BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 40, Number 1, Pages 125–129
S 0273-0979(02)00959-X
Article electronically published on October 16, 2002

The search for mathematical roots, 1870-1940: Logics, set theories, and the foundations of mathematics from Cantor through Russell to Gödel, by I. Grattan-Guinness, Princeton University Press, Princeton, NJ, 2000, 690 pp., $24.95, ISBN 0-691-05857-1

Ask a mathematician of the age of Gauss "What is mathematics?" and you could expect a stock answer somewhat along the following lines: "Mathematics consists of arithmetic and geometry, arithmetic being the science of quantity, just as geometry is the science of space." A few philosophers (Berkeley or Kant, for instance) might raise questions about the nature and sources of mathematical knowledge or about the legitimacy of certain forms of mathematical reasoning, but those questions existed on the fringes of mathematics and seemed to have little to do with the core of the discipline.

Then, over the course of the nineteenth century, under the pressure of developments within mathematics itself, the accepted answer dramatically broke down. In analysis, Bolzano, investigating the foundations of the calculus, gives his "purely analytic proof" of the intermediate value theorem; Weierstrass and his students independently rediscover his results and attempt to put the calculus on a rigorous arithmetical foundation. In algebra, Gauss and Hamilton provide geometric interpretations of the complex numbers; Hamilton widens the number concept, introducing quaternions, which fail to obey the commutative law of multiplication. Grassmann introduces new vector algebras; his work and the work of algebraists like Sylvester, Benjamin Peirce, DeMorgan, and Boole put algebra on a new footing: no longer can it be assumed that algebraic equations must behave like the operations of elementary arithmetic. A new school of algebraically minded logicians (Boole, Charles Peirce) then seek to apply the new techniques to the study of the laws of logic.

In geometry, Gauss hits on the idea of non-Euclidean geometry; the posthumous publication of his ideas and the work of Riemann, Lobachevsky, Bolyai, and Helmholtz raise deep questions about the familiar space of the *Elements*. The discoveries come tripping over one another, in mutual interaction, with revolutionary implications for the theory that algebra is the science of quantity and geometry the science of space. Cantor, working on problems in the tradition of Weierstrass and Riemann, discovers the existence of non-denumerable collections and of the transfinite number classes; his work is bound up with Dedekind's set-theoretic analyses of the real and the natural numbers, which in turn stands in close relationship to Dedekind's research in number theory and algebra. Dedekind, Peirce, Frege, and Peano all undertake studies of the foundations of the natural numbers and of their relationship to logic; in the process, logic is widened far beyond its old Aristotelian bounds. Even the character of the natural numbers now seems to have been called into question, and there are other unsettling novelties, such as Peano's discovery of space-filling curves and of continuous, nowhere-differentiable functions. And indeed it turns out that the unrestricted acceptance of infinite totalities leads to paradox. Some mathematicians (like Kronecker and, in certain moods, Poincaré) reject the

2000 *Mathematics Subject Classification.* Primary 01A55, 01A60, 03-03.

infinitary set-theoretic approach altogether and call for a grounding of mathematics in the natural numbers; others (notably Russell) attempt to find certainty by grounding mathematics in the basic principles of logic itself.

These are just a few of the high points; the list could be extended much further. Not since the ancient Greeks, if then, had there been such an irruption of philosophical ideas into the very heart of mathematics. (Some of the basic texts are collected and translated in [1].) Mathematicians of the first rank—Cantor, Dedekind, Poincaré, Hilbert, Brouwer, Weyl—found themselves obliged to confront questions about the nature of mathematics, the status of geometry, the relationship between logic and arithmetic, and the character of the infinite, and to subject these questions to intense mathematical investigation. The answers they gave did much to shape the mathematics of the twentieth century, as well as, of course, to spawn the modern, technical sub-discipline of foundations.

What intellectual forces caused this burst of activity? What problems did thinkers like Dedekind and Hilbert and Poincaré address, and how were their foundational efforts related to their wider researches? What was the impact on mathematics, and why has the intensity of the foundational questions diminished? *The search for mathematical roots* attempts to give a systematic history of the foundations of mathematics from the days of Cantor and Dedekind in the 1870s, through Russell in the 1900s, to the work of Gödel in the 1930s, by which time the modern discipline of foundations was solidly established. It is a long book, nearly 700 pages, with an extremely helpful 75 pages devoted to bibliographical references.

A comprehensive intellectual history on this scale is much needed, and Grattan-Guinness announces his intention to improve on the existing literature in two important ways: first, by considering the links between foundational research and research in mainstream algebra, geometry, and analysis; second, by calling attention to numerous minor or forgotten figures who were influential at the time but whose contributions have been overshadowed by subsequent developments. Much of the existing literature has been philosophically motivated and preoccupied with the exegesis of individual thinkers, notably Frege and Russell, who are widely (and rightly) viewed as founding giants of analytical philosophy. But the wider mathematical context has in the process often been lost from sight. Grattan-Guinness's insistence that Peano and Schröder, Grassmann and Peirce be given their due and that the mainstream developments in nineteenth-century mathematics be treated as central to the foundational story is surely correct, and the list (594-5) of some fifty major archival sources gives an indication of how vast are the resources for such a study.

The book begins with an account of the contributions to logic of the British algebraists (especially DeMorgan and Boole)—a good place to start, since the new algebra was responsible for shaking loose traditional conceptions about "the science of quantity". The next chapter treats the work of Cantor and (more briefly) of Dedekind, sketching the early development of transfinite number theory and Dedekind's construction of the real and the natural numbers. This material is well known and has been the object of several detailed studies (e.g. [2] and [6]), but the following chapter, which summarizes the logical contributions of Peirce, Grassmann, and Schröder, brings into the story important figures whose contribution is often overlooked. The same is true for Peano, who, with his school, receives an entire chapter: it is easy to forget that, for Russell, Peano was a figure of comparable importance to Cantor or Frege. In general the work of the Italian logicians and geometers of the late nineteenth century tends to be neglected, so it is good to have Peano's career and principal writings summarized here.

The next two chapters, both long, are devoted to Whitehead and Russell's *Principia mathematica*. These chapters are the heart of the book: where the earlier chapters describe the principal influences leading up to *Principia*, the later chapters will describe *Principia's* influence. Grattan-Guinness covers Russell's writing of the *Principles of mathematics* and his 1901 discovery of the paradox of the class of all classes that do not belong to themselves, then describes the progress of his thinking through the writing of *Principia*, his various attempts to deal with the paradoxes, the eventual adoption of ramified type theory and of the axiom of reducibility, and (a matter not often discussed) the development of higher cardinal and ordinal arithmetic in the later volumes of *Principia*. The book concludes with three chapters tracing the influence of Russell's views on a wide range of philosophers and logicians, among them Wittgenstein, C. I. Lewis, Ramsey, Dewey, Hilbert, Lesniéwski, Carnap, Ayer, Schlick, Tarski, Quine, Gödel and Piaget.

But although this book provides a useful assemblage of materials and information necessary to any history of this period, it does not provide a satisfying overview of this critical period in the history of mathematics. What one wants is a history of mathematical ideas to know what problems a Cantor or a Hilbert faced, how they looked to him at the time, through what course of reasoning he reached his attempted solutions, and how those solutions have stood the test of time. Names, dates, institutional facts, biographical tidbits are a useful starting point, but they do not get to the heart of things; and in its analysis of the underlying ideas this history disappoints.

Here are some examples. Dedekind's philosophy of mathematics is disposed of in three pages (107-9, mostly composed of direct quotation or of paraphrase); elsewhere he is identified as "a follower of Dirichlet," but his supplements to Dirichlet's *Lectures on the theory of numbers*, which laid the foundation for modern abstract algebra and which stand in close relation to his foundational work on the number systems, are not even mentioned, let alone given the close analysis they deserve. Cantor's broad views on mathematics get five pages (119-23). We are told that he was a "formalist," that he "exhibited traits of Platonism," and also that he "drew on idealist elements." But how Cantor attempted to reconcile these divergent positions we are not told, beyond the unhelpful observation that "he did not exhibit a very clear position." (Those who seek a detailed, mathematically informed analysis of Cantor's thought and of the origins of set theory should turn instead to the classic study [3].)

Or consider the treatment of Hilbert's 1900 "Hilbert Problems" address, which gets nearly a page of discussion, most of it devoted to organizational matters. We are told the date, the occasion, the location, that the talk was held in the morning, that Moritz Cantor was in the chair, that Hilbert discussed only the first ten of his published 23 problems. This is all very interesting, but what, one wonders, about the substance? After all, the Hilbert Problems address is not just a list of challenging games and puzzles: Hilbert, in response to deep developments in nineteenth-century mathematics, was trying to sketch a novel conception that would pull together into a unified whole work in geometry, in abstract axiomatics, in mathematical physics, in algebra and number theory and logic. His views on these matters exerted an enormous influence on the mathematics of the twentieth century: a careful examination of what he was up to and an analysis of the role played by foundational concerns would be most welcome. But all we get is a single sentence (p. 135): "Strikingly, and doubtless bearing order in mind, he placed Cantor's continuum problem as the first problem (with the well ordering principle as an associated question), and 'the consistency of the arithmetical axioms' as the second."

The book's subtitle purports to offer a history of the foundations of mathematics from 1870 to 1940, but in fact most of the attention focuses on Russell in precisely the middle decade of this seventy-year span. We get a lot of information about ramified type theory and about the influence of Russell on insignificant logicians of the twenties, but the fact does not seem to be adequately appreciated that ramified type theory was a dead end and that the crucial foundational developments in the three decades after *Principia* came at the hands of Hilbert, Brouwer, and their followers. (A collection of primary materials with detailed historical commentaries is provided by [5].) But the intuitionists Brouwer and Weyl do not come in here for any kind of close scrutiny. The work of Gentzen is ignored; so, except in passing, is that of Herbrand. Skolem's name crops up a few times, always superficially. (Indeed, Skolem, Herbrand, and Gentzen, three of the greatest logicians of the century, get fewer references in the index than Keyser, Hawtrey and Dingler. "Keyser, Hawtrey, and Dingler?" Exactly.)

The treatment of the foundational contributions of Hilbert is most surprising. His work during the years 1917-1930 receives four pages of discussion (471-5), which is not exactly lavish. But Hilbert, more than anybody else in this period, reoriented foundational studies and laid the groundwork for everything that came after. In 1917-18 he held a remarkable series of lectures on logic; a formal protocol was prepared by his assistant, Paul Bernays, and much of the text was later incorporated wholesale into the book by Hilbert and Ackermann [4]. In those lectures Hilbert did two things: first, he developed a series of increasingly strong logical calculi; second, he for the first time clearly posed, and in certain cases solved, the metamathematical questions of consistency and completeness (to which were soon added independence and decidability). The lectures present a radical simplification of the *Principia*, with much of the baroque detail pared away, but, more importantly, Hilbert decisively shifted foundational research away from the investigation of logic tout court to the comparative study of formal systems. His lectures and their metamathematical point of view mark the birth of modern mathematical logic and, with modest annotations, could still today provide an adequate introduction to the subject—not something one can say of the *Principia* or of any earlier text. The questions posed and the techniques developed to answer them are thoroughly modern. Out of the researches that began with these lectures, and that then continued with his proof-theoretic attempts to establish the consistency of arithmetic, were to emerge the subjects of model theory, recursion theory, and proof theory. The work of Bernays, Gentzen, Herbrand, Gödel, and Tarski, depending as it does on the concepts of metamathematics and of formal system, would not have been possible without this radical rethinking. For a sophisticated description and analysis of these lectures and of the development of Hilbert's views in 1917-22, see [7].

Grattan-Guinness is aware of the 1917-18 lectures and devotes to them almost an entire page. The discussion contains numerous technical slips, but more importantly fails to recognize their historical importance and does not attempt to plumb their connection to Hilbert's other mathematical work (something that Hilbert himself attempts to explain in the first half of the lectures). This failure to appreciate the novelty of the work being done on the Continent is not a trivial defect and throws out of balance the discussion of the entire second half of the seventy-year period: the truly seminal developments are scanted in favor of a somewhat undiscriminating heaping up of facts about minor figures. In sum: anybody looking for a careful history of the ideas at the heart of this exceptional period will go away hungry. It is, however, useful to have so much information on it gathered in one place.

References

[1] W. Ewald, *From Kant to Hilbert: A source book in the foundations of mathematics.* Oxford University Press, 1996. MR **98i:**01040

[2] Ferreirós, J., *Labyrinth of thought: A history of set theory and its role in modern mathematics.* Birkhäuser Verlag, 1999. MR **2000m:**03005

[3] Hallett, M., *Cantorian set theory and limitation of size.* Oxford University Press, 1984. MR **86e:**03003

[4] Hilbert, D. and Ackermann, W., *Grundzüge der theoretischen Logik.* Springer Verlag, 1928.

[5] Mancosu, P., *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s.* Oxford University Press, 1998. MR **2000j:**01122

[6] Moore, G., *Zermelo's axiom of choice: its origins, development, and influence.* Springer Verlag, 1982. MR **85b:**01036

[7] Sieg, W., *Hilbert's programs: 1917-1922.* Bulletin of Symbolic Logic **5** (1999), 1-44. MR **2000d:**01016

William Ewald
University of Pennsylvania Law School
E-mail address: wewald@law.upenn.edu