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Vertex algebras and algebraic curves, by Edward Frenkel and David Ben-Zvi, Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, 2001, xii+348 pp., $55.00, ISBN 0-8218-2894-0

Vertex algebras and algebraic curves by E. Frenkel and D. Ben-Zvi is an excellent introduction to the theory of vertex algebras and their connection with the geometry of algebraic curves. This superb book contains the first published systematic exposition of the algebro-geometric approach to vertex algebras and its relation with the algebraic approach. To provide the reader with a broad background, I will first review the mathematical development of vertex operator algebras and conformal field theories, in relation to the main theme of the book. The book under review has fine historical discussions in its own style, including a section of bibliographic notes at the end of each chapter.

Vertex algebras are a class of algebras underlying a number of recent constructions in mathematics and physics. A vertex algebra is an "algebra" of vertex operators, or equivalently, an "algebra" whose operations are given by vertex operators. Vertex operators, operators describing the interaction of particles at a vertex, were first introduced and studied in the early 1970's by physicists in dual resonance theory, which led physicists to string theory. In the representation theory of affine Lie algebras, examples of vertex operators were rediscovered in the late 1970's by J. Lepowsky and R. Wilson and vertex operators were further exploited and developed by I. Frenkel and V. Kac. In a breakthrough in 1983 by I. Frenkel, Lepowsky and A. Meurman, vertex operators were used to give a construction, incorporating a vertex operator realization of the Griess algebra, of what they called the "moonshine module", an infinite-dimensional representation of the Fischer-Griess Monster finite simple group. This work in fact proved a conjecture of J. McKay and J. Thompson, a part of the monstrous moonshine conjecture of J. Conway and S. Norton. In 1986, motivated partly by this construction of the moonshine module, R. Borcherds initiated a general theory of vertex operators and, in particular, axiomatized the notion of vertex algebra. He constructed vertex algebras associated to even lattices and stated that the moonshine module carries a vertex algebra structure. The latter statement was proved by I. Frenkel, Lepowsky and Meurman based on their earlier construction of and their results on the moonshine module. Since then, the theory of vertex (operator) algebras has been rapidly developed and has found a number of applications in a wide range of branches of mathematics, including Borcherds' beautiful proof of the remaining parts of the full monstrous moonshine conjecture.

Meanwhile, in physics, A. Belavin, A. Polyakov and A. Zamolodchikov systematized the basic properties of the operator product structure of (two-dimensional) conformal field theory, a physical theory arising in both condensed matter physics and string theory. In the study of these algebraic structures, physicists arrived at a physical notion of "chiral algebra". Physically, the underlying vector space of such a chiral algebra is the space of "meromorphic fields" in a conformal field theory,

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and the operation is defined using the operator product expansion for these fields. This physical notion of chiral algebra can be easily rigorized and can be proven to be essentially equivalent to the notion of vertex (operator) algebra.

Roughly speaking, a vertex algebra is a vector space V, equipped with a distinguished vector, called the vacuum vector, and a linear map, called the vertex operator map, from V to the space of formal Laurent series with linear operators on V as coefficients, satisfying some natural axioms. The main axiom, which first appeared in the work of I. Frenkel, Lepowsky and Meurman and which was further elaborated by I. Frenkel, Lepowsky and the reviewer, is a far-reaching analogue of the Jacobi identity in the definition of Lie algebra and was correspondingly called the "Jacobi identity" (for vertex operator algebras) by these authors. Borcherds also discovered the Jacobi identity, but in his original published definition, he chose instead an "associator formula" and "skew-symmetry" as the main axioms. Besides the Jacobi identity, vertex algebras have a number of other fundamental properties and axioms, including what are called "commutativity" ("locality") and "associativity", introduced and studied by Frenkel-Lepowsky-Meurman, Frenkel-Huang-Lepowsky and P. Goddard; in particular, commutativity, together with some other minor axioms, implies associativity and the Jacobi identity. C. Dong, Lepowsky and H. Li found that a particularly simple-looking variant, "weak commutativity", of commutativity, together with some other minor axioms, can be used to replace the Jacobi identity as the main axiom for vertex algebras. The book under review also uses commutativity as the main axiom for vertex algebras. However, the (weak) commutativity axiom, together with the other minor axioms, are not the correct axioms for modules and intertwining operators, as the work just mentioned makes clear. The Jacobi identity is indeed the "correct" main axiom because of, among a number of other reasons, a natural S_3-symmetry of this identity and the unified axiom systems for vertex algebras, modules and intertwining operators in terms of this identity. Another important property is associativity, which implies a basic form of the operator product expansion used extensively by physicists. Together with skew-symmetry, (weak) associativity can also be used to replace the Jacobi identity in the definition of vertex algebra. It can be shown that these definitions are in fact equivalent to Borcherds' original definition.

Vertex algebras form a very general class. In many cases, a grading by the integers on a vertex algebra, satisfying natural conditions, can be very useful and important. In the book under review, a vertex algebra is required, for simplicity, to have a grading by the nonnegative integers, satisfying the corresponding conditions. Another structure playing a fundamental role in the theory of vertex algebras is the Virasoro algebra, the (essentially unique) central extension of the Lie algebra of Laurent polynomial vector fields on the complex plane. In the work of Frenkel-Lepowsky-Meurman on vertex operator algebras and the Monster, their "moonshine module" was what they called a "vertex operator algebra", which is a vertex algebra together with a grading by the integers and a distinguished element, called a conformal element, such that the natural compatibility conditions, including the Virasoro bracket relations, hold. In their book, E. Frenkel and Ben-Zvi consider "conformal vertex algebras", by which they mean vertex operator algebras without nonzero elements of negative weight. In fact, for the main results of the book, not all axioms for conformal vertex algebras are needed. To state the theorems in their most general forms and to apply the constructions to the vertex algebras associated to representations of affine Lie algebras at critical levels, the authors also consider "quasi-conformal vertex algebras", which are vertex algebras without nonzero elements of negative weight equipped with a suitable "half" of the Virasoro operators satisfying all the properties that still make sense.

These definitions are purely algebraic. But all the algebraic data and axioms have deep geometric meanings, reflecting the conformal-field-theoretic meaning and interpretation of vertex (operator) algebras. In fact, starting from the work of D. Friedan and S. Shenker in 1985, physicists first realized the importance of moduli spaces of Riemann surfaces or algebraic curves with punctures in the study of conformal field theories. Around the same time, Y. Manin pointed out that the moduli space of algebraic curves of all genera should play a role in the representation theory of the Virasoro algebra analogous to the role of the space G/P (P is a parabolic subgroup) in the representation theory of a semisimple Lie group G. Soon M. Kontsevich, A. Beilinson and V. Schechtman found a relationship between the Virasoro algebra and the determinant line bundles over the moduli spaces of curves with punctures. Around 1986, motivated by the path integral formulation of string theory, I. Frenkel initiated a program to construct geometric conformal field theories, in a suitable sense, from vertex operator algebras.

In 1987, also motivated by the path integral formulation, Kontsevich and G. Segal gave a precise definition of the notion of conformal field theory itself. Roughly speaking, a conformal field theory in the sense of Kontsevich-Segal is a projective representation of a suitable geometric category whose objects are disjoint unions of finitely many standard circles and whose morphisms are Riemann surfaces with oriented and analytically parametrized boundary components. To explain the rich structure of "left and right parts" of conformal field theories such as conformal blocks, Segal further introduced the notions of "modular functor" and "weakly conformal field theory". Motivated by the operator formalism for the theory of free bosons and free fermions, C. Vafa also gave, on a physical level of rigor, a formulation of conformal field theories using Riemann surfaces with punctures and local coordinates vanishing at these punctures.

Segal's modular functors and weakly conformal field theories were defined in terms of the sewing operation of Riemann surfaces with parametrized boundaries. In fact, all the examples, still only partially constructed, of conformal field theories have an additional property: There are "correlation functions" defined on the moduli spaces of curves with punctures such that the boundary points of the compactified moduli spaces are "regular singular points" (in a certain sense) of these correlation functions. This property, together with the axioms for a weakly conformal field theory, implies an important "factorization property" that naturally relates the space of conformal blocks on a smooth curve with punctures to the spaces of conformal blocks on smooth curves of lower genus or with fewer punctures by letting the original curve approach a boundary point of the moduli space. In fact, around the same time that Segal gave his definitions, A. Tsuchiya, K. Ueno and Y. Yamada constructed the algebro-geometric portions of the holomorphic weakly conformal field theories associated to affine Lie algebras (the Wess-Zumino-Novikov-Witten models) and proved this factorization property in this case. Together with the diagonalization of the fusion rules by the action of the modular transformation $\tau \mapsto -1/\tau$, conjectured by E. Verlinde and proved by G. Moore and N. Seiberg, the factorization property implies the famous Verlinde formula.

The geometry of vertex (operator) algebras can be understood completely in this framework. In his program to construct geometric conformal field theories from vertex operator algebras, I. Frenkel obtained a geometric interpretation of vertex operators and their associativity property in terms of discs with small discs deleted. Using genus-zero Riemann surfaces with punctures and local coordinates, the sewing operation and determinant line bundles, the reviewer gave a geometric definition of the notion of vertex operator algebra. A theorem establishing the equivalence between the algebraic and geometric definitions, incorporating the geometry of the infinite-dimensional Virasoro algebra and nonzero central charges (the hard parts), was also formulated and proved by the reviewer.

The definitions of the notions of conformal field theory, modular functor and weakly conformal field theory are simple, beautiful and conceptually satisfactory. Moreover, *assuming the mathematical existence of such theories*, many surprising and beautiful consequences have been derived by physicists and mathematicians. The Verlinde formula, quantum knot and three-manifold invariants, and mirror symmetry for Calabi-Yau manifolds are among the best-known examples. It is thus a most urgent and important problem to *mathematically* construct examples of (weakly) conformal field theories. Since the definition of (weakly) conformal field theory involves algebra, geometry and analysis, it is not surprising that the construction of examples is very hard. Up to now, it is only for the free fermion theories that a complete construction has been sketched, by Segal in an unpublished manuscript; this construction has been further clarified recently by I. Kriz.

There are a number of difficulties one will encounter when one tries to construct conformal field theories. One of the main difficulties comes from the analytic nature of Segal's definition of (weakly) conformal field theory. So an obvious strategy is to first construct some substructures of (weakly) conformal field theories which do not involve any infinite-dimensional analysis. Indeed, the algebro-geometric portions of the Wess-Zumino-Novikov-Witten models constructed by Tsuchiya, Ueno and Yamada are examples of such substructures. Also, Beilinson, B. Feigin and B. Mazur have constructed the algebro-geometric portions of the "minimal models", which are based on certain classes of representations of the Virasoro algebra. These algebro-geometric substructures are in fact enough for some of the important applications of conformal field theories. To construct the algebro-geometric portions of general weakly conformal field theories, one first has to formulate these algebro-geometric portions in general. In particular, one needs to define conformal blocks in general in the framework of algebraic geometry.

Y. Zhu first defined conformal blocks on Riemann surfaces with punctures, in the case of a vertex operator algebra which is assumed to be generated by lowest weight vectors for the Virasoro algebra (when viewed as a module for the Virasoro algebra). He needed this restriction on the vertex operator algebras in order to define conformal blocks without using local coordinates near the punctures. In the book under review, using a change-of-variable formula for vertex operators first obtained by the reviewer, E. Frenkel and Ben-Zvi give a coordinate-free formulation of the notion of quasi-conformal vertex algebra. This coordinate-free formulation allows them to define conformal blocks for a quasi-conformal vertex algebra.

In order to understand the operator product expansion in physics in the language of algebraic geometry, Beilinson and V. Drinfeld introduced a notion of what they called "chiral algebras" (different from "chiral algebras" in the sense of physicists). An important aspect of this notion is that it is defined on arbitrary smooth curves (i.e., Riemann surfaces), not just on genus-zero ones. It was proved by Lepowsky and the reviewer that locally, this notion of chiral algebra is equivalent to the notion of vertex algebra. By using the coordinate-free formulation of quasi-conformal vertex algebra mentioned above, E. Frenkel and Ben-Zvi construct in their book a chiral algebra on an arbitrary smooth curve starting from a quasi-conformal vertex algebra and consequently establish the equivalence between the notion of chiral algebra on a smooth curve and the notion of quasi-conformal vertex algebra. One expects that the study of such algebras in the algebro-geometric setting would, among other things, eventually lead to a generalization of the works of Tsuchiya-Ueno-Yamada and Beilinson-Feigin-Mazur mentioned above, especially a proof of the factorization property in general.

Since the axioms in Kontsevich-Segal's definition of conformal field theory are nothing but rigorously stated properties of path integrals in physics, a complete conformal field theory must satisfy all the axioms in this definition. Certainly the construction of the algebro-geometric portions of conformal field theories would not construct conformal field theories in this desired sense. For example, to sew Riemann surfaces, one needs to cut holes from closed Riemann surfaces and use parametrizations on the boundaries of the holes to identify points on the surfaces. Cutting holes and identifying points using parametrizations of course cannot in general be formulated in algebraic geometry; in general the sewing operation is simply not an operation in algebraic geometry. Also, a conformal field theory must have a complete locally convex topological vector space with a nondegenerate bilinear form, but the algebro-geometric construction does not give such a space. From vertex algebras and modules, one can in fact construct a space which, with a suitable topology, should be dense in the correct complete locally convex topological vector space. But the construction of the topology and consequently the topological completion would need much more sophisticated geometric and analytic information than the algebro-geometric portion of the theory.

Research toward the construction of general conformal field theories beyond the algebro-geometric portions has been done in the genus-zero and genus-one cases: The geometry and topological completions of suitable vertex operator algebras obtained by the reviewer in fact give holomorphic genus-zero conformal field theories. The reviewer has also constructed genus-zero modular functors and genus-zero weakly conformal field theories from representations of suitable vertex operator algebras. Certain special types of genus-one correlation functions were constructed by Zhu using the sewing operation from representations of suitable vertex operator algebras. In all these constructions, there are steps that can be reinterpreted as proofs of parts of the factorization property. It is therefore clear that any construction of conformal field theory needs, explicitly or implicitly, the algebro-geometric portions, especially the factorization property for conformal blocks. In addition, the algebro-geometric method and formulation, including the factorization property, are expected to have generalizations to theories over fields other than the complex numbers and to have applications to problems in algebraic geometry.

The book by E. Frenkel and Ben-Zvi introduces the reader to the theory of vertex algebras and their connection with the geometry of algebraic curves. The authors have made a great effort to make the book accessible to readers without extensive background. I always feel that in this field, it is not easy for a beginner to learn the geometry of conformal field theories and to find out how much has been done, because some of the most important works have still not been published. Therefore, the publication of this book is very timely, and the authors have done a great service to the mathematical community.

In addition to the well-written exposition, the book, as we have mentioned, also contains new material, for example, the definition of conformal blocks on higher-genus curves for a quasi-conformal vertex algebra and the construction of chiral algebras on higher-genus curves from quasi-conformal vertex algebras.

The book can be roughly divided into four parts. The first part, consisting of Chapters 1 to 4, is a self-contained introduction to the basics of vertex algebras. The material presented in this part includes the definition and basic properties of vertex algebras; examples based on certain representations of Heisenberg algebras, affine Lie algebras and the Virasoro algebra; and from lattices, a detailed discussion of associativity and the operator product expansion, and an introduction to rational vertex algebras. Only standard undergraduate algebra is needed to understand the material in this part.

In the second part, Chapters 5 to 9, the authors start to change the setting from algebra to algebraic geometry. In fact, this is the central part of the whole book; the algebro-geometric theory of vertex algebras and conformal blocks is developed here. Although the authors have tried to make this part as self-contained as possible, some familiarity with basic notions in algebraic geometry would be helpful. Vertex algebra bundles are constructed and conformal blocks are defined here. Examples of vertex algebra bundles and conformal blocks are also presented. This part is based on a coordinate-free formulation of quasi-conformal vertex algebras. This formulation can be roughly understood as follows: Vertex algebras need, explicitly or implicitly, a formal or complex variable which, geometrically, depends on a coordinate. Given a vertex algebra and a smooth curve, one can choose a local coordinate near any point and attach the vertex algebra to the curve locally. To obtain a structure depending only on the vertex algebra and the curve, one has to know how the vertex operators change when the coordinate changes. Such a change-of-variable formula was previously obtained by the reviewer as a consequence of the sewing axiom in the geometric definition of vertex operator algebra, and the idea is that coordinate changes can be obtained using the sewing operation. Conceptually, this is the connection between the geometric definition of vertex operator algebra and the algebraic geometry of vertex algebras studied in this book. Of course, working in the setting of algebraic geometry, the authors use (abstract) discs in the sense of algebraic geometry rather than discs in the sense of analytic geometry.

Chapters 10 to 15 form the third part. Here the authors present some important constructions and applications, including free field realizations of affine Lie algebras, the Knizhnik-Zamolodchikov equations, quantum Drinfeld-Sokolov reduction, vertex Lie algebras and vertex Poisson algebras. This part contains many interesting structures, constructions, examples and results related to vertex algebras. The material in this part is mostly algebraic.

In the last part, consisting of Chapters 16 to 19, the authors return to the geometry but at a more advanced level. Sheaves of coinvariants of vertex algebras on the moduli spaces of curves with punctures and on the moduli spaces of bundles are constructed. The relationship established originally by Beilinson and Drinfeld between the sheaves of coinvariants corresponding to representations of affine Lie algebras at critical levels and the geometric Langlands correspondence are discussed. An introduction to the chiral algebras of Beilinson-Drinfeld and a construction of chiral algebras on an arbitrary smooth curve from quasi-conformal vertex algebras are presented in the last chapter. Much of the material in this part can serve as a starting point for further research on the algebraic geometry of vertex algebras and conformal field theories, and on applications of vertex algebras to algebraic geometry.

The book also includes an appendix containing elementary material in algebraic geometry and Lie algebras. At the end of each chapter, there is a section of very helpful bibliographical notes on the material presented in the chapter.

In summary, the book by E. Frenkel and Ben-Zvi should be very useful for both beginners and active researchers in the field of vertex algebras, conformal field theories, algebraic geometry and other related areas of mathematics. The geometric part of the material presented in this book should be particularly valuable to mathematicians and physicists working on geometric conformal field theories and their applications in geometry and physics. I believe that in this beautiful and exciting field, this book will have a long-lasting influence.

Yi-Zhi Huang
Rutgers University
E-mail address: yzhuang@math.rutgers.edu