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The quantum theory of fields. III. Supersymmetry, by Steven Weinberg, Cambridge Univ. Press, Cambridge, 2000, xxii + 419 pp., $49.95, ISBN 0-521-66000-9

Supersymmetry is an idea that has played a critical role in many of the recent developments in theoretical physics of interest to mathematicians. The third volume of *The quantum theory of fields* by Steven Weinberg [1] is an introduction to supersymmetric field theory and supergravity. The first two volumes of the series treat the essentials of quantum field theory. In this third volume, Weinberg has created the most complete introduction to supersymmetry to date. Although the text is aimed squarely at physicists, it should prove useful to mathematicians interested in learning about supersymmetry in its natural physical setting. As a supplement, to help bridge the cultural and language differences between physics and mathematics, I would suggest the short lecture series by Freed [2]. My goal, in the course of this review, is to convey both the essential idea behind supersymmetry and some of the background needed to peruse the physics literature.

What is supersymmetry? The basic notion behind supersymmetry (SUSY) can be described in the setting of quantum mechanics. As a starting point, let us consider a particle in one dimension with position x moving in a potential well, $V(x)$. The time evolution of this system is governed by a Hamiltonian, H, which takes the form

$$H = \frac{1}{2}p^2 + V(x). \tag{1}$$

The momentum, p, is a vector field that satisfies the commutation relation

$$[x, p] = xp - px = i. \tag{2}$$

To avoid unimportant technical issues, let us assume that $V \to \infty$ as $|x| \to \infty$. The states of this theory describe the possible quantum mechanical configurations for the particle. Each state is described by a square normalizable function of x. We also note that our Hamiltonian is Hermitian with respect to the standard inner product on the Hilbert space of states.

To make this system supersymmetric, we need to add additional degrees of freedom of a quite different kind. Particles in nature come in two distinct flavors: there are bosons and there are fermions. While bosons are described by conventional commuting variables, fermions are described by variables that take values in a Grassmann algebra. A collection of M real fermions in quantum mechanics satisfies the quantization conditions:

$$\{\psi_a, \psi_b\} = \psi_a\psi_b + \psi_b\psi_a = \delta_{ab}, \quad a = 1, \dots, M. \tag{3}$$

By contrast, additional bosonic degrees of freedom, labelled say $y_1, y_2, \dots,$ would satisfy commutation relations:

$$[x, y_i] = 0, \quad [y_i, y_j] = 0. \tag{4}$$

That fermions satisfy anti-commutation rather than commutation relations is their hallmark. In a system with bosons and fermions, we can define a *conserved* $\mathbb{Z}_2$

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charge measured by the operator $(-1)^F$,

$$(-1)^F = \prod_a \psi_a. \tag{5}$$

A purely bosonic operator $\mathcal{O}_B$ satisfies

$$\big[(-1)^F, \mathcal{O}_B\big] = 0, \tag{6}$$

while a purely fermionic operator $\mathcal{O}_F$ satisfies

$$\big\{(-1)^F, \mathcal{O}_F\big\} = 0. \tag{7}$$

That $(-1)^F$ is conserved is the statement that it commutes with H, so H is a purely bosonic operator. The Hilbert space of the theory can therefore be organized into states with definite eigenvalue under $(-1)^F$. Bosonic states have eigenvalue $+1$, while fermionic states have eigenvalue -1.

We might now imagine modifying H by adding additional interactions which preserve the condition that H be Hermitian. These interactions are operators which are constructed from the fermions, ψ_a, and the boson, x. The Hamiltonian must continue to commute with $(-1)^F$, so these interactions can involve only even numbers of fermions. Beyond this constraint, these interactions can be essentially whatever we choose. For supersymmetry, however, we demand that the resulting Hamiltonian satisfy the algebraic relations

$$\{Q_a, Q_b\} = H\delta_{ab}, \quad a, b = 1, \ldots, N. \tag{8}$$

Our supercharges, Q_a, are also Hermitian operators, and the parameter N determines the degree of supersymmetry. A system with $N = 1$ has simple supersymmetry. Systems with $N > 1$ have extended supersymmetry. Typically, our control over a theory increases with the number of supersymmetries.

To supersymmetrize the Hamiltonian of equation (1), we can add an interaction coupling two fermions to the boson, x, giving a new Hamiltonian,

$$H_{\text{susy}} = \frac{1}{2}p^2 + V(x) - i\frac{\partial}{\partial x}\sqrt{2V(x)}\psi_1\psi_2. \tag{9}$$

This Hamiltonian obeys the algebra

$$H_{\text{susy}} = Q^2, \tag{10}$$

with supercharge

$$Q = \psi_1 p + \psi_2\sqrt{2V(x)}. \tag{11}$$

Note that Q is fermionic. It is, in essence, a 'square-root' of H. We can now begin to see why supersymmetry is so powerful. From the algebra given in (8), we see that the spectrum of any supersymmetric Hamiltonian, H_{susy}, is bounded from below (by zero). Further, supersymmetry requires that eigenstates of H with non-zero energy eigenvalue appear in degenerate pairs. Given any state $|E\rangle$ with energy eigenvalue E, we can construct a degenerate state, $\frac{Q}{\sqrt{E}}|E\rangle$. Since Q itself is fermionic, one of these states is bosonic while the other is fermionic. In this way, supersymmetry maps bosons to fermions and vice-versa. Although arrived at in a simplified setting, these essential features of supersymmetry generalize from quantum mechanics to higher-dimensional field theories.

Weinberg begins his text not with supersymmetric quantum mechanics, but with a short historical note. Supersymmetry has its roots in the early development of string theory. It was soon realized, however, that supersymmetry could be implemented in conventional four-dimensional quantum field theories. The Standard Model of particle physics constitutes the most important example of a theory that can be supersymmetrized. The resulting theory, known as the 'minimal supersymmetric standard model' (MSSM), remains one of the more promising candidates for describing particle physics beyond the current Standard Model. There is a common terminology associated to the pairing of bosons and fermions under supersymmetry. In most cases, to an observed fermion of a supersymmetric theory, we associate a 'sparticle' superpartner. For example, electrons are leptons observed in the world around us. In the context of a supersymmetric theory of leptons, the superpartner of an electron is a 'selectron'. Likewise, the superpartner of a quark is a 'squark'. For observed bosons, the appellation for the superpartner is different. We append 'ino' to the name of the boson; for example, the superpartner of a graviton – the particle that we believe mediates the gravitational interaction – is a gravitino, while the superpartner of a photon is a photino.

Clearly, supersymmetry is not an exact symmetry of the world around us. If this were the case, we would have already observed the superpartners of light or massless particles like the photon. Nevertheless, many physicists believe that supersymmetry will be observed as a fundamental symmetry of particle interactions as we probe higher energy scales. It is also worth noting that our only consistent theory of quantum gravity – namely, string theory – requires supersymmetry in its formulation.

The initial discovery of supersymmetry was particularly surprising because it avoids a famous 'no-go' theorem by Coleman and Mandula [3]. Quantum field theories in $D+1$ space-time dimensions consist of states that transform irreducibly under the Poincaré group of rotations, boosts, and translations.[1] Under reasonable conditions, Coleman and Mandula argued that the only possible symmetries of quantum field theory consist of the Poincaré group together with possible internal symmetries that commute with Poincaré. However, the theorem assumes that all symmetries preserve the $\mathbb{Z}_2$ grading by fermion number. By contrast, supersymmetry relates bosons to fermions, and hence evades the theorem. Weinberg's discussion of these points is remarkably detailed. He gives complete arguments which are often simpler than those given in the original papers. The thoroughness with which Weinberg develops his arguments is perhaps my favorite feature of this text.

Supersymmetry algebras are the next topic of discussion. The $N=1$ supersymmetry algebra (10) of quantum mechanics generalizes to four space-time dimensions ($D=3$) in the following way:

$$\begin{aligned} \{Q_\alpha, Q^*_\beta\} &= 2\sigma^\mu_{\alpha\beta} P_\mu, \qquad \alpha,\beta = 1,2 \\ \{Q_\alpha, Q_\beta\} &= 0. \end{aligned} \tag{12}$$

A word on notation is in order: the σ^μ are the Pauli matrices with $\sigma^0 = -\mathbf{1}$. In a convenient basis,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13}$$

[1] It is unreasonable for me to attempt an explanation of quantum field theory here. Fortunately, there has been a substantial effort devoted to making quantum field theory accessible to mathematicians. The proceeds of this effort appear in [4].

The P_μ are space-time momenta, while the Q_α are complex two component space-time spinors. The space-time metric is the Minkowski metric with diagonal entries, $\{-1, 1, 1, 1\}$. One of the annoying features of the literature on supersymmetry is the bewildering array of notations and conventions. Whether one likes or dislikes his choices, Weinberg is considerate enough to clearly state his conventions, which are consistent with his first two volumes on quantum field theory.

In quantum field theory, there is a rather important correlation between the statistics of a particle (Bose versus Fermi) and its spin. A particle described by a quantum field theory is characterized by its mass and spin. The notion of spin is an important one and merits an explanation. These two characteristics, mass and spin, come naturally from representation theory in the following way: the single particle states of the quantum field theory essentially describe a particle moving with momentum P. These states form an irreducible representation of the Poincaré group. Irreducible representations of Poincaré are labelled by two Casimir invariants. The first Casimir is $P \cdot P = -M^2$, where M is the mass of the particle.

The second invariant can be described as follows: while $P \cdot P$ is invariant under the action of the Lorentz group, any particular momentum P is only left invariant by a subgroup of the Lorentz group, $Spin(3, 1)$. This subgroup is called the little group of $Spin(3, 1)$. In determining the spin of a particle, we need to consider two distinct cases: in the case of massive particles where $M \neq 0$, it is not hard to see that the little group is $SU(2)$. The single particle states with a fixed momentum P transform irreducibly under the little group. Let the dimension of this $SU(2)$ representation be $2j + 1$. The quantum number j is the spin of the particle, and $-M^2 j(j + 1)$ is the second Casimir of the Poincaré group. If we boost to a frame where the particle is stationary so only $P^0 = M$ is non-zero, we see that spin indeed describes how the particle rotates in three space.

For massless particles where $P \cdot P = 0$, the situation is different. The little group which leaves any particular P invariant has three generators which we label J_3, B_1 and B_2. In a convenient frame where the particle is moving along the third axis, J_3 generates rotations around this axis, while B_1 and B_2 generate particular boosts. These generators satisfy the Lie algebra relations

$$[B_1, B_2] = 0, \quad [J_3, B_1] = iB_2, \quad [J_3, B_2] = -iB_1. \tag{14}$$

The representations of this algebra which correspond to physical particles are quite restricted: B_1 and B_2 must act trivially on allowed representations. Further, the eigenvalue of J_3, known as the helicity of the particle, is quantized. The helicity can be either integral or half-integral.

There is, however, a standard abuse of notation under which a massless particle is assigned spin, as if it were massive. For example, a massless photon is a spin 1 particle even though it consists only of helicity ± 1 states. Likewise, a graviton is a spin 2 particle. With this caveat in mind, we can now state the spin-statistics theorem which follows from quite general properties of quantum field theory: bosons must have integral spin, while fermions must have half-integral spin.

As we see from (12), minimal supersymmetry in four dimensions requires four real supercharges. Our next step, in tandem with Weinberg, is to study representations of the minimal supersymmetry algebra. From the representations, we can determine the combinations of particles needed to build supersymmetric field theories. There are again two distinct cases. If we consider a multiplet of massive particles with mass M, we can always boost to a frame where the particles are stationary. The only non-vanishing component of P^μ is again $P^0 = M$. In this convenient frame, it is easy to construct representations of the resulting algebra,

$$\{Q_\alpha, Q^*_\beta\} = 2\delta_{\alpha\beta} M. \tag{15}$$

Take a Fock vacuum $|0\rangle$ satisfying $Q_\alpha|0\rangle = 0$. By acting with Q^*, we build a representation with states

$$|0\rangle, \quad Q^*_\alpha|0\rangle, \quad \epsilon^{\alpha\beta} Q^*_\alpha Q^*_\beta|0\rangle.$$

Note that the Fock vacuum need not be invariant under the Lorentz group. This representation of the supersymmetry algebra is reducible under the Lorentz group. On decomposition to irreducible representations under Lorentz, we discover the particle content of this supermultiplet. For example, if the Fock vacuum $|0\rangle$ is invariant under the Lorentz group, we obtain the following fields: two scalar fields with spin 0 corresponding to the states $|0\rangle$ and $\epsilon^{\alpha\beta} Q^*_\alpha Q^*_\beta|0\rangle$, and one fermion with spin 1/2, $Q^*_\alpha|0\rangle$.

Because this supermultiplet contains massive particles, there are no particularly strong physical constraints on the permitted representations. For sufficiently large masses, the spins of the constituent particles can be arbitrarily high without a physical inconsistency appearing in the low-energy observed theory. The situation is quite different for massless particles. We can still boost to a special frame where $P^\mu = (E, 0, 0, E)$. The algebra of (12) becomes

$$\{Q_\alpha, Q^*_\beta\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta}. \tag{16}$$

We see that half the supersymmetry generators are represented trivially. The size of our representations is therefore reduced in comparision with the case of a massive particle. On massless particles, there are strong physical constraints. Theories of massless particles with spins greater than 2 are not believed to be consistent. We should note that a theory with a spin 2 massless particle is necessarily a theory of gravity. The spin 2 particle is the graviton.

We therefore impose the constraint that the spins of massless particles in a supermultiplet not exceed this bound. However, if we increase the number of supersymmetries beyond the minimal four supercharges, the maximum spin of particles in a supermultiplet increases. As Weinberg describes in more detail, we can have an extended supersymmetry algebra with at most 32 real supercharges. This maximally supersymmetric theory is quite unique. It has a single permitted representation. Among the particles of this supermultiplet is a spin 2 particle, so this maximally supersymmetric theory includes gravity. A theory of supersymmetry and gravity is known as a theory of supergravity (SUGRA).

In four dimensions, theories with sixteen or fewer real supercharges need not contain gravity. There is a profound difference between supersymmetric theories with and without gravity. Without gravity, supersymmetry is a global symmetry: we must perform the same supersymmetry transformation at each point in space-time. With gravity, the situation is quite different. Supersymmetry becomes a local, or gauge symmetry, in which the parameters of our supersymmetry transformation are allowed to vary over space-time. The construction of supergravity theories is quite involved, but Weinberg's discussion in chapter 31 is a reasonable place to start.

To complete this brief introduction to supersymmetry, let us turn momentarily to the Lagrangian formulation of supersymmetric theories. This is, by far, the most common formalism for discussing field theory. Our quantum mechanical Hamiltonian of equation (9) will serve as an example. This Hamiltonian can be obtained from the action

$$\begin{aligned} S &= \int dt\, L(x,\psi) \\ &= \int dt \left\{ \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{i}{2}\sum_i \psi_i \frac{d\psi_i}{dt} - V(x) + i\frac{\partial}{\partial x}\sqrt{2V(x)}\psi_1\psi_2. \right\}, \end{aligned} \tag{17}$$

by a Legendre transformation. In the Lagrangian formalism, where L is the Lagrangian, we view $x = x(t)$ and $\psi_i = \psi_i(t)$ as fields rather than operators. As fields in the Lagrangian, fermions obey the anti-commutation relations

$$\{\psi_a, \psi_b\} = 0. \tag{18}$$

Supersymmetry transformations are parametrized by a Grassmann variable ϵ and act on the fields in the following way:

$$\delta_\epsilon x = -i\epsilon\psi_1, \qquad \delta_\epsilon\psi_1 = \epsilon\frac{dx}{dt}, \qquad \delta_\epsilon\psi_2 = \epsilon\sqrt{2V(x)}. \tag{19}$$

Under the action of any symmetry, the Lagrangian must vary into a total derivative. It is not hard to check that the variations given in (19) define a symmetry of the action (17) with ϵ time-independent. Closure of the supersymmetry algebra implies that

$$[\delta_\epsilon, \delta_{\epsilon'}] = 2i\epsilon\epsilon'\frac{d}{dt}, \tag{20}$$

when acting on any of the fields. It is a quite general feature of supersymmetry that satisfying (20) on all the fields typically requires the use of the equations of motion. For a theory defined by a Lagrangian, the equations of motion are the Euler-Lagrange equations. In such situations, we say that the symmetry algebra closes only on-shell, i.e., for fields satisfying their equations of motion. This is true even for theories of free particles.

At this point, we should systematize the procedure for constructing supersymmetric Lagrangians and extend it to higher-dimensional field theories. One way to do this is by introducing the notion of superspace. Unfortunately, fascinating topics like superspace, supersymmetry breaking, and applications like Seiberg-Witten theory, are beyond the scope of this review. Fortunately, these topics and many more can be found in Weinberg's quite comprehensive text. These modern topics, particularly the derivation of the Seiberg-Witten solution [5] and the discussion of non-perturbative physics, largely differentiate Weinberg's text from other classic texts on supersymmetry and supergravity, namely, the book by Wess and Bagger [6] and the book by West [7]. I hope the reader is sufficiently enticed to further explore this beautiful subject.

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