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Stable groups, by Bruno Poizat, translated by Moses Gabriel Klein, Math. Surveys Monogr., vol. 87, Amer. Math. Soc., Providence, RI, 2001, xiii + 129 pp., $49.00, ISBN 0-8218-2685-0

This is a book that I like. So says its author in the foreword to this English edition, and the reviewer concurs. Indeed, the book is well-written, it is well-organized, and it contains a beautiful development of the theory of stable groups.

The book under review, an English translation of Poizat's self-published *Groupes Stables*, retains a fresh perspective on a subject that has undergone substantial development since the publication of the French original a decade and a half ago. The mathematical ideology propounded in this book has been largely confirmed, even though some of the conjectures have been refuted.

Poizat argues, sometimes explicitly, but generally through practice, that the theory of stable groups grows organically from the study of models and from the internal robustness of its own mathematical theory. This orientation is reflected in the organization of the book in that, with the exception of the first few sections of the first chapter which concerns easy consequences of the definition of a stable group, the theory is developed from the special to the general and the axiomatic approach is eschewed.

The introduction opens with the assertion that the book under review is a mathematics book for logicians to complement the author's earlier book [3], now also in an English translation [4], which is a logic book for mathematicians. The introduction is addressed primarily to nonlogicians with the intent of inviting them to study stable groups. However, even though the author cautions that only certain sections should trouble nonlogicians, anyone unfamiliar with stability theory would find all of the book except, perhaps, parts of the first and fourth chapters, impenetrable. A mathematician interested in this subject should work through a text on model theory such as [4] or [2] before delving into this book.

The title of the book under review does describe its subject matter exactly. However, while the author is justified in saying *I hope you know what a group is* to explain the absence of a definition, since the word *stable* is overused in mathematics, it requires some explanation. Strictly speaking, a stable group is a group whose theory in some language extending the language of groups is stable in the sense of model theory. This definition is meaningful only to a logician, but the book under review is aimed at a wider audience, and Poizat demonstrates in the introduction that the relevant notions from logic may be expressed without a lengthy discussion of formal languages, well-formed formulas, theories, and the rest of the syntactic apparatus of formal logic.

A *structure* is a set M given together with a family $\mathcal{D}_n$ of distinguished subsets of M^n (called the *definable* subsets of M^n) for each $n \in \mathbb{N}$. We assume basic closure properties on the definable sets. Namely, each $\mathcal{D}_n$ is a sub-Boolean algebra of $\mathcal{P}(M^n)$, every singleton is definable, the projection of a definable set is definable, Cartesian products of definable sets are definable, the diagonal is definable,

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and the class of definable sets is closed under coordinate permutations. Poizat includes another closure condition related to the existence of definable quotients in his definition of a structure.

Any group may be considered as a structure by taking for the class of definable sets the smallest class of sets satisfying the closure conditions for definable sets and containing the graph of the function $(x, y) \mapsto xy^{-1}$. We call such a structure a *pure group*. It is also possible to consider a group as an enriched structure in which there are other definable sets. This extra generality occurs in practice and is not introduced merely for its own sake. For example, when the additive group of the complex numbers is considered as a pure group, the definable sets are just finite Boolean combinations of affine spaces while every Zariski constructible set is definable once the graph of multiplication is adjoined to the class of definable sets.

We say that a structure $(M, \{\mathcal{D}_n\}_{n=0}^{\infty})$ is *stable* if for each definable set $X \in \mathcal{D}_{n+m}$ there is a natural number N such that if $a_0, \dots, a_N \in M^n$ and $b_0, \dots b_N \in M^m$, then $(a_i, b_j) \in X$ for some $i > j$ or $(a_i, b_j) \notin X$ for some $i \leq j$. (This is really the definition of stability for the *theory* of a structure, but this distinction is immaterial to the subject at hand.) At this point we have a formal definition of a stable group: A *stable group* is a stable structure $(G, \{\mathcal{D}_n\}_{n=0}^{\infty})$ having a distinguished element $e \in G$ and distinguished definable sets $\Gamma \in \mathcal{D}_3$ and $I \in \mathcal{D}_2$ for which I is a graph of a function $\iota : x \mapsto x^{-1}$ and Γ is the graph of a function we denote by $\mu : (x, y) \mapsto xy$ for which (G, μ, ι, e) is a group.

The reader may very well wonder what the condition of stability could possibly have to do with group theory. The theory developed in this book answers this query admirably.

Poizat introduces a class of groups with a clear geometric provenance before delving into the full intricacies of general stable groups. A *ranked universe* is a structure $(M, \{\mathcal{D}_n\}_{n=0}^{\infty})$ given together with a function d from the class of definable sets to $\mathbb{N} \cup \{-\infty\}$ for which

- $d(X) = -\infty$ if and only if X is empty;
- $d(X) \geq n+1$ if and only if there is a countable sequence $X_0, X_1, \dots$ of pairwise disjoint subsets of X for which $d(X_i) \geq n$ for each i;
- if $f : X \to Y$ is a definable function and $e = d(f^{-1}\{y\})$ for all $y \in Y$, then $d(X) = e + d(Y)$; and
- for each definable function $f : X \to Y$ there is a natural number N such that if $|f^{-1}\{y\}| > N$, then $f^{-1}\{y\}$ is infinite.

If M is an algebraic variety over an algebraically closed field and one takes the class of Zariski constructible sets in the various powers of M as the definable sets and algebro-geometric dimension for d, then M is a ranked universe. In particular, an algebraic group considered with all the structure coming from constructible sets is a ranked group.

The notion of a ranked universe is closely connected with, though in general distinct from, that of a structure of finite Morley rank. A structure has *finite Morley rank* if there is a function R defined on the class of definable sets taking values in $\mathbb{N} \cup \{-\infty\}$ for which $R(X) = -\infty$ if and only if $X = \varnothing$ and $R(X) \geq n+1$ just in case for every natural number m there are m pairwise disjoint subsets $X_1, \dots, X_m \subseteq X$ of X each with $R(X_i) \geq n$. A structure with finite Morley rank is necessarily stable and for a group having finite Morley rank is equivalent to being a ranked universe.

Thus, ranked groups provide examples of stable groups. Before the reader concludes that this class of groups includes many groups beyond algebraic groups, I should remark that the main conjecture for groups of finite Morley rank is that there are no exotic groups of finite Morley rank. More precisely, the Cherlin-Zilber conjecture asserts that every (infinite) simple group of finite Morley rank is an algebraic group over an algebraically closed field. Despite the concerted efforts of teams of group theorists and model theorists, the Cherlin-Zilber conjecture remains wide open. The second and third chapters of the book under review concentrate on groups of finite Morley rank. The book [1] goes deeper into the theory of groups of finite Morley rank, but it owes a debt to Poizat.

Let us now discuss the contents of this book in some detail.

In the first chapter, the basic consequences of stability for a group are proved. These propositions have very simple proofs, but their implications are strong. For example, it is shown that a stable associative cancellative monoid must be a group, a fact implying that a stable integral domain must be a field and a nonempty definable subset of a group which is closed under multiplication must be a subgroup. This result is followed by a series of propositions on descending chain conditions on subgroups of stable groups. The notion of the connected component (of the identity) of a stable group is generalized from algebraic groups. With algebraic groups the connected component of the identity is usually defined topologically, but we need a more abstract definition for general stable groups. We say that a group G (considered as a structure) is *connected* if there is no proper definable subgroup of G of finite index. In general, even for a stable group, there need not exist a minimal subgroup of finite index. However, for each formula $\phi(x, y)$, where x ranges over G and y ranges over some Cartesian power of G, there is a local connected component of G associated to ϕ. We define $G^0(\phi)$ to be the intersection of all subgroups of G of the form $H_a := \{g \in G : \phi(g, a)\}$ of finite index in G. For a stable group G, the group $G^0(\phi)$ is necessarily definable. While the local connected component makes sense in general, the main application of this idea concerns the formula defining the centralizer of an element: $\phi(x, y) := xyx^{-1}y^{-1} = 1$. A number of structural theorems on the centralizer connected component of a stable group are proved in this chapter. The first chapter ends with a proof that an $\aleph_0$-categorical group of finite Morley rank is abelian-by-finite.

The second chapter deals, mostly, with groups of finite Morley rank. General stable groups have generic types, but the full development of this theory is delayed until the fifth chapter. In the context of groups of finite Morley rank, we may define a type p to be generic in the group G if the Morley rank of p is equal to that of G itself. Some fruitful consequences of this notion are developed in this chapter. We say that a definable subset X of a group G is *indecomposable* if for any definable subgroup $H \leq G$ either X/H is infinite or X/H is a singleton. Zilber's Indecomposability Theorem, which states that the group generated by a collection of indecomposable sets (each of which contains the identity element of the group) is definable, is proven in this chapter as an application of the work on generic types. The third and fourth sections of this chapter consist of a proof of the equivalence of being ranked and of having finite Morley rank for groups. The key to this proof is a straightforward application of Zilber's Indecomposability Theorem to show that a simple group of finite Morley rank is almost strongly minimal.

Binding groups, or definable automorphism groups, appear in the fourth section of the second chapter. While at first blush this topic may seem out of place in the development of the subject, it serves to introduce (or recall) some ideas from stability theory, such as orthogonality, internality, analysis, *etc.*, that are essential to Hrushovski's more refined analysis of stable groups whose explication begins at the end of chapter two but about which we will speak later in this review. The theorem on binding groups and its corollaries justify the assertion of the naturality of stable groups, as binding groups arise from purely stability theoretic considerations even in theories in which there is no obvious algebraic content. In turn, the theory of binding groups has been applied to the algebraic study of differential equations to produce a differential Galois theory properly generalizing the Picard-Vessiot theory for linear differential equations and even Kolchin's theory for so-called strongly normal extensions.

The third chapter is entitled "Fields", but its contents are better described by its subtitle "Algebraic properties of groups of finite Morley rank". The first section opens with Macintyre's theorem that an infinite ω-stable field is necessarily algebraically closed, a result which implies that only algebraically closed fields and finite fields eliminate quantifiers in the language of fields. Some propositions on the possibilities for definable nonlinear additive maps in fields of finite Morley rank close out this section. The open problem about whether there is a field K with an automorphism $\sigma : K \to K$ which is *not* some integral power of the Frobenius automorphism such that the structure $(K, +, \cdot, \sigma)$ has finite Morley rank is mentioned in this section and some steps towards a solution are taken. It should be remarked that this question is still open even if one assumes that the structure has Morley rank one. In the second section, it is shown that from certain definable groups of automorphisms of an abelian group (all in a structure of finite Morley rank) a field is definable. Thus, the analysis of fields of finite Morley rank is directly relevant to the study of definable groups of automorphisms of groups of finite Morley rank. The third section concerns Reineke's theorem on minimal ω-stable groups and its consequences. We say that an infinite group G is *minimal* if every proper definable subgroup is finite. Reineke showed that a minimal superstable group is abelian. It follows that every infinite superstable group contains an infinite definable abelian subgroup. The main point of the fourth section is that the various groups associated to commutators (for example, the various derived groups and the groups in the central series) of a group of finite Morley rank are definable. Basic properties of solvable and semisimple groups of finite Morley rank are developed in the fifth and sixth sections. The seventh section concerns a result on definable actions of finite Morley rank groups on strongly minimal sets (definable sets of Morley rank and Morley degree one). That is, if a finite Morley rank group G acts faithfully and transitively on a strongly minimal set A, then G has Morley rank at most 3 and either G is abelian and A is a principal homogeneous space for G or A is the affine or projective line over an algebraically closed field and G is a group of fractional linear transformations.

The third chapter ends with an analysis of *bad groups.* A *bad group* is a nontrivial connected nonsolvable group G of finite Morley rank in which every proper definable subgroup is nilpotent by finite. It is shown in this section that no bad group can be elementarily equivalent to an ultraproduct of locally finite groups. In particular, a bad group cannot be isomorphic to any algebraic group. It is also shown that every bad group is a definable extension of a simple bad group. Thus, if there are any bad groups, then the Cherlin-Zilber conjecture is false. The notion of a *Borel group* in a group G of finite Morley rank—namely, a maximal definable, connected, proper subgroup—is introduced in the proof of the main theorem of this section, and these Borels play the rôle of their more classical counterparts in the theory of algebraic groups.

The fourth chapter, "Geometry: Introduction to algebraic groups", should be understood as a model theorist's introduction to algebraic groups. The first four sections consist of a model theoretic treatment of Weil-style algebraic geometry. For example, the ideas of *varieties*, the *Zariski topology*, and *generic point* in a variety are presented and then interpreted model theoretically. The fifth section contains a proof of the Weil-Hrushovski group chunk theorem: if a constructible set has the structure of a group with the group operations given by definable functions, then, in fact, it is definably isomorphic to an algebraic group. The chapter ends with a proof of Rosenlicht's theorem that a connected algebraic group is an extension of a linear algebraic group by its center and with a couple of consequences about the structure of groups definable in a pure algebraically closed field, including a version of the Borel-Tits theorem that every group isomorphism between simple algebraic groups over algebraically closed fields is given by an isomorphism of the underlying fields followed by an isomorphism of algebraic groups.

In the fifth chapter, we leave the comfort of finite Morley rank groups to study stable groups proper, but the theory continues to work even without the crutch of a good dimension function. In a group G considered as a structure in some language extending the language of groups, we say that the definable set $X \subseteq G$ is *(right-)generic* if there is some natural number n and elements $g_1, \ldots, g_n \in G$ such that $G = \bigcup_{i=1}^{n} g_i X$. We say that a type is *(right-)generic* if for each formula in the type, the set it defines is (right-)generic. Directly from the definition of Morley rank, it is easy to prove that any type of maximal rank is generic in this new sense. It is something of a miracle that in any stable group there are generic types. Moreover, in a strong sense, the generics control the group. For instance, every element of a stable group may be expressed as the product of two realizations of generic types.

Unlike in the case of ω-stable groups, ∞-definable groups appear in the theory of general stable groups. An ∞-*definable set* X in a structure M is given by a definable equivalence relation E on some power M^n of M and a collection $\{X_i\}_{i \in I}$ of E-invariant definable subsets of M^n. We think of X as $\bigcap_{i \in I} X_i/E \subseteq M^n/E$, but this identification makes sense only if M is sufficiently saturated ($|I|^+$-compact would do). By an ∞-*definable group* we mean a group whose universe is an ∞-definable set and whose group operations are given by the restriction of definable functions to that set. A general stable group G does not have a definable connected component, but it does have an ∞-definable connected component $G^0 \leq G$ with the property that G/G^0 is profinite and if $H \leq G$ is a definable group of finite index, then $G^0 \leq H$. Remarkably, G^0 may be defined without any parameters, it has a unique generic type, and the generic types of G are parametrized by G/G^0.

As applications of the work on generics, a theorem of Hrushovski in pure model theory, that every unidimensional theory is superstable, and another theorem of Hrushovski on generically presented group actions are described at the end of the chapter on generics.

With the sixth chapter we return to the context of stable groups with a rank on the class of definable sets, though in this case we have the ordinal-valued Lascar or U-rank. An important feature of U-rank for groups is that if $H \leq G$ is a definable subgroup, then $U(H) + U(G/H) \leq U(G) \leq U(H) \oplus U(G/H)$ where $\oplus$ is Cantor's natural sum of ordinals. Using this inequality and properties of generic types, one can show that if $U(G) = \omega^\alpha n + \gamma$ with $0 < n < \omega$ and $\gamma < \omega^\alpha$, then there is a definable normal subgroup $H \leq G$ with $U(H) = \omega^\alpha n$. This result allows one to analyze superstable groups in terms of groups with monomial Lascar ranks and to prove several results on superstable groups, including the fact that an infinite superstable division ring is an algebraically closed field.

The book ends with a chapter on local weights in stable groups. A type p is *regular* if for every extension $q \supseteq p$ either q is a nonforking extension of p or $q \perp p$. For example, if $U(p) = \omega^\alpha$ for some α, then p is regular. We say that an ∞-definable set X is *p-internal* if there is a definable function $u(x_1, \ldots, x_n)$ such that for each element $x \in X$ there are realizations $a_1, \ldots, a_n$ of p such that $x = u(a_1, \ldots, a_n)$. Associated to a regular type p there is a dimension function w_p, the local p-weight, which takes values in $\mathbb{N}$ and is defined on p-internal sets. A more refined version of the subgroup existence theorem for groups with $U(G) = \omega^\alpha n + \gamma$ is proved through an analysis of local p-weight, and a number of analogues of results about groups of finite Morley rank are proved with w_p standing in for Morley rank.

This translation serves its intended purpose of making the contents of *Groupes Stables* accessible to Anglophones, but there are three general defects with the present translation. First, some technical terms, for which there are generally accepted French and English versions, have been translated neologistically. For example, the stability theoretic stabilizer of a type p is usually denoted by $\mathrm{Stab}(p)$ and is called the *stabilizer of p*. In French, this group is denoted by $\mathrm{Fix}(p)$ and is called the *fixateur de p*. The translator chose to denote this group by $\mathrm{Fix}(p)$ and to call it the *fixer of p*. Notably, the translator was happy to call the set theoretic stabilizer by its usual name even though its French appelation is also *fixateur*. *Fixer* is defined in the text so that the reader is left with no doubt as to its meaning. When *quasi-strongly minimal* sets appear, the reader must correctly deduce that *almost strongly minimal* was meant and the reader should be aware that a *son* of a type is an extension. Secondly, because of the difficulty of mathematical typesetting before the wide availability of TEX, Latin characters were used for logical formulas in the original edition. Even though it is now no more difficult to typeset $\exists$ than $\mathbf{E}$, logical formulas are still typeset with Latin characters. Finally, the original edition was actually written in French, and not the dessicated mathematical French of Bourbaki. There are puns, idiomatic expressions, and literary references throughout the text (sometimes in the course of a proof). The literal translations of some of this material can make very little sense. For example, it is remarked that a regular type p may be indifferent to a group whose generic type is p-internal. The last line of the justification for this remark is *a regular type wants nothing to do with its forking sons*, a comment which in French is amusing rather than bemusing as *un type régulier ne veut avoir aucun commerce avec ses fils déviants*.

The reader susceptible to provocation is advised to ignore the foreword to the English edition of the book under review. While the reviewer finds the comments on the use of English in mathematics objectionable on several grounds, there is no need to insist that the author's acquiescence to this English translation be accompanied by a corresponding surrender of his principles. The self-serving remarks including the favorable comparison by the author between himself and Oscar Wilde and Charles Baudelaire demand a response, but in the interests of preventing the author from claiming his crown of martyrdom, the response will be brief. The apology proffered *I am ready to offer my apologies to any person that my irresponsible behavior has offended, hoping that I have caused to them no harm more serious than a superficial irritation* may seem genuine enough, but it is expressed in a way and couched in a passage that allows the author to claim the mantle of the victimized defender of free thought. As the author is no doubt aware, his tasteless joke in the French edition of this book crossed the line between offensive and insulting. Crowing about his *touch of jubilation* at *join[ing] the restricted club of cursed writers* compounds the insult. A sincere apology had been in order, and any future printing of this book should contain such an apology with the smug passages about the author's stunt excised.

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