BOOK REVIEWS

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p-Adic L-functions and p-adic representations, by Bernadette Perrin-Riou, translated by Leila Schneps, Amer. Math. Soc., Providence, RI, 2000, xx + 150 pp., \$49.00, ISBN 0-8218-1946-1

This research monograph presents a significant advance in an active and important field. To describe its contents and put them into perspective, we begin with a classical example.

If χ is a Dirichlet character, the Dirichlet L-function $L(\chi, s)$ is defined to be the meromorphic continuation to $\mathbf{C}$ of the Euler product $\prod_q (1 - \chi(q)q^{-s})^{-1}$, product over prime numbers q. When χ is the trivial character this is the Riemann zeta function, analytic except for a simple pole at $s = 1$, and otherwise $L(\chi, s)$ is an entire function.

These functions have many important properties, but we will concentrate on their special values. For example, if $d < -3$ is a squarefree integer and χ_d is the quadratic Dirichlet character with the property that for all primes q not dividing d,

$$\chi_d(q) = \begin{cases} 1 & \text{if } d \text{ is a square modulo } q \\ -1 & \text{if } d \text{ is not a square modulo } q, \end{cases}$$

then

$$L(\chi_d, 0) = h_d,$$

where h_d is the class number of the imaginary quadratic field $\mathbf{Q}(\sqrt{d})$.

This "analytic class number formula" is a special case of what is expected to be a very general connection between special values of L-functions and arithmetic invariants. The Birch and Swinnerton-Dyer conjecture is one such generalization, relating the behavior of the L-function of an elliptic curve at $s = 1$ with the arithmetic of the elliptic curve. Bloch and Kato [BK] have conjectured such relations in vast generality, for L-functions attached to motives, but these have been proved only in very special cases.

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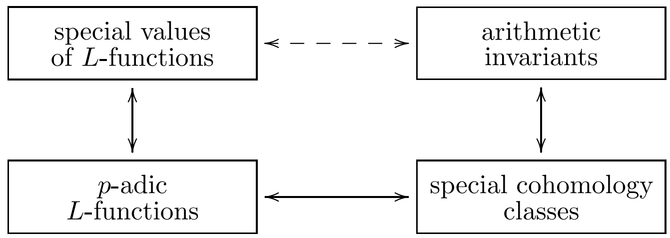
So how can one hope to prove such relations, relating analytic and algebraic objects? The most promising route is described in the following diagram.

(1)



One can try to prove the desired relations (the dashed upper arrow) by understanding the other three arrows. We first describe this picture in some detail, in the special case of a Dirichlet character χ.

Fix a prime number $p > 2$, write $\mathbf{Q}_p$ for the p-adic field (the p-adic completion of $\mathbf{Q}$), and write $\mathbf{Z}_p$ for the ring of p-adic integers. Suppose that χ is an even Dirichlet character (i.e., $\chi(-1) = 1$). Fix embeddings of the algebraic closure $\bar{\mathbf{Q}}$ in both $\mathbf{C}$ and $\bar{\mathbf{Q}}_p$ (the algebraic closure of $\mathbf{Q}_p$), so that we can view χ as taking values both in $\mathbf{C}^\times$ and in $\bar{\mathbf{Q}}_p^\times$.

The (classical, complex) L-function $L(\chi, s)$ was defined above. The p-adic L-function attached to χ was first constructed by Kubota and Leopoldt, and then a different construction was given by Iwasawa. This p-adic L-function, which we will denote by $L_p(\chi, s)$, is a p-adic analytic function from $\mathbf{Z}_p$ to the subring $\mathcal{O}_\chi$ of $\bar{\mathbf{Q}}_p$ generated over $\mathbf{Z}_p$ by the values of χ, which interpolates special values of $L(\chi, s)$:

$$(2) \qquad L_p(\chi, k) = (1 - \chi(p)p^{-k})L(\chi, k) \quad \text{for integers } k \le 0,\ k \equiv 1 \pmod{p-1}.$$

Since these k form a dense subset of $\mathbf{Z}_p$, these interpolation formulas determine the continuous function $L_p(\chi, s)$ on all of $\mathbf{Z}_p$. The analyticity is expressed by the fact that there is a power series $g_\chi(T) \in \mathcal{O}_\chi[[T]]$ such that $L_p(\chi, s) = g_\chi((1+p)^s - 1)$. Once we know the existence of an analytic function $L_p(\chi, s)$ satisfying (2), its connection with special values of the classical L-function (the left-hand arrow of (1)) is clear.

If the conductor of χ is mp^r, with m prime to p and $r \ge 0$, then the "special cohomology classes" in (1) are the cyclotomic units $\zeta_{mp^n} - 1$ for $n \in \mathbf{Z}^+$, where $\zeta_{mp^n} = e^{2\pi i/(mp^n)}$ is a primitive mp^n-th root of unity. Iwasawa showed that these units are related to $L_p(\chi, s)$; this relation is an example of the bottom arrow of (1). We will describe a slightly different version of this connection, a construction of Coates and Wiles [CW] that uses these cyclotomic units to construct $L_p(\chi, s)$.

For simplicity, we will restrict ourselves to the case where the conductor of χ is p, so χ is a nontrivial (even) character of $\mathrm{Gal}(\mathbf{Q}(\boldsymbol{\mu}_p)/\mathbf{Q})$ where $\boldsymbol{\mu}_p$ denotes the group of p-th roots of unity. Suppose that $\mathbf{u} = \{u_n\} \in \varprojlim \mathbf{Z}_p[\boldsymbol{\mu}_{p^n}]^\times$, the inverse limit with respect to the norm map (i.e., u_n is a unit of the ring $\mathbf{Z}_p[\boldsymbol{\mu}_{p^n}]$ for each n, and the norm of u_{n+1} is u_n). Coates and Wiles showed that there is a power series $h_{\mathbf{u}}(X) \in \mathbf{Z}_p[[X]]$ such that $h_{\mathbf{u}}(\zeta_{p^n} - 1) = u_n$ for every n. (Soon afterwards, Coleman [Cn] generalized the map $\mathbf{u} \mapsto h_{\mathbf{u}}$, and it has come to be known as the Coleman map.) Leopoldt's "Γ-transform" shows that there is a (necessarily unique) p-adic analytic function $\mathcal{L}_{\mathbf{u}}$ satisfying

$$(3) \qquad \mathcal{L}_{\mathbf{u}}(k) = \left(\left((X+1)\frac{d}{dX}\right)^k \tilde{h}_{\mathbf{u}}(X)\right)\Big|_{X=0}$$

for every $k \geq 1$, where $\tilde{h}_{\mathbf{u}}(X) = \log(h_{\mathbf{u}}(X)) - p^{-1} \sum_{\zeta \in \boldsymbol{\mu}_p} \log(h_{\mathbf{u}}(\zeta X + \zeta - 1))$.

Now take $\mathbf{u}^{\chi} = \{u_n\}$ with

$$u_n = \prod_{a=1}^{p-1} (\zeta_{p^n}^{\omega(a)} - 1)^{\chi^{-1}(a)},$$

where ω is the Teichmuller character, the Dirichlet character $(\mathbf{Z}/p\mathbf{Z})^{\times} \to \mathbf{Z}_p^{\times}$ satisfying $\omega(a) \equiv a \pmod{p}$. Then $\mathbf{u}^{\chi} = \{u_n\} \in \varprojlim \mathbf{Z}_p[\boldsymbol{\mu}_{p^n}]^{\times}$, and one sees easily that $h_{\mathbf{u}^{\chi}} = \prod_{a=1}^{p-1} ((X+1)^{\omega(a)} - 1)^{\chi^{-1}(a)}$. A direct calculation now shows (see for example [La], Chapter 7, §5), using (3) and the definition (2) of the p-adic L-function, that $\mathcal{L}_{\mathbf{u}^{\chi}}(s) = L_p(\chi, s)$. In other words, the Γ-transform of the Coleman power series of the cyclotomic unit $\mathbf{u}^{\chi}$ is the p-adic L-function. This is the fundamental example of the bottom arrow of (1).

The right hand arrow of the diagram (1) is provided, at least in certain cases, by Kolyvagin's theory of Euler systems, which in this case relates cyclotomic units with ideal class groups of cyclotomic fields. Since this part of the picture plays no direct role in the book under review, we will not discuss this here. See [Ru1] for more information in this special case, and [Ru2] in general.

The book by Perrin-Riou develops a vast generalization of the example discussed above, which we now attempt to describe. Because the general picture is very technical and complex, we will limit this discussion to a broad and necessarily incomplete (and frequently oversimplified) overview.

In the general case, the fundamental object of study is a p-adic Galois representation, which we will take to mean a free $\mathbf{Z}_p$-module T of finite rank with a continuous action of $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. Given such a T, its (Cartier) dual $T^* = \mathrm{Hom}(T, \mathbf{Z}_p(1))$ also plays an important role, where $\mathbf{Z}_p(1) = \varprojlim \boldsymbol{\mu}_{p^n}$. If χ is a Dirichlet character of conductor p as above (which we identify with a character $G_{\mathbf{Q}} \to \mathbf{Q}_p^{\times}$ in the usual way), we associate to it the one-dimensional Galois representation $T = \mathbf{Z}_p(1) \otimes \chi^{-1}$ on which $\gamma \in G_{\mathbf{Q}}$ acts via multiplication by $\chi^{-1}\varepsilon(\gamma)$, where $\epsilon : G_{\mathbf{Q}} \to \mathrm{Aut}(\boldsymbol{\mu}_{p^{\infty}}) \cong \mathbf{Z}_p^{\times}$ is the cyclotomic character. (For a general Dirichlet character we need to adjoin the values of χ to $\mathbf{Z}_p$). We then get $T^* = \chi$, a one-dimensional representation with $G_{\mathbf{Q}}$ acting via χ.

Another important collection of examples comes from the theory of elliptic curves: if E is an elliptic curve, then its p-adic Tate module $T_p(E)$ is the free, rank-two $\mathbf{Z}_p$-module $\varprojlim E[p^n]$, where $E[p^n]$ denotes the kernel of multiplication by p^n in $E(\bar{\mathbf{Q}})$. The Weil pairing shows that $T_p(E)^* = T_p(E)$.

In order to make sense of the diagram (1) in this general context, we need to make some assumptions on the representation T. Perrin-Riou assumes that T comes from a "motivic structure" that is "crystalline at p". (Roughly speaking, this means that T looks like it comes from the action of $G_{\mathbf{Q}}$ on a piece of the cohomology of some variety that has good reduction at p.) Given such a T, let $A^* = T^* \otimes (\mathbf{Q}_p/\mathbf{Z}_p)$, the divisible version of the dual representation.

Using ideas pioneered by Greenberg [Gr] and developed by many people, we can associate to A^* a Selmer group $\mathrm{Sel}(\mathbf{Q}, A^*)$, a subgroup of $H^1(\mathbf{Q}, A^*)$ cut out by "local conditions". If $T = \mathbf{Z}_p(1) \otimes \chi^{-1}$ as above, then A^* is a copy of $\mathbf{Q}_p/\mathbf{Z}_p$ on which $G_{\mathbf{Q}}$ acts via χ, and the Selmer group is essentially the χ-part of the ideal class group of the abelian extension of $\mathbf{Q}$ corresponding to χ. If T is the Tate module of an elliptic curve E, then $A^* = E[p^{\infty}]$ and the Selmer group is the classical p-power

Selmer group attached to E, which lies in an exact sequence

$$0 \longrightarrow E(\mathbf{Q}) \otimes (\mathbf{Q}_p/\mathbf{Z}_p) \to \mathrm{Sel}(\mathbf{Q}, E[p^\infty]) \longrightarrow Ш_{E/\mathbf{Q}}[p^\infty] \longrightarrow 0$$

where $E(\mathbf{Q})$ is the group of rational points on E and $Ш_{E/\mathbf{Q}}$ is the Tate-Shafarevich group of E. Extrapolating from these examples, for a general representation T we will view $\mathrm{Sel}(\mathbf{Q}, A^*)$ (and $\mathrm{Sel}(\mathbf{Q}(\boldsymbol{\mu}_{p^n}), A^*)$ for every n) as the interesting "arithmetic invariants" in diagram (1).

On the other hand it has been known for some time how to (attempt to) define an L-function attached to T. Namely, $L(T, s)$ is given as an Euler product $L(T, s) = \prod_q \ell_q(T, s)$ where for all primes $q \neq p$,

$$\ell_q(T, s) = \det(1 - \mathrm{Fr}_q^{-1} q^{-s} \mid T^{I_q})^{-1}.$$

Here $I_q \subset G_{\mathbf{Q}}$ is an inertia group of q (well-defined up to conjugation), T^{I_q} is the subgroup of T fixed by I_q, and $\mathrm{Fr}_q \in G_{\mathbf{Q}}$ is a Frobenius automorphism for q, well-defined modulo I_q and conjugation. The Euler factor $\ell_p(T, s)$ is considerably more difficult to define, and we omit the definition. When $T = \mathbf{Z}_p(1) \otimes \chi^{-1}$, we get $\ell_q(T^*, s) = (1 - \chi^{-1}(q) q^{-s})^{-1}$ for every q, so $L(T^*, s) = L(\chi^{-1}, s)$.

It is a standard conjecture that, with our assumptions about the representation T, the L-function $L(T, s)$ has a meromorphic continuation to all of $\mathbf{C}$, which is analytic away from $s = 1$, and there is a functional equation relating $L(T, s)$ and $L(T^*, s)$. This conjecture is proved only in very special cases, including the cases of Dirichlet characters and elliptic curves mentioned above. (For elliptic curves this is a consequence of the Shimura-Taniyama conjecture, whose proof was begun by Wiles [Wi] and recently completed by Breuil, Conrad, Diamond and Taylor [BCDT].) The "special values of L-functions" in diagram (1) are the values $L(T^* \otimes \rho, k)$ for integers k and characters $\rho : G_{\mathbf{Q}} \to \bar{\mathbf{Q}}_p^\times$ of finite order.

The Bloch-Kato conjecture [BK] relates these special values of L-functions with orders of Selmer groups. This is the fundamental relation which we would like to understand. The diagram (1) is a proposed road map of a promising approach, and we would like to try to fill in the rest of the diagram.

How can one define a p-adic L-function $L_p(T)$ in this generality? This question is at the heart of Perrin-Riou's book. It is an exceedingly hard question, and we will only indicate some of the difficulties that need to be overcome. Any answer must be highly conjectural, because even the existence of the special values (of the complex L-function) that we hope to interpolate is conjectural.

The first problem is that the values $L(T \otimes \rho, k)$, assuming they exist, are complex numbers. Before we can interpolate them p-adically we need to make them into algebraic numbers so that we can view them inside $\bar{\mathbf{Q}}_p$. Deligne has defined a "period" $\Omega(T \otimes \rho, k) \in \mathbf{C}^\times$ and conjectured that $L(T \otimes \rho, k)/\Omega(T \otimes \rho, k) \in \bar{\mathbf{Q}} \subset \bar{\mathbf{Q}}_p$. But before these numbers can be interpolated p-adically, one must first multiply by

- a p-adic period, to replace the complex period which was removed,
- an analogue of the factor $(1 - \chi(p)p^{-k})$ of (2),
- an appropriate "Euler factor at infinity".

Once this has been done (none of these factors are obvious, and there is no room for error) Perrin-Riou conjectures the existence of a p-adic analytic function interpolating these correctly normalized L-values with a formula analogous to (2) but much more complicated.

The above discussion skips over many subtleties. The general definition of "p-adic analytic" is weaker than in the example above of a Dirichlet character, where we required that the analytic function correspond to a power series in $\mathbf{Z}_p[[X]]$. In the general case we allow elements of the field of fractions $\mathcal{K}$ of the subring of $\mathbf{Q}_p[[X]]$ consisting of power series whose growth is bounded by a power of $|\log(1+X)|_p$ as $|X|_p \to 1^-$, $X \in \bar{\mathbf{Q}}_p$.

Also, in the general case the p-adic period associated to T^* lies in a $\mathbf{Q}_p$-vector space $\wedge^{d^-}\mathbf{D}$, where $\mathbf{D}$ is a $\mathbf{Q}_p$-vector space of dimension $d = \mathrm{rank}_{\mathbf{Z}_p}(T)$ associated to T by Fontaine's theory of p-adic periods, and d^- is the $\mathbf{Z}_p$-rank of the subgroup of T where complex conjugation acts by -1. In the end, "the" p-adic L-function $L_p(T^*)$ is an element of $\mathcal{K}\otimes\wedge^{d^-}\mathbf{D}$ that can be viewed as a $\binom{d}{d^-}$-dimensional family of p-adic analytic functions. If we work through all of these complications and accept all of these conjectures, we obtain both the lower left corner and the left-hand map of (1).

The key to the bottom of the diagram (1) is a generalization, constructed by Perrin-Riou in [PR1], of the cyclotomic Coleman map $\mathbf{u} \mapsto h_{\mathbf{u}} \mapsto \mathcal{L}_{\mathbf{u}}$ discussed above. The oversimplified version of the general Coleman map is a map

$$\mathcal{L} : \varprojlim H^1(\mathbf{Q}_p(\boldsymbol{\mu}_{p^n}), T) \longrightarrow \mathcal{K} \otimes \mathbf{D}.$$

This Coleman map has the same flavor as the cyclotomic one, as it takes a norm-compatible sequence of elements $z_n \in H^1(\mathbf{Q}_p(\boldsymbol{\mu}_{p^n}), T)$ to a power series that interpolates the "logarithms" of the elements z_n. In this case the logarithm in question, from $H^1(\mathbf{Q}_p(\boldsymbol{\mu}_{p^n}), T)$ to $\mathbf{Q}_p(\boldsymbol{\mu}_{p^n}) \otimes \mathbf{D}$, is the dual of the Bloch-Kato exponential map.

Combining this Coleman map with the localization-at-p map, we get

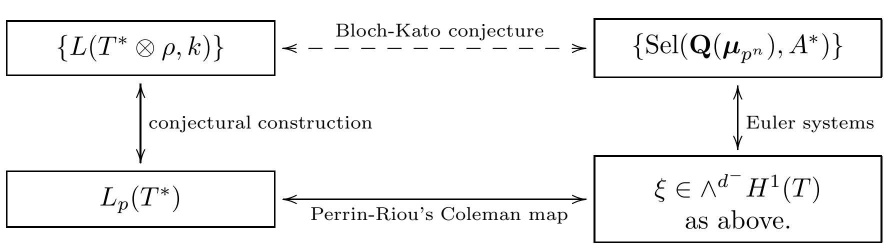
$$\mathcal{L}^{\otimes d^-} : \wedge^{d^-} \varprojlim H^1(\mathbf{Q}(\boldsymbol{\mu}_{p^n}), T) \longrightarrow \mathcal{K} \otimes \wedge^{d^-}\mathbf{D}.$$

Let ι denote the automorphism of $\mathcal{K} \otimes \wedge^{d^-}\mathbf{D}$ induced by $X \mapsto -X/(X+1)$ (that is, $X+1 \mapsto (X+1)^{-1}$) on $\mathcal{K} \subset \mathbf{Q}_p((X))$.

Conjecturally, $\varprojlim H^1(\mathbf{Q}(\boldsymbol{\mu}_{p^n}), T)$ is a module of rank d^- over the Iwasawa algebra, so its d^--th exterior power has rank one.

Question. *Is there an element* $\xi \in \wedge^{d^-} \varprojlim H^1(\mathbf{Q}(\boldsymbol{\mu}_{p^n}), T)$ *such that* $\mathcal{L}^{\otimes d^-}(\xi) = L_p(T^*)^\iota$*?*

In this generality one can conjecture that the answer is often "yes" and always "almost". (For example, Perrin-Riou asks a weaker question and conjectures that the answer is always "yes".) The element ξ satisfying $\mathcal{L}^{\otimes d^-}(\xi) = L_p(T^*)^\iota$, if it exists, provides the "special cohomology classes" of diagram (1), and Perrin-Riou's Coleman map provides the bottom arrow. Thus, given a p-adic representation T, we can redraw the diagram (1)



In summary, thanks to this book one has a fairly complete *conjectural* version of the diagram (1), in tremendous generality. As one might guess, a book that works out the details of (and removes the imprecision from) the discussion above is necessarily very technical and not easy to read. Even experts would benefit from reading two other papers by the author, which carry out all the constructions of this book in two special cases: [PR2], for the case of Dirichlet characters, and [PR3], for the case of elliptic curves.

In addition, since the original French version was published, Colmez has been able to weaken the assumption that T is crystalline at p and to prove one of the conjectures stated in Perrin-Riou's book. His paper [Cz] provides another written version which can be useful in studying this important work of Perrin-Riou.

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