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Cohomology of number fields, by Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, Grundlehren der mathematischen Wissenschaften, vol. 323, Springer-Verlag, 2000, 720 pp., $109.00, ISBN 3-540-66671-0

The story begins in the mid-1930s, with the work of Witold Hurewicz on algebraic topology. More specifically, the question was how the fundamental group of a complex was related to its homology and cohomology groups. After defining the higher homotopy groups in 1935, Hurewicz considered *aspherical* complexes, that is, complexes where all higher homotopy groups are zero. For such complexes, he proved in [10] that all the homology and cohomology groups were entirely determined by the fundamental group.

At that point, it was natural to ask whether one could find an *algebraic* description of the connection between the fundamental group and the homology groups, one that did not depend on considering the topological space at all. A first step in this direction came from a theorem of Heinz Hopf in [9] that gave an explicit way to construct the second homology group of an aspherical complex in terms of its fundamental group. The construction is non-obvious, and at first glance appears to be non-canonical, since it is described in terms of a presentation of the fundamental group by generators and relations. Hopf proved that in fact the group he constructed did not depend on the choice of presentation, suggesting that there must be a more "invariant" way of describing it.

The interest generated by Hopf's paper is well-described in Saunders Mac Lane's historical account [11]. In short order, several authors – Samuel Eilenberg and Mac Lane, Hopf, Hans Freudenthal, and Benno Eckmann – independently defined what is now known as the homology and cohomology of groups. Together, these papers showed how to obtain, for any group G and any G-module A (i.e., any abelian group A with a G-action), the homology groups $H_n(G, A)$ and the cohomology groups $H^n(G, A)$. Even the ring structure on the cohomology $H^*(G, A)$ is already discussed in these papers. Theorems relating the construction to algebraic topology showed that this construction was indeed the answer to the question raised by the work of Hurewicz and Hopf.

Of course, algebraists were interested in interpreting the groups obtained by this new construction in algebraic terms. Some of the cohomology groups turned out to boil down to objects that had been studied for a while already. For example, $H^0(G, A) = A^G$, the group of elements of A that are fixed under the action of G. This in itself must have suggested that the situation where G is the Galois group of some field extension L/K and A is (some subset of) L should be of interest. Looking at $H^1(G, A)$ immediately confirms this impression.

Here's the story. In his 1897 *Zahlbericht* (recently published in English translation – see [5]), David Hilbert gave an account of algebraic number theory as it stood at the time. Given its author, it is not surprising that the *Zahlbericht* is not merely an expository account, but a masterful reorganization of the theory. The 90th theorem in the book considers an extension K/k of number fields (that

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is, of finite extensions of $\mathbb{Q}$) that is Galois and whose Galois group is cyclic with generator s. The norm function $N : K \to k$ is defined as usual. Theorem 90 then says that any element $\alpha \in K^\times$ such that $N(\alpha) = 1$ is of the form $\alpha = \beta/s(\beta)$ for some element $\beta \in K^\times$.

Attempting to extend this result to other fields shows that a naïve extension is false, so that one has to look deeper. The correct extension is usually attributed to Emmy Noether. To explain it, let's write the group operation on A multiplicatively and the action of $g \in G$ as $a \mapsto a^g$. Define a "crossed homomorphism" $f : G \to A$ to be a function such that $f(gh) = f(g)f(h)^g$. It isn't hard to check that for each $b \in A$ the function f_b defined by $f_b(g) = b^g b^{-1}$ is a crossed homomorphism. Noether's theorem says that if $G = \mathrm{Gal}(L/K)$ is the Galois group of an extension L/K of number fields and $A = L^\times$ with the Galois action of G, then any crossed homomorphism is equal to f_b for some $b \in L^\times$. To see that this is indeed a generalization of Hilbert's theorem, just note that if G is cyclic, then a crossed homomorphism is completely determined by the image of a generator of G, and that this image can (and must) be chosen to be any element of $L^\times$ whose norm is 1.

And here we get back to cohomology, since it turns out that $H^1(G, A)$ is exactly the quotient of the group of all crossed homomorphisms from G to A (now known as 1-cocycles) by the subgroup consisting of the crossed homomorphisms of the form f_b (now known as 1-coboundaries). This allows us to read Hilbert's Theorem 90 as it is stated in most modern textbooks: if $G = \mathrm{Gal}(L/K)$ is the Galois group of a finite extension of number fields, then $H^1(G, L^\times) = 0$.

In fact, there is a further step to take. Taking the inverse limit over all finite extensions L/K yields the infinite topological group

$$G_K = \mathrm{Gal}(\overline{K}/K) = \varprojlim \mathrm{Gal}(L/K)$$

which is the Galois group of the algebraic closure $\overline{K}^\times$ of K, and one can then interpret Theorem 90 as saying that $H^1(G_K, \overline{K}^\times) = 0$.

Even this simple bit of reinterpretation has some payoff, because group cohomology comes with so much structure. Consider, for example, the homomorphism $\overline{K}^\times \to \overline{K}^\times$ given by $x \mapsto x^p$. This is a homomorphism of G_K-modules, it is surjective (since $\overline{K}^\times$ is algebraically closed), and its kernel is clearly the subgroup μ_p of p-th roots of unity. So we have an exact sequence

$$0 \to \mu_p \to \overline{K}^\times \to \overline{K}^\times \to 0.$$

Assume that all the p-th roots of unity already belong to K, so that μ_p is fixed under G. The long exact sequence of cohomology then gives us an exact sequence

$$0 \to H^0(G_K, \mu_p) \to H^0(G_k, \overline{K}^\times) \to H^0(G_K, \overline{K}^\times) \to H^1(G_K, \mu_p) \to H^1(G_K, \overline{K}^\times),$$

which boils down to

$$0 \to \mu_p \to K^\times \to K^\times \to H^1(G_K, \mu_p) \to 0,$$

where the final 0 comes from Theorem 90. Since G_K acts trivially on μ_p, we have $H^1(G_K, \mu_p) = \mathrm{Hom}(G_k, \mu_p)$, so that in particular we get that $\mathrm{Hom}(G_k, \mu_p) \cong K^\times/(K^\times)^p$. This isomorphism is the basic content of "Kummer theory", which gives a description of all cyclic extensions L/K of degree p. Thus, the cohomological point of view gives us Kummer theory "for free" once we know Hilbert's Theorem 90.

Just as happened with H^1, the second cohomology group $H^2(G, A)$ turned out to be, in a certain sense, known. It could be interpreted in terms of extensions of A by G, that is, of groups E such that there is an exact sequence

$$1 \to A \to E \to G \to 1.$$

Basically, if one takes a set-theoretic section of the quotient homomorphism $E \to G$, the product in E of the lifts of two elements of G must differ from the lift of their product in G by an element of A. This gives a function $f : G \times G \to A$; writing out the associative law in E shows that f must satisfy certain conditions. A function satisfying such conditions was known as a "factor set", and $H^2(G, A)$ turns out to be the quotient of the group of all factor sets (which get renamed to 2-cocyles) by an appropriate subgroup.

For number theorists, the most interesting aspect of this was that factor sets were also known to be important in the theory of central simple algebras. In fact, given a Galois extension L/K with Galois group G, one can use a factor set $f : G \times G \to L^\times$ to construct a simple algebra of dimension n^2 as a vector space over K whose center is K. Hence, $H^2(G, L^\times)$ turns out to be connected to the *Brauer Group* of all central simple algebras over K. Specifically, it is isomorphic to the subgroup corresponding to central simple algebras that split over L (i.e., that become isomorphic to a matrix algebra when tensored with L). This was interesting, because central simple algebras played a significant role in one of the approaches to class field theory, which is the next important part of our story.

Class field theory was born in the late nineteenth century in the work of Leopold Kronecker, Heinrich Weber, and David Hilbert, who together laid out a research program that became the main thrust of algebraic number theory through the 1930s. It is concerned with extensions L of a number field K which are Galois with an abelian Galois group; we call them *abelian extensions*. The main goal was well summarized by Claude Chevalley in [3]:

> L'objet de la théorie du corps de classes est de montrer comment les extensions abeliennes d'un corps de nombres algébriques K peuvent êtres déterminées par des éléments tirés de la connaissance de K lui-même; ou, si l'on veut présenter les choses en termes dialectiques, comment un corps possède en soi les éléments de son propre dépassement (et ce, sans aucune contradiction interne!). [1]

By the end of the 1930s, all the fundamental theorems of class field theory had been proved (by many mathematicians, including Hilbert, Teiji Takagi, Philipp Furtwängler, and Emil Artin),[2] but there were still several lingering issues. First of all, there was the fact that the original proofs involved "transcendental methods" (that is, complex analysis). Second, the congruence class groups that were at the heart of the theory were complicated to define, making the statements of the basic theorems less elegant than they should be. Finally, Hasse had shown that there was also a *local* class field theory (where one starts with a finite extension of the

[1] Roughly, "The object of class field theory is to show how the abelian extensions of an algebraic number field K can be determined by objects taken from our knowledge of K itself; or, if one wishes to present things in dialectical terms, how a field contains within itself the elements of its own surpassing (and this without any internal contradiction!)."

[2] The historical article [4], by Helmut Hasse, gives a useful account of the development of class field theory until the mid-1940s.

p-adic numbers instead of with a number field), and it seemed natural to attempt to base the full "global" theory on the local theory.

All of this was addressed by the work of Chevalley. Beginning in 1933 and culminating in his 1940 paper [3], he showed that if one replaced the ideal classes of the original theory by the *idèle class group*, one obtained a smoother formulation of the whole theory. Using the theory of central simple algebras, he established the local theory and then showed how to construct the global theory on that basis. Finally, he observed that one should formulate the whole theory in terms of the maximal abelian extension of K instead of working with finite extensions L/K.

In the late 1940s and early 1950s, this whole edifice was reinterpreted from a cohomological point of view. The crucial role played by central simple algebras in the local theory was probably the major motivating factor. Central simple algebras were determined by factor sets, and factor sets were 2-cocycles, which suggested that in the local case the thing to do was to study the second cohomology group $H^2(G_K, \overline{K}^\times)$. Furthermore, 2-cocyles still make sense in the global situation, where $\overline{K}^\times$ had to be replaced by the idèle class group (so that the connection to simple algebras was lost). In particular, André Weil discovered around 1950 that the "fundamental class" that plays a central role in the local theory could be obtained also in the global case, and this turned out to be a fundamental insight.

The reconstruction was largely accomplished in a flurry of papers by Gerhard Hochschild (see [6], [7], [8]), Tadasi Nakayama (see [8], [13], [14]), Weil (see [21]), and John Tate (see [20]), and in the Artin-Tate seminar on class field theory at Princeton (1951–52, since published as [1]).

Two features on the new theory were to influence future developments significantly. First, in their seminar Artin and Tate formulated a general theory of "class formations" that was basically an axiomatization for class field theory. The idea was to separate what was "algebra" from what was "number theory". Thus, one abstracted the input from number theory into a few crucial results and showed that if one took those as axioms one could prove purely algebraically the theorems of class field theory. As we will see, the book under review uses the same strategy.

The second important feature was the attention to higher-order cohomology groups. On the one hand, there was much interest in understanding the third cohomology groups H^3 and figuring out how they fit into the picture (see [14], for example, and the comments in section 5 of [11]). On the other, Tate's article [20] showed that one could use the cup product to study the higher cohomology groups, and hinted at a construction of an extended cohomology theory in which the reciprocity isomorphism at the center of class field theory could be obtained in this way. (The final paragraph of this paper is one of the greatest cliffhangers in the history of mathematics, particularly since the "subsequent publication" it mentions never appeared! See, however, [19] and the expository accounts of class field theory mentioned below.) Tate's approach to the reciprocity isomorphism is the one found in most modern treatments of class field theory.

During the next decade, Tate continued to investigate the cohomology of Galois groups and obtained many fundamental insights. At the International Congresses of Mathematicians in 1954 and in 1962, he gave lectures explaining fundamental new ideas. One of Tate's most important insights was that class field theory can be understood as a consequence of a duality theorem for Galois cohomology, and that one could obtain far more general theorems of this type. The extent of Tate's influence can be seen in the bibliographic remarks for the first two chapters in Jean-Pierre Serre's monograph [18], which note that almost all of the results are due to Tate. This book also reproduces (pages 61–65) a letter from Tate to Serre, dated 1963, in which he discusses several duality theorems for Galois cohomology. Unfortunately, much of Tate's work was never published, and supplying the details is one of the major goals of the book under review.

Cohomological ideas and techniques continue to be an important part of algebraic number theory. They played a role in the determination of the structure of the absolute Galois group of a local field, in the proof that every solvable group can be realized as a Galois group of an extension of $\mathbb{Q}$, and in many other significant recent results, several of which appear (and in many cases get their first treatment in an expository style) in this book.

Cohomology of number fields is in many ways a sequel to *Algebraic number theory*, by Jürgen Neukirch, which was originally published (in German) in 1992 (the English translation [16], part of the same series from Springer, appeared in 1999). The authors have two main goals. First, they hope to "provide a textbook for students, as well as a reference book for the working mathematician." Second, they want to provide full proofs for some crucial results:

> Central and frequently used theorems, such as the global duality theorem of G. Poitou and J. Tate, as well as results such as the theorem of I. R. Shafarevich on the realization of solvable groups as Galois groups over global fields, had been part of algebraic number theory for a long time. But the proofs of statements like these were spread over many original articles, some of which contained serious mistakes, and some even remained unpublished. It was the initial motivation of the authors to fill these gaps ... (Introduction, page vi)

The result is a very valuable book. While not easy to read, it does provide access to the theory to students, and the inclusion of full proofs of several important theorems will make it very useful to researchers in the area as well.

The authors restrict themselves to Galois cohomology proper, avoiding (in contrast to [12], for example) both étale and flat cohomology theories. They also restrict themselves to the Galois modules that are "of dimension one or less," that is, to those that are either well-chosen submodules of the fields themselves or are closely related to them. So, for example, they do not consider the Galois cohomology of elliptic curves or the cohomology groups that arise in the deformation theory of Galois representations.

The structure of the book reflects the method of the Artin-Tate seminar. A first section, entitled "Algebraic Theory", treats the cohomology of profinite groups in as much generality as possible. The first chapter defines the various kinds of cohomology groups to be considered, plus the basic structures that come with them, such as the long exact sequence for cohomology and the cup product. The second chapter deals with ideas from homological algebra: spectral sequences, derived funtors, and continuous cochain cohomology. The fourth and fifth chapters of the first section deal with free products of profinite groups and with Iwasawa modules, setting up the algebraic background for topics to be discussed in the second part.

Chapter III is the crucial chapter of this part of the book, since it develops the duality theory. Here the reader will find the notion of a class formation, the definition of a "level-compact" G-module, and an alternative description of the theory in terms of "G-modulations". The whole thing is done in an abstract, axiomatized setting and held in reserve for the applications in the second part of the book.

The separation of the "algebraic" results from the "arithmetical" applications is in many ways a good idea. It certainly has a clarifying effect for those who know the theory, since it makes it easier to see precisely what number-theoretical facts make it all work. For the student, however, the lack of applications in these chapters may also result in a lack of motivation, and it might be a good idea to jump to the first few chapters of the second part to see why all the algebraic machinery is being set up.

The second part, entitled "Arithmetic Theory", is where we find the payoff for all the work done in the first part. After a preliminary chapter on "Galois Cohomology", we have chapter VII, on "Cohomology of Local Fields", and chapter VIII, on "Cohomology of Global Fields". Here we get the local and global versions of duality and class field theory, the local and global Euler-Poincaré characteristic, and the connection between the local and global theory. It is here that we find the complete proof of the Poitou-Tate duality theorem that is promised in the introduction.

Chapter IX deals with "The Absolute Galois Group of a Global Field". The main results are the Grunwald-Wang theorem and Shafarevich's theorem that every solvable group can be realized as a Galois group over the rationals. Chapter X, entitled "Restricted Ramification", looks at the cohomology of the Galois group G_S of the maximal extension of a number field K that is unramified outside a finite set of primes S. This is the group that appears naturally in many applications (particularly to arithmetical algebraic geometry). The chapter covers several interesting and difficult issues, from Leopoldt's Conjecture to class field towers, the Fontaine-Mazur conjecture, and the structure of G_S as a profinite group.

Chapter XI is an introduction to the Iwasawa theory of number fields, ending with a discussion (without proofs) of (several versions of) the "Main Conjecture" and its applications. Finally, chapter XII is a brief introduction to Grothendieck's "Anabelian Geometry".

The portion of Galois cohomology that is directly related to class field theory has been discussed in many textbooks, beginning with [1] and including [2], [15], and [17]. The treatment of the theory in such books tends to be utilitarian: one develops only what is needed for the task at hand. (Weil argued in [22] that it was not worth the effort to develop group cohomology solely to obtain class field theory. Most authors do not agree with him, but they still try to limit the cohomological theory as much as possible.) Readers familiar with any of these books will find the more thorough treatment in *Cohomology of number fields* useful and interesting.

Serre covers much of the duality theory in [18], along with some material on non-abelian Galois cohomology. Serre's book takes a more advanced perspective and does not try to be self-contained. The duality theory is also developed by Milne in [12]. Milne's book goes well beyond *Cohomology of number fields*; it includes the étale and flat theories and considers higher-dimensional Galois modules as well. The book under review is more accessible than either Serre or Milne. It is more detailed and more complete on the topics that it shares with the other books, but of course it does not cover the same range of material.

The book contains an extensive bibliography. There are occasional historical notes, but this reviewer would have liked to see much more along these lines, and in more predictable places. The main thing, though, is the mathematics, which is done well, clearly, and thoroughly. Overall, the authors' attention to detail and their determination to produce a full account of the theory has resulted in a book that will serve as an important reference for years to come.

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