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Isoperimetric inequalities: Differential geometric and analytic perspectives, by Isaac Chavel, Cambridge Univ. Press, Cambridge, UK, 2001, xii+268 pp., $74.95, ISBN 0-521-80267-9

Perhaps the first isoperimetric inequality one encounters in mathematics is the problem of minimizing the perimeter of a rectangle in the plane bounding a fixed area. One can easily solve this problem by expressing the perimeter as a function of x and taking the derivative to find the minimum. But if one uses the method of Lagrange multipliers, the answer in geometric form pops out immediately: both sides must have equal length. The question of which length this is may then be answered easily by plugging back into the original equation.

The generalization of this result to arbitrary domains in the plane has a strong appeal to any budding geometric analyst. I first encountered it in my teens while reading Polya's charming book *Mathematics and Plausible Reasoning*. The theorem is that if D is a plane domain with area A and perimeter L, then

$$L^2 \geq 4\pi A,$$

with equality if and only if D is a disk.

The reason that L appears squared is because if we multiply the plane by a constant k, one multiplies L by k but A by k^2.

One can attempt to prove this in a number of ways: first, one can try the Lagrange multipliers approach to see the geometrically satisfying solution "the boundary of D is a circle" pop out. Or, one can guess in advance that the minimizer must be as symmetric as possible, to show that if D is not a disk, it can be made more symmetric in a way that lowers L^2/A. This is the approach that appears in Polya.

Both of these approaches have their drawbacks. In the Lagrange multiplier approach, one must assume that a minimizer exists. Polya's approach is elegant and completely elementary, but its intricacies rely very strongly on the fact that one is in the plane. In addition, there are analytic arguments, three of which are presented in the book under review, which connect the isoperimetric inequality with more advanced areas of mathematics, but which also do not prepare one for higher dimensions.

At the end of his very vigorous argument, Polya writes, "The isoperimetric theorem, deeply rooted in our experience and intuition, so easy to conjecture, but not so easy to prove, is an inexhaustible source of inspiration."

Where does one go from here? The answer depends only on your imagination and, to a lesser extent, your mathematical taste. A natural question is to ask what happens in higher dimensions. Here the answer is clear beforehand– an n-dimensional disk. But the technique is less clear.

Another direction is to ask what happens in the hyperbolic plane. Here, one expects that the answer is a disk, but the question changes, since not all disks in hyperbolic space are the same. One finds that for large disks, the ratio L/A rather than L^2/A is the one that behaves correctly, so that one has the isoperimetric

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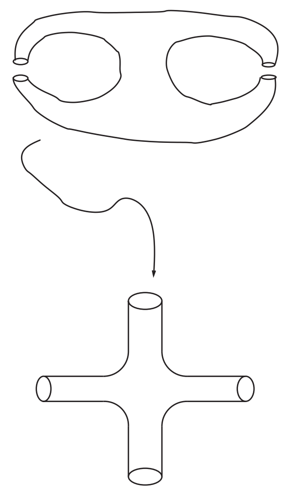

FIGURE 1. Opening up a hyperbolic surface

inequality

$$\frac{L}{A} > 1,$$

with equality reached only asymptotically for disks of large radius. A formulation which gives one equality on the case of a disk is

$$L^2 \geq 4\pi A + A^2.$$

Another direction is to investigate more general manifolds, rather than simply connected spaces of constant curvature. For instance, let S be a compact Riemann surface (constant curvature -1) of genus two. Open up the handles to get a cross-shaped figure as shown in Figure 1 above. Then paste infinitely many copies of this along a 2-dimensional grid, to get the surface $\widetilde{S}$ shown in Figure 2 below.

Should we think of $\widetilde{S}$ as hyperbolic because it has constant curvature -1? Or should we think of it as basically Euclidean because it is stretched out over a grid?

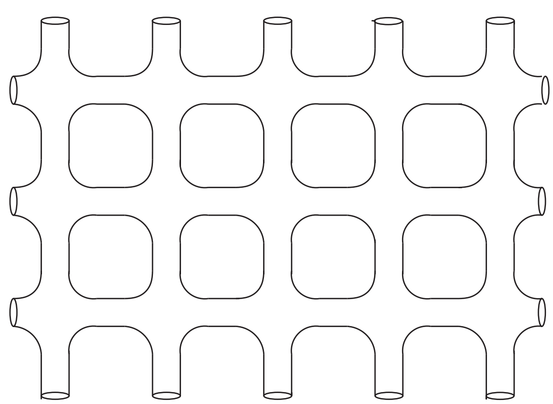

FIGURE 2. Stretching a surface over a 2-dimensional grid

Put another way, do we expect an isoperimetric inequality of the form $L/A \geq$ (const), or one of the form $L^2/A \geq$ (const), for (const) a positive constant?

And what about surfaces of higher genus, copies of which can be pasted together over an n-dimensional lattice, for arbitrary n? Do the pasted surfaces obey a 2-dimensional isoperimetric inequality or an n-dimensional one?

A more contemporary direction would be to forget about manifolds entirely, or at least put them in the background, and look at graphs, particularly Cayley graphs of groups. We already encountered the Cayley graph of $\mathbb{Z} \oplus \mathbb{Z}$ as the two-dimensional grid discussed above.

Given an infinite k-regular graph and a surface of large enough genus (depending on k), one may cut the surface open as above and paste copies of it together, as above, over the graph. The resulting surface looks like the graph if you stand far enough away and blur your vision. This suggests that defining and understanding isoperimetric problems on graphs might be a good idea.

A further direction would be to relate analysis to isoperimetric inequalities. We encountered this idea already in a serious way in the case of plane domains, but it does not take long before the fascinating world of Sobolev constants, eigenvalues, and heat flow starts opening up. To see the connection with heat flow, at least heuristically, imagine that you are investigating heat flow on some non-compact manifold M– our surface $\widetilde{S}$ will do for an example. If the heat starts out lying in some compact region K with small boundary relative to its volume, then it is hard for the heat to escape out of K– it will take a while. But if every such compact region has large boundary, then one would expect the heat to escape readily outside larger and larger compact sets. Since it is somewhat easier to conceptualize isoperimetric inequalities than large-time heat flow, one is very much tempted to make use of the former to study the latter.

Then there is my personal favorite– taking two or more of the directions given above, and passing back and forth between them.

Chavel wisely decides to break his book into two parts. The first part is devoted to the isoperimetric inequality in Euclidean space. The emphasis here is on developing technique. The problem is attacked for increasingly more general domains, with the view towards developing several different points of view, comparing and contrasting them. The target audience is clearly graduate students, who will be pulled in by the intrinsic interest of the material, but will stay for the development of technique.

The second half of the book is devoted to isoperimetric inequalities on non-compact manifolds. The goal is the development of techniques for studying the geometry of heat flow, an area of high current research interest. As Chavel puts it, "the best may be yet to come."

Along the way, the reader is introduced to the broad array of techniques and approaches alluded to above. This material is more demanding than the first half, but the grad student who stays on after the first half will find a solid introduction to an area which is very active and which lies at the heart of contemporary geometric analysis. There is also a great deal of material here for the expert as well.

One of the central themes of the second half of the book is the notion of *discretization.* Here, Chavel shows how one can study problems such as large-time heat flow on a general manifold by building a graph which models this behavior– the inverse of the approach sketched above with the surface $\widetilde{S}$. This is a very powerful technique in the subject, which is treated here for the first time in book form.

The book is written with Chavel's well-known attention to detail. It is expected that the reader will have Chavel's previous book, *Riemannian Geometry: A Modern Introduction*, at hand or is at least comfortable with the basics of Riemannian geometry. Each chapter includes a section of bibliographic notes, which this reviewer certainly found interesting and helpful.

Robert Brooks
Technion-Israel Institute of Technology
E-mail address: rbrooks@tx.technion.ac.il