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Holomorphy and convexity in Lie theory, by Karl-Hermann Neeb, de Gruyter Expositions in Mathematics, vol. 28, de Gruyter, Berlin and New York, 2000, xxi + 778 pp., \$298.00, ISBN 3-11-015669-5

Among the very basic characteristics of mathematics (and mathematicians) we find the ambition to seek connections between various fields and to integrate new results into their proper place in the already established theories. This also means that old concepts expand their radius of activity and are seen in an ever widening perspective thus proving their value. In the book *Holomorphy and Convexity in Lie Theory* by Karl-Herman Neeb we see two such old and central concepts expand their role in modern mathematics, namely on the one hand *convexity* and on the other *holomorphy*. Convex sets (here always in a finite-dimensional real vector space V) are well-known, and they appear in many interesting and useful places, sometimes in surprising ways; corresponding to this one defines convex functions

$$f : V \mapsto \mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$$

to be those with non-empty domain $D_f = f^{-1}(\mathbb{R})$ and convex epigraph

$$\mathrm{epi}(f) = \{(v, t) \in V \times \mathbb{R} | f(t) \leq t\}.$$

For such a function (with mild extra conditions) f, the differential df maps the domain diffeomorphically onto an open convex domain in the dual space V^* to V - this is the classical Fenchel Convexity Theorem, and it is used at key places in the book by Neeb. Holomorphy is taken to be the study of holomorphic functions on complex manifolds, and again there is a large body of mathematics to relate to; the main focus by Neeb is the construction of canonical Hilbert spaces of holomorphic functions - and to see how convexity and holomorphy illuminate the theory of Lie groups, and vice versa.

Lie groups have over the years developed two particular sides of their personality, if one may use such a metaphor here: they have a classical side associated with classical mechanics in physics, where their place is as symmetry groups of symplectic manifolds, and they have a quantum side corresponding to unitary actions on complex Hilbert spaces. The correspondence between these two sides is sometimes called quantization, and more specifically one has the program of geometric quantization, which seeks to establish rather direct and constructive ties between these two sides of a Lie group G. Under the rubric of "Holomorphic Representation Theory", this is exactly what is achieved by Neeb in the monumental and almost encyclopedic book under review, dealing with a certain class of Lie groups, certain convex domains and functions associated with this, and a certain class of unitary representations in Hilbert spaces of holomorphic functions. In applications to physics, these representations are quite important, since they are associated with the requirement of positivity of the energy in quantum systems modeled by the representations. This theory is also intimately connected with the rich theory of bounded symmetric domains in $\mathbb{C}^n$ and the geometry and function theory of these domains.

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Consider the following example: let D be the unit disc in the complex plane and let $G = \mathrm{SU}(1,1)$ be the group of biholomorphic Möbius transformations of D, i.e.

$$g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, z \in D$$

with the condition that $ad - bc = 1$ and

$$\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the space of holomorphic functions f on D the following defines an action of G:

$$(g.f)(z) = (cz+d)^{-n} f(g^{-1}.z), \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $n = 2, 3, 4, ...$, and in fact this preserves the integral

$$||f||^2 = \int_D |f(z)|^2 (1-|z|^2)^{n-2} dxdy, \quad z = x + iy.$$

Since the holomorphic functions f with $||f||^2 < \infty$ form a complete space $\mathcal{H}_n$ with respect to this norm, we here have for each n a unitary representation of G in the Hilbert space $\mathcal{H}_n$; it is called the holomorphic discrete series of G. In this case the classical side of G is seen in its action on D, this being a Kähler manifold with respect to the usual hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{(1-|z|^2)^2}.$$

Let us now turn to the basic general setting which starts with a Lie group G, for simplicity say a closed subgroup of the invertible real $n \times n$ matrices, and its Lie algebra $\mathfrak{g}$, also consisting of matrices. Consider the adjoint action

$$\mathrm{Ad} : G \mapsto \mathrm{Aut}(\mathfrak{g}), \quad g \mapsto (X \mapsto gXg^{-1}), \quad g \in G, X \in \mathfrak{g}$$

which is a homomorphism, and the corresponding coadjoint action on $\mathfrak{g}^*$ denoted Ad^* and given by

$$(\mathrm{Ad}^*(g)f, X) = (f, \mathrm{Ad}(g^{-1})X), \quad g \in G, X \in \mathfrak{g}, f \in \mathfrak{g}^*.$$

With these definitions in place, we may state a little more precisely the goals and results of holomorphic representation theory. First of all, the relevant class of Lie groups are the admissible ones, meaning that $\mathfrak{g}$ should contain a G-invariant convex cone C (in the non-trivial sense, that it should have non-empty interior and contain no affine lines). A basic result then says that there is an appropriately compact Cartan subalgebra $\mathfrak{t}$ (so-called compactly embedded, a notion developed by Hilgert and Hofmann) i.e. $\mathfrak{t}$ is maximal abelian, and there is a corresponding root decomposition of the complexification of $\mathfrak{g}$. Similarly a coadjoint orbit $\mathcal{O}_f = \mathrm{Ad}^*(G)f$ is said to be admissible, provided it is closed and its convex hull $\mathrm{conv}(\mathcal{O}_f)$ does not contain any affine line. Now convexity makes an appearance in the following result: Consider the natural projection (restriction mapping)

$$pr : \mathfrak{g}^* \mapsto \mathfrak{t}^*;$$

then for an admissible orbit the image $pr(\mathcal{O}_f) = \mathrm{conv}(\mathcal{W}.f) + D$, where D is a polyhedral cone, and $\mathcal{W}$ a finite Weyl group acting on $\mathfrak{t}$ and by duality on $\mathfrak{t}^*$; here $f \in \mathfrak{t}^*$.

This type of convexity from the classical side of Lie theory we encounter in analogous form on the quantum side. For this, consider a unitary representation of G in a complex Hilbert space $\mathcal{H}$, i.e. a continuous homomorphism

$$\pi : G \mapsto \mathcal{U}(\mathcal{H})$$

from G into the unitary operators on $\mathcal{H}$, and let $\mathcal{H}^\infty$ denote the dense subspace of smooth vectors (v such that $g \mapsto \pi(g)v$ is smooth). Now we define the moment map Φ of this representation to be

$$\Phi : \mathbb{P}(\mathcal{H}^\infty) \mapsto \mathfrak{g}^*, \quad \Phi([v])(X) = \frac{1}{i}\frac{(d\pi(X)v, v)}{(v, v)}$$

with the differential of the representation denoted

$$d\pi(X)v = (d/dt)|_{t=0}\pi(\exp(tX))v, \quad v \in \mathcal{H}^\infty, X \in \mathfrak{g}$$

and (v, v) the inner product in the Hilbert space. This moment map is then G-equivariant from the projective space of $\mathcal{H}^\infty$ to the dual of the Lie algebra. The closed convex hull of the image $I_\pi = \overline{\text{conv}}(\text{Im}\Phi)$ of this moment map is an important invariant, called the convex moment set.

I_π allows us to characterize the admissible class of representations that are of interest, namely the unitary highest weight representations - these are precisely those for which I_π contains no affine lines, and there will be associated with each of these (in the irreducible case) an admissible coadjoint orbit $\mathcal{O}_f$, where $f = -i\lambda$ where this λ is the so-called highest weight. In this case we have $I_\pi = \text{conv}(\mathcal{O}_{-i\lambda})$, and the extreme points of this are the orbits themselves.

Finally we reach one of the major tools in holomorphic representation theory, pioneered in [4] and [5], and to which Neeb has contributed significantly himself, namely the complex Ol'shanskii semigroup. This is a holomorphic Lie semigroup Γ of the same complex dimension as the real dimension of G and having G as its natural Shilov boundary, just as the real line is the boundary of the upper half-plane. It is the curved analog of tube-type domains $V + iW$ where W is an open convex cone in the real vector space V, and it is a wonderful complex manifold in which to do analysis of holomorphic functions, sometimes referred to in connection with the celebrated Gel'fand-Gindikin program. It has the form

$$\Gamma = G \exp(iW)$$

with W the G-invariant cone in $\mathfrak{g}$ associated the unitary highest weight representation π via the condition

$$-\infty < \inf(I_\pi, X) \quad (\forall X \in W).$$

Since W is G-invariant we have a natural action of $G \times G$ on Γ by $(g_1, g_2).\gamma = g_1\gamma g_2^{-1}$. Note how the cone W is playing the role of the cone C earlier considered when we defined admissible Lie algebras. Hence we have explained the three types of admissibility, namely for the Lie algebra, for the orbit, and for the representation. To a unitary highest weight representation we associate on the one hand a cone in the Lie algebra, which therefore is admissible, and on the other hand an admissible orbit in the dual of the Lie algebra; and these correspondences go in the other directions as well.

Now unitary highest weight representations are further characterized by the property that they extend from G to holomorphic representations of the semigroup Γ. In this way one is brought to yet another of the great classics in mathematics, namely C^*-algebras, now appearing as certain quotients of the group C^*-algebra $C^*(G)$, modeling the holomorphic representations of Γ. Again there is a story of classical versus quantum at this point or at least between operators and geometric objects; any C^*-algebra A has associated with it the compact convex set of its states $S(A)$; and the extreme points of $S(A)$ correspond to irreducible representations of A, as was shown by Gel'fand, Naimark, and Segal.

The above are the main themes of the book by Neeb, only sketched here to give an impression of the ever-present interplay between on the one hand geometry of orbits, cones, and states, and on the other hand representation theory of both groups and semigroups. Some of the more advanced results are reached in the final chapters on complex geometry and representation theory. Let me just mention a few: Suppose we want to find a $G \times G$ - invariant subdomain of the holomorphic semigroup Γ of the form $\Omega = G\exp(iB)$ with B an invariant open subset of W, such that Ω is a Stein manifold (i.e. it has sufficiently many holomorphic functions to separate points, roughly speaking). Then the nice answer (proved by Neeb) is that this happens if and only if B is convex. Similarly, suppose ϕ is a $G \times G$ - invariant function on Ω; then ϕ is a plurisubharmonic function if and only if $X \mapsto \phi(\exp(iX))$ is a locally convex function on B. Thus very natural conditions for the complex geometry on Γ are reflected in convexity on W. More complex geometry shows up in the concept of coherent state representations, a notion also of interest in physics; these are unitary representations where the projective space of smooth vectors has an orbit which is a complex submanifold, and they in turn correspond to unitary highest weight representations. Finally on Γ one may define a Hardy space $\mathcal{H}^2(\Gamma)$, generalizing the classical Hardy space for the upper half-plane or the unit disc; it is a Hilbert space of holomorphic functions on Γ, invariant under $G \times G$, and the irreducible constituents in its decomposition will all be unitary highest weight modules.

The book requires a solid background as far as the Lie-theoretic part is concerned; however, the beginning and in particular the part about convex sets and functions is self-contained and can serve as a nice introduction to these theories, reaching a rather high level. The overlap with [2] is small; indeed, the book by Neeb is a logical continuation of the initial and groundbreaking work by Hilgert, Hofmann and Lawson, and it is nice to see how functional analysis, complex analysis, and representation theory have developed within the framework of Lie semigroups. One might also want to keep in mind [1], where different aspects of the theory of cones and representation theory are explained. The main parts of the book are labeled: A. Abstract Representation Theory, B. Convex Geometry and Representations of Vector Spaces, C. Convex Geometry of Lie Algebras, D. Highest Weight Representations of Lie Algebras, Lie Groups, and Semigroups, E. Complex Geometry and Representation Theory. In addition there is an appendix with useful surveys and even some new proofs; this is the case for example in the nice discussion of the Stone - von Neumann Theorem about the uniqueness of the Schrödinger representation in Appendix VIII.

Neeb's book does not contain any exercises, but on the other hand there are many carefully worked out examples and meticulous notes about the history and literature in the field. Which brings me to the main impression about this book: It guides the reader to the absolutely most general statements (at present), and does so very patiently with great care taken to provide clarity of both notation and explanation of ideas. For this mixture of almost Bourbaki style and more informal insights to the reader, I think the book has its greatest strength. It is an important book taking its well-prepared (and energetic!) reader on a grand tour of the geometry of unitary highest weight representation. Furthermore, it contains several new proofs and improvements on earlier work; for example it gives a new proof of Lawson's general theorem on the existence of Ol'shanskii semigroups. On the negative side one can hardly count the omission of topics (because otherwise the book would have been just too long) such as those mentioned by Neeb himself: ordered symmetric spaces, automorphic forms, wavelets, infinite-dimensional domains for example. In summary then, the book by Neeb has given the subject a solid platform from which to build many more advances in the mathematics of convexity and holomorphy as it applies to representation theory of Lie groups.

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