BOOK REVIEWS

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Global solutions of nonlinear Schrödinger equations, by J. Bourgain, Amer. Math. Soc., Providence, RI, 1999, viii+182 pp., \$35.00, ISBN 0-8218-1919-4

In the last ten years incredible progress has been made in the theory of nonlinear dispersive equations. The term dispersive describes the fact that the solutions of these equations are waves that tend to spread out spatially, when no boundary constraints are imposed. Two well known equations belong to this class: the nonlinear Schrödinger equation (NLS), and the Korteweg-de-Vries equation (KdV). These equations, together with combinations of them and the wave equation, have been proposed as models for many basic wave phenomena in physics. Examples of these phenomena include the propagations of signals in optic fibers, nonlinear ionic-sonic waves in plasma in a magnetic field, and long waves in plasma. A satisfactory analysis of these phenomena could be accomplished by answering questions like long time existence and uniqueness for the solution of the associated Cauchy problem, regularity properties of the solution, continuity with respect to the initial profiles,[1] scattering and soliton stability, possible blow-up of some norms in finite time, rate of blow-up and stability of blow-up profiles.

The book *Global solutions of nonlinear Schrödinger equations* by J. Bourgain focuses on the treatment of the nonlinear Schrödinger initial value problem

$$\left\{ \begin{array}{l} i\partial_t u + \Delta u + \sigma u|u|^{p-2} = 0, \quad \text{for} \quad \sigma = \pm 1, \\ u(x,0) = \phi(x) \in H^s(\mathbb{R}^d) \quad \text{or} \quad H^s(\mathbb{T}^d). \end{array} \right. \tag{1}$$

It should be said that the questions addressed in the book are prototypical for all types of dispersive equations.

As for many problems that come directly from investigating the real world, the literature that treats the Schrödinger equation in one way or another is enormous. The size of the bibliography is also due to the fact that often the same question, for example well-posedness, can be addressed purely analytically or numerically. It is also interesting to notice that sometimes the pure mathematical approach goes as far as relating certain aspects of this equation to sophisticated concepts in algebra and geometry (see for example [20]).

Bourgain's book does not have the pretence to give a complete overview of the Schrödinger equation; this would be practically impossible. It presents some of the most important recent findings for, and addresses several open questions related to,

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[1] All these properties are summarized under the expression *local* (if proved for short time) or *global* (if proved for arbitrarily long time) *well-posedness for the Cauchy problem.*

the initial value problem (1), using a pure mathematical approach. It is in a way very fortunate that the book was published almost at the same time when C. Sulem and P.-L. Sulem successfully managed in their survey work [45] to put in a clear perspective the rigorous theory of the NLS equation and the physical understanding of some of the phenomena described by it. In fact the "physical" motivation for some of the questions that Bourgain addresses in his work are extensively explained in [45].

In this review I chose to address only some aspects of the dense material that Bourgain collects for us in this relatively short book. The choice I made is purely dictated by the direction of my research and not at all by any sort of abstract ranking of the subjects presented by the author.

Going back a couple of decades, the question of well-posedness for dispersive equations was usually answered by proving a priori estimates in some Sobolev norms for the solution, and then using these estimates and some compactness arguments to actually prove the existence of the solution and various properties for it. Unfortunately, this general argument, now known as the *energy method*, required quite a lot of regularity to start with; see [44] for an overview. A much stronger tool was found when it became clear that the solution of any linear dispersive initial value problem in $\mathbb{R}^d$ can be viewed as the adjoint of the operator that restricts the Fourier transform to an appropriate hypersurface.[2] The tools were the strong and beautiful theorems on L^p estimates for these operators [43]. In the context of the well-posedness theory, these theorems came to be known as Strichartz estimates [16], [22], [24]. These estimates control $S(t)\phi$, the solution of the linear dispersive equation, with respect to the initial data ϕ. One could then look at the nonlinear term as a "small" perturbation of the linear equation, in particular when the initial data are "small", in a certain sense, or the time of existence is "short". In the case of the NLS (1) one uses the Duhamel principle to rewrite (1) as the integral equation

$$u(x,t) = S(t)\phi(x) + \int_0^t S(t-t')\sigma u|u|^{p-2}(x,t')\,dt'. \tag{2}$$

Then a solution becomes a fixed point in the space of functions determined by the norms involved in the Strichartz inequalities (see again [16] for an overview). More refined results in $\mathbb{R}^d$ were obtained once it became clear that for generic initial data ϕ, the linear solution $S(t)\phi$ has a better smoothness than ϕ itself. This property, now known as the *smoothing effect*, was first observed in a weak version by Kato [21] for the KdV equation, then was presented in its strongest version in [25] for KdV type equations, and in [15], [24], [40], [48] for the Schrödinger equation. A companion of the smoothing effect estimate is the *maximal function* estimate, in which the maximum in time of the solution is considered [25], [47]. The smoothing effect was definitely the missing tool in order to be able to consider general Schrödinger equations with nonlinearity involving not just u as in (1), but also the first derivatives of u, (see [8], [9], [26], [27]). The situation for periodic initial value problems is very different in view of the fact that Fourier transforms are actually Fourier series and integration by parts, on which oscillatory integrals estimates strongly rely, is not available. A decade ago Bourgain [1], using a discrete analysis based on concepts in combinatorics and number theory, was able to prove

[2]The curvature of the hypersurface is directly related to the dispersive character of the equation.

strong a priori estimates for the periodic KdV and Schrödinger linear operators and solve, also in this case, the initial value problem using a fixed point theorem in spaces with low regularity.

In his book Bourgain addresses two questions in the realm of well-posedness: well-posedness for the H^1-critical equation in (1), and well-posedness below the energy norm. To understand the depth of these questions we should introduce two concepts: conservation laws and scaling. The equation in (1) enjoys two conservation laws, the L^2 norm and the Hamiltonian:

$$\int |u|^2(x,t)\,dx = \int |\phi|^2(x)\,dx, \tag{3}$$

$$H(u) = \int \frac{1}{2}|\nabla u|^2(x,t) - \sigma\frac{1}{p}|u|^p\,dx = H(\phi). \tag{4}$$

If in (1) we take $\sigma = -1$ (defocusing), then combining (3) and (4), one obtains the uniform bound $\|u(t)\|_{H^1} \leq C$. As we will see below, uniform bounds in Sobolev norms are very important when one wants to prove well-posedness for data of any size and on an interval of arbitrarily long time. As for the scaling, it's easy to check that if u solves (1), then $u_\lambda(x,t) = \lambda^{-\frac{2}{p-2}}u(\lambda^{-1}x,\lambda^{-2}t)$ is the solution for the initial value problem with initial data $\phi_\lambda(x) = \lambda^{-\frac{2}{p-2}}\phi(\lambda^{-1}x)$. It is easy to check that

$$\|\phi_\lambda\|_{\dot H^s} \sim \lambda^{-\frac{2}{p-2}+\frac{d}{2}-s}, \tag{5}$$

where $\dot H^s$ is the homogeneous Sobolev space[3] of order s. If $s > -\frac{2}{p-2} + \frac{d}{2}$ and $\lambda \gg 1$, one can then use the rescaled initial value problem to be in the advantageous position of having small initial data; see [7], [46]. As mentioned above, the nonlinear part of the equation will behave as a "small" perturbation in the appropriate Sobolev space, and well-posedness can be proved in an arbitrarily large interval. If one goes back to the original, nonscaled equation, then the result translates into well-posedness in an interval of time $[0,T]$ that is inversely proportional to $\|\phi\|_{H^s}$; that is

$$T \sim \|\phi\|_{H^s}^{-\alpha}, \tag{6}$$

for some $\alpha > 0$. If $s = -\frac{2}{p-2} + \frac{d}{2} = s_c$, we say that $\dot H^s$ is a critical Sobolev space for the problem and clearly the argument above cannot be used. Bourgain considers the case when $p = 2 + \frac{4}{d-2}$, $d = 3$, and hence $\dot H^1$ is the critical Sobolev space. On the other hand, as recalled above, at least in the defocusing case, we know that the H^1 norm of the solution is uniformly bounded, so it is reasonable to conjecture global well-posedness in time for the initial value problem. The first part of the book is dedicated to the proof of this conjecture, under the restriction of radially symmetric initial data. It is not known whether the result is true once the radial symmetry assumption is removed.[4] The proof that Bourgain presents is conducted by contradiction and is based on an original induction process on the size of the

[3]More precisely $f \in \dot H^s$ if and only if $\left(\int |\xi|^{2s}|\hat f|^2(\xi)\,d\xi\right)^{1/2} < \infty$, where $\hat f$ is the Fourier transform of f.

[4]We should point out that in the equivalent problem for the wave equation, the radial symmetry assumption has been removed [18], and one expects that this could be the case also for the Schrödinger equation.

Hamiltonian. The main ingredient of the proof is a concentration mechanism that involves the gradient of the solution.[5]

Answering the question of global well-posedness in a positive way opens the doors to the investigations of new phenomena, scattering being one of these. In general terms we say that, given a nonlinear dispersive initial value problem that is globally well-posed in some Sobolev space H^s, scattering occurs if as t tends to infinity, the nonlinear solution $u(x,t)$ approaches a linear solution in the norm H^s. Clearly as the regularity required decreases[6] the scattering result becomes more complicated to prove. A fundamental inequality global in time, the Morawetz inequality, plays a special role in critical situations, like the one described above. Unfortunately this inequality has a radial symmetry that forces a restriction of the problem to radial initial data. It is highly desirable to look for other inequalities that, although weaker, do not present the radial restriction.

Another question that one can address, once the global existence of a solution $u(x,t)$ is established, is the time asymptotic of the higher Sobolev norms $\|u(t)\|_{H^s}, s \gg 1$. It is interesting to estimate this norm in time because, as s increases, the value of the integral describes how much energy gets moved to high frequencies during the evolution process.[7] The expectation is that if the dispersive initial value problem is periodic, then one should not expect a bound better than polynomial in time. If instead one does not impose any boundary condition, the bound should be independent of time. In the book some partial results are presented ([2], [3], [41], [42] and the more recent work [10]), but the whole picture of this phenomena is still far from being completed. A much stronger and in fact sharp result is proven [4] for the periodic linear Schrödinger equation with a bounded and smooth real potential $V(x,t)$, in $\mathbb{T}^d$. Here the result is that $\|u(t)\|_{H^s} \lesssim |t|^\epsilon$, for any $s \in [1,\infty]$ and for any $\epsilon > 0$. It is conjectured that such a result should also hold when the periodic boundary conditions are removed, but the proof that Bourgain presents is not applicable directly in this case, so the problem in $\mathbb{R}^d$ is still open.

When one considers questions on global well-posedness it is natural to also address the even more complicated questions of blow-up, blow-up rate, continuation after blow-up time and stability. The book recalls for example the standard result of blow-up of the energy norm for the focusing cubic Schrödinger equation in two dimension. This is obtained by using a viriel inequality, which provides the result in an indirect fashion; see [51], [17]. Also some explicit solutions with critical mass which blow-up at a fixed time T and with prescribed blow-up points are presented. These solutions were introduced by Merle [36], [37] and are constructed by using the groundstate function Q. The precise rate of blow-up on the other hand has been "described" only using numerical computations; see for example [31]. After the book was published, Martel and Merle [33], [34], [35], [38] addressed the question of blow-up, and related phenomena, for the generalized KdV equation[8] in a much more systematic and direct way. Thanks to their work, at least for this equation, the blow-up phenomena are now much better understood. In a very recent work [39] Merle and Raphael proved the optimal conjectured [31] upper bound for

[5]The basic tool used to identify concentration is a Morawetz-type inequality that will be mentioned again below.

[6]That is as s becomes smaller.

[7]This is one of the questions addressed in weak turbulence theory [52].

[8]One should say that for this equation a viriel inequality does not lead directly to blow-up.

blow-up of a solution to the cubic focusing 2D Schrödinger equation[9] with negative Hamiltonian (4).

All the questions and phenomena introduced above are considered in a regime of smoothness that is at least at the level of the energy norm H^1. So one question, to which a large part of the book is dedicated, is well-posedness for rough initial data, that is in H^s, for $s < 1$. We recalled above the critical exponent $s_c = -\frac{2}{p-2} + \frac{d}{2}$. Even though this is not always the case, (see [32], [28] for counterexamples), one expects in general that for $s > s_c$, local well-posedness can be proved. For example if $d = 2$ and $p = 4$, local well-posedness is expected for $s > 0$. This was in fact proved to be the case [6], [1]. In order to obtain local well-posedness in this low regime of regularity, one needs to sharpen the classical theorem on restriction of the Fourier transform and the related Strichartz and maximal function estimates. In doing so a variety of interesting and difficult problems, which one may view as purely harmonic analysis questions, need to be considered.[10] Once the local well-posedness is proved also for rough data, the next obvious question to ask is whether the local result could be iterated to a global one. It was remarked above that conservation laws are the classical tool to perform the iteration. As an example let's consider the cubic defocusing nonlinear Schrödinger equation [11] in $\mathbb{R}^2$. As recalled, local well-posedness can be proved in this case for $s > 0$. More precisely for any $\phi \in H^s(\mathbb{R}^2)$ there exist $T_0 \sim \|\phi\|_{H^s}^{-\alpha}$ (for some $\alpha > 0$) and a unique solution $u(x,t)$ that belongs to $C([0,T_0],H^s)$. Of course one can repeat the argument at time T_0 and advance further on an interval $T_1 \sim \|u(T_0)\|_{H^s}^{-\alpha}$. Clearly, if $\|u(t)\|_{H^s}$ grows too fast in time, this process will never define the solution at an arbitrary time far in the future. Certainly an appropriate conservation law would give us control for these norms. In the case we are considering we only have (3) and (4), which together bound the H^1 norm, and iteration would give global well-posedness in this space. So the question remains open for $0 < s < 1$, where no conservation laws are available. Bourgain presents a very general method to attack this problem. He splits the initial data ϕ into a small frequency part ϕ_0 and a large frequency part ψ_0. Clearly ϕ_0 is as regular as one wants and has a large norm, and ψ_0 is still rough, but with small rough norms. Then he lets ϕ_0 evolve following the given IVP, and he obtains a solution $u_0(t)$ that lives in the space H^1, for all $t \in [0,\delta]$, $\delta \sim \|\phi_0\|_{H^1}^{-\alpha}$, and for which a uniform bound is provided. He lets ψ_0 evolve with respect to a difference equation to obtain a solution v_0 such that the desired solution $u = u_0 + v_0$. He writes $v_0(x,t) = S(t)\psi_0(x) + w(x,t)$, where $S(t)$ is the group associated to the Schrödinger equation and $w(x,t)$ is the nonlinear part of the solution. It is not hard to see also that $v_0(t)$ is defined for all $t \in [0,\delta]$. He is now able to iterate. In the second step he defines the new initial data

$$\psi_1(x) = S(\delta)\psi_0(x) \text{ and } \phi_1(x) = u_0(\delta,x) + w(\delta,x).$$

The heart of the matter in Bourgain's argument is in proving that $w(\delta,\cdot) \in H^1$, even though it comes from rough data, and the error $\|w(\delta,\cdot)\|_{H^1}$ is small. This allows the iteration to be continued and to obtain global well-posedness for any $\phi \in H^s, s > 3/5$, leaving now the smaller gap $s \in (0,3/5)$. After the book was published, with J. Colliander, M. Keel, H. Takaoka and T. Tao [14], we introduced

[9] This is in fact the equation in (1) when $\sigma = 1$, $p = 4$ and $d = 2$.

[10] One of the most notorious is the calculation of the dimension of the Kakeya set [5], [49], [50], [22].

[11] This is the equation in (1) when $\sigma = -1$ and $p = 4$.

a new method to extend local well-posedness to global well-posedness. As I will discuss later, this method can be implemented for a variety of dispersive equations. Here I will only discuss the procedure for the particular Schrödinger initial value problem (1), with $p = 4$, $\sigma = -1$ and in $\mathbb{R}^2$. Also in our argument we use the strongest feature of the equation in hand: the conservation of the Hamiltonian. Clearly if we start with initial data $\phi \in H^s$, $0 < s < \frac{3}{5}$, the solution $u(x,t)$ has infinite Hamiltonian. We then look at the low frequency part of the solution u by defining the energy $E(u(t)) = H(Iu(t))$, where I is the multiplier operator given by a smooth multiplier that satisfies

$$m(\xi) = \begin{cases} 1, & |\xi| < 1/2N \\ N^{1-s}|\xi|^{s-1} & |\xi| \geq N; \end{cases} \tag{7}$$

in other words, $\widehat{If}(\xi) = m(\xi)\hat{f}(\xi)$. Clearly $H(Iu(t)) \sim \|u(t)\|_{H^s}$. Because the definition of $E(u(t))$ depends on the Hamiltonian and this is conserved, we expect that $E(u(t))$ is not "too far" from being conserved itself. By using the equation in the appropriate way and the fact that I commutes with derivatives, we find that in a fixed interval $[0, \delta]$

$$|E(u(t)) - E(\phi)| \lesssim N^{-3/4}. \tag{8}$$

The decay in the parameter N justifies the name of *almost conservation law* that we give to $E(u(t))$. The estimate (8) can be used to extend global well-posedness for all initial data in H^s, $s > 4/7$. The method described above, and in particular a refinement of it that improves the decay in (8), can be used also to investigate global well-posedness for KdV type equation [13] and one dimensional Schrödinger equations with derivatives [11], [12]. In both cases we proved that local well-posedness and global well-posedness coincide. We also use a similar procedure to investigate scattering and stability below the energy norm.

As Bourgain explains in his book, one of the reasons why one may be interested in proving the existence of a global flow for rough data is that some of these rough Sobolev spaces present a well known geometric structure associated with the flow: they are symplectic spaces of infinite dimension. An example is the periodic, cubic defocusing Schrödinger equation in $L^2(\mathbb{T})$. In this space the equation has a global flow [1], and one may investigate whether certain classical questions in finite dimension symplectic spaces still hold in this infinite dimensional setting. An example of these questions is the validity of Gromov's symplectic non-squeezing theorem [19]. By extending earlier work of Kuksin [29], [30], Bourgain proved this theorem for the Schrödinger flow in $L^2(\mathbb{T})$. His method consists in projecting the equation into the finite dimensional space of the first N Fourier modes, using the well established finite dimensional theory here, and then taking the limit as $N \to \infty$. Another example of a symplectic space is $H^{-1/2}(\mathbb{T})$ with the periodic KdV equation. With Colliander, Keel, Takaoka and Tao, we recently proved, using almost conservation laws, that indeed this flow is global [13]. Unfortunately Bourgain's approach, of projecting into the finite dimensional space of the first N modes, does not work in the KdV context because of some "unpleasant" interactions of large frequencies that "migrate" to small frequencies.

Two more topics are presented in this book as subjects to investigate once the flow of the equation has been proved to be global. Both of them are again generalizations in spaces of infinite dimension of classical concepts in finite dimension

Hamiltonian settings.[12] The first topic is the definition of a Gibbs measure; the second is the extension of the KAM theory to these global PDE flows. Bourgain presents a variety of results, many of them providing a complete theory when the initial value problem is defined on the circle $\mathbb{T}$, but almost completely open in higher dimensions. It is clear that also in this case the theory is not even nearly settled, and many questions have just started to be considered.

I highly recommend this book to all those people who want to learn about the recent advances in the theory of dispersive equations (not only the Schrödinger equation), and to all those researchers in analysis and PDE that want to focus their effort on problems that are at the same time interesting and challenging. The number of unresolved questions posed in this book is quite large, so in a way I consider this work a gold mine of open problems. Bourgain's style is often a bit too concise, but it is also true that the goal of the book was to summarize in one single volume many different and new results that in many cases had already appeared in published papers.

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