Poizat, who explicated and refashioned much of Shelah's formidable machinery. The second part of the book under review is an exposition of the basics of stability theory from the Parisian point of view. There is some degree of overlap with Daniel Lascar's book *Stability in model theory* [3]. The book under review culminates with the determination of the possible functions $I_T(-)$ when T is a countable totally transcendental non-multidimensional theory (as in Lascar's book, and the first edition of Shelah's book). The material should be known by anybody actively working around the purer side of model theory, and Poizat's book is an excellent source. Rather than review the second part of the book in detail, I will just point out a few crucial notions as well as some special contributions of Poizat.

Maybe the most characteristic notion in model theory is that of a *type*. One cannot do too much on the purer side of the subject without a thorough understanding of what types are and how they are manipulated. I defined above when two tuples in a structure have the same type. Alternatively, if M is a structure, A a subset of (the underlying set of) M and b a finite tuple from (the underlying set of) M, the *type* of b over A in M, $tp_M(b/A)$, is the set of formulas $\phi(x)$ with parameters from A which are true in M when b is substituted for x. If M is sufficiently saturated, we can identify $tp_M(b/A)$ with its locus, the orbit X of the tuple b under $Aut_A(M)$, the group of automorphisms of M which fix A pointwise. An extension of $tp_M(b/A)$ to a superset $B \supset A$ of parameters from M is, from this point of view, an orbit $Y \subset X$ under $Aut_B(M)$. One can ask whether, among extensions, there are "privileged" ones. For example working in a large algebraically closed field K, if $V \subseteq K^n$ is an irreducible variety defined over a subfield k of K, the "generic type" of V is simply $tp_K(b/k)$ where b is a generic point of V over k (in the sense of Weil). If k' is an extension of k, then this type *does* have a natural privileged extension, namely $tp_K(c/k')$ where c is some (any) generic point of V over k'. Among the essential aspects of stability theory is a theory of such privileged extensions, called nonforking extensions, in a very general context. This is entirely the work of Saharon Shelah. The general context is that of a *stable theory.* One of the ways of defining stability is via counting types. For κ an infinite cardinal, T is said to be κ-stable if for any model M of T and subset A of M of cardinality $< \kappa$ there are at most κ types over A. T is said to be stable if it is κ-stable for some κ. The totally transcendental theories mentioned in the previous paragraph are precisely the $\aleph_0$-stable ones. The connection of this notion with the problem of counting models comes from a theorem of Shelah that an *unstable* theory has the maximum number possible (2^κ) of models of cardinality κ for any uncountable κ. Thus, for Shelah's purposes, attention can be restricted to stable theories. Among the main contributions of the Parisian school was a novel approach to Shelah's theory of nonforking extensions, via the *fundamental order* of a theory, introduced by Poizat in his thesis.

Among other distinctive contributions of Poizat expounded in the book are imaginaries and their elimination, as well as a model-theoretic approach to the Galois theory of differential equations. (Even the material on imaginaries has a Galois-theoretic aspect.) An *imaginary element* in a structure M is simply an equivalence class with respect to a definable equivalence relation. Shelah provided a formalism for treating imaginary elements in M on a par with ordinary finite tuples from M. Poizat raised the question of when imaginary elements can be simply *identified* with ordinary tuples, that is when a theory or structure admits *elimination of imaginaries.* This is a model-theoretic analogy of the geometric question of when the quotient of some kind of *space* (smooth manifold, algebraic variety,...) by a "closed" equivalence relation has the structure of a space belonging to the same category. In some situations this is more than an analogy. The fact (proved in Chapter 16) that algebraically closed fields admit elimination of imaginaries yields a quick proof that the quotient of an algebraic group by a normal algebraic subgroup has itself the structure of an algebraic group.

The Galois theory of differential equations is examined in Chapter 18 (on prime models in the stable context). It was Poizat who first noticed that the Picard-Vessiot theory and Kolchin's generalization are special cases of a model-theoretic construction. This model-theoretic generalization is as follows: in a structure M we are given sets X and Y, definable without parameters, and for some tuple b from M we have a definable function $f(-,-)$ such that for all $a \in X$ there is a tuple c of elements of Y such that $a = f(b,c)$. In this situation, possibly assuming stability, the group of elementary permutations of X which fix Y pointwise is shown to have the structure of a *definable group*. In the Picard-Vessiot context, X is the solution set of a linear differential equation (over $\mathbf{C}(t)$ say), and $Y = \mathbf{C}$. The solution set X is read in the differential closure of $\mathbf{C}(t)$. The tuple b is simply a fundamental system of solutions for the differential equation. The fact that the relevant automorphism group has naturally the structure of a complex algebraic group can be recovered from the general context above, together with quantifier-elimination in and stability of differentially closed fields.

The job of translating Poizat's French text into English was not an enviable one. The translator, Moses Klein, has done well overall. However, there are a few bad choices of technical words as well as some clumsy expressions and ambiguities. "Dope" is chosen rather than "dop" for Shelah's "dimensional order property". What in English is usually called the "average type" becomes the "mean type". "Fraction" is used to mean "function" in a couple of places.

Poizat wrote the preface to this English edition himself (in English). In this preface, as well as throughout the book, Poizat's sense of humour, peculiar even for a Frenchman, is much in evidence. Each chapter begins with a quotation having some tenuous relation to the chapter's contents. Among the authors of these quotations are Baudelaire, Evariste Galois, Chuck Berry, and Alistair Lachlan. The bulk of the preface, on the other hand, is a quite serious lament about what Poizat considers to be the intolerance shown by the mathematical and scientific establishment to writing and publishing in languages other than English. I could elaborate on what one might consider to be an old-fashioned French chauvinism underlying these complaints, but I would rather not.

There are a number of other excellent books on model theory in print, each of which has something special to offer. Among those which start more or less from scratch are [1], [2] and [6], all of which are recommended. More books are expected in the next few years. Poizat's book is unique among these in the extent to which advanced stability theory is treated as well as elementary model theory and logic. The book belongs on the shelf of anybody with a more than passing interest in the subject. The French original is recommended even more.

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