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Selected papers of S. A. Amitsur with commentary, Avinoam Mann, Amitai Regev, Louis Rowen, David J. Saltman, and Lance W. Small (Editors), American Mathematical Society, Providence, RI, 2001, Part 1, xx+583 pp., $62.50, ISBN 0-8218-2924-6; Part 2, xx+615 pp., $63.50, ISBN 0-8218-2925-4; Set, $103.00, ISBN 0-8218-0688-2

These volumes collect most of the published papers of Shimshon Avraham Amitsur, preceded by a brief biographical sketch by Avinoam Mann. I had the privilege of knowing Amitsur, whom I greatly respected not only as a mathematician but as a person. Much of today's ring theory has its origins in his work, and there is much that the algebraists of today can gain from these volumes.

One product of the computer age is *webofscience*, a database which lists the citations of scientific articles, thus giving an approximation of their influence. In areas of ring theory familiar to me, my opinions of Amitsur's most important work were confirmed by the statistics. However, I was surprised to discover how influential his work was in areas where he only worked for a short time, such as general radical theory and cohomology of rings.

The editors have divided his work into four areas and provided commentary on each. The commentary is useful and welcome, but the division into areas is not completely successful, because there is so much overlap between them. I will discuss them in order, combining the two sections on polynomial identities.

I. General Ring Theory - commentary by Louis H. Rowen.

This section collects all of Amitsur's work in ring theory which is not connected with either polynomial identity rings or division rings. As one would expect, there are papers on diverse topics.

He wrote several papers on radicals and semisimplicity. Three of his earliest papers [A2], [A4], [A5], published in 1952-54, develop general radical theory. Except for a survey article in 1971 [A20], he did not return to this topic. The "general" nature of these papers is atypical of Amitsur - nearly all of his work is much more concrete. Nevertheless, his papers are among the most cited in the area - but the citations lie far in the past, a reflection of the fact that general radical theory has faded into the background of ring theory.

On the other hand, his three short papers on semisimplicity contain ideas which keep reappearing and theorems which keep being applied. The first [A10] shows how various finiteness questions can be attacked for algebras over nondenumerable fields. The second [A11] contains the much used result that the Jacobson radical of a polynomial ring $R[x]$ is a nil ideal, where R is a ring and x is a central indeterminate. The third [A14] shows that for any group G and any field F which is transcendental over the rationals, the Jacobson radical of the group algebra $F[G]$ is zero. Although it is quite certain that the Jacobson radical of $F[G]$ is zero for any field of characteristic zero and any group G, no further progress has been made on this question. All in all, Amitsur's results on the behavior of the Jacobson radical under field extensions and polynomial extensions are fundamental.

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Other noteworthy papers in this section are [A17], a seminal paper on generalized polynomial identities, and [A19], which uses Morita contexts to give proofs of Goldie's Theorem and the Wedderburn-Artin Theorem.

II. Rings Satisfying a Polynomial Identity - commentary by Lance W. Small. III. Combinatorial Polynomial Identity Theory - commentary by Amitai Regev.

Let K be a field, and let $K\langle \mathbf{X} \rangle = K\langle x_1, x_2, \dots \rangle$ be a free associative algebra over K in countably many variables. A K-algebra R is said to satisfy a *polynomial identity* (or, R is a *PI-ring*) if there is a nonzero polynomial $f(x_1, \dots x_n) \in K\langle \mathbf{X} \rangle$ such that $f(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in R$. For example, commutative rings satisfy $x_1x_2 - x_2x_1$ and $M_2(K)$ (2×2 matrices over K) satisfies $(x_1x_2 - x_2x_1)x_3 - x_3(x_1x_2 - x_2x_1)$, an identity first given by W. Wagner [W] in 1937. The set of all $f \in K\langle \mathbf{X} \rangle$ which vanish under substitutions from R forms an ideal of $K\langle \mathbf{X} \rangle$ which is closed under endomorphisms of $K\langle \mathbf{X} \rangle$. Such ideals are called $T - ideals$.

Polynomial identities do not have a long history. The term *polynomial identity* first occurs in I. Kaplansky's seminal paper [K], and polynomial identity theory only emerged as a recognized area of mathematics afterward. There are only a handful of earlier results which belong to the theory, and they can be found in a survey article by Amitsur himself [A22].

The earliest polynomial identity paper is apparently a 1922 article of M. Dehn [D] motivated by projective geometry. As in his widely known pioneering work in group theory and topology, he raised questions which could not really be attacked until decades later, after enough machinery and techniques had been developed. In [D], Dehn was looking for analogues of Pappus' Theorem. Recall that a projective plane is said to be *Desarguesian* if it can be coordinatized by a division ring. Pappus' Theorem [HP, Theorem 2.6, p. 26] says that the division ring is commutative if and only if certain geometric conditions are satisfied. Amitsur carries out Dehn's program in one of his most brilliant papers [A18]. He shows that for each positive integer n there is a geometric condition $\mathcal{G}_n$ such that a Desarguesian projective plane satisfies $\mathcal{G}_n$ if and only if its coordinate division ring has dimension $\leq n^2$ over its center. In order to do this, he first introduces rational identities and constructs what P. M. Cohn later called [C, p. 281] the *universal field of fractions*, a basic object in his theory.

I singled out [A18], not only because it is an outstanding article, but also because the solution of Dehn's problem is a consequence of ideas from PI-theory rather than part of its development. In the sequel, I will give a chronological treatment of PI-theory and Amitsur's role in it.

Soon after Kaplansky's paper, it was realized that $M_n(K)$, the ring of $n \times n$ matrices over a field K, satisfied a *standard polynomial* $S_r(x_1, \dots, x_r)$ for some integer r, and it was conjectured that $r = 2n$ is the least such r. This was answered affirmatively by the Amitsur-Levitzki Theorem [AL], which is Amitsur's first significant paper on PI-rings. It is also (by a small margin) his most cited paper, in part because it has received so many different proofs.

It is hard to imagine that something like this could happen today, but from 1948 to 1966, the development of PI-theory was done almost single-handedly by Amitsur. In [A1] he shows that PI-s are preserved under extension of scalars and establishes other basic facts which show up in any presentation of PI-theory, and in [A3] he proves the important result that any PI-ring satisfies a power of the standard identity. In [A7] he proves that a PI-domain has an Öre ring of fractions - the generalization of this result to prime PI-rings by E. C. Posner [P] in 1960 is one of the two important PI-results of this period not due to Amitsur. (The other, A. I. Shirshov's theorem on module finiteness of PI-algebras [S], was overlooked in the West and did not have a major influence on the development of PI-theory until the 1970's.) In [A8], he develops the theory of T-ideals, determines all prime T-ideals, and proves (using terminology introduced a decade later) that the ring of generic matrices is a domain. In [A12] he proves a version of the Nullstellensatz for PI-rings, and in [A15] he obtains partial results toward chararacterizing groups whose representations have bounded degree. The full characterization was accomplished by I. M. Isaacs and D. S. Passman [IP].

In contrast to the tiny repetitious steps into the known one sees in most mathematics, it is remarkable how much Amitsur accomplished with no road map telling him what to prove or how to prove it, and almost as remarkable that most of his proofs have not been improved.

Starting in 1966, there was a series of new developments and breakthroughs in PI-theory: The introduction of noncommutative affine rings and the ring of generic matrices by C. Procesi in 1966 [Pr1], the characterization of Azumaya algebras in terms of polynomial identities by M. Artin in 1969 [Ar] (extended from algebras to rings by Procesi [Pr2]), A. Regev's 1972 theorem that the tensor product of PI-rings is a PI-ring [Re], the construction of central polynomials in 1972-73 by E. Formanek [F] and Y. P. Razmyslov [R1], the introduction of trace identities in 1974-76 by Razmyslov [R2] and Procesi [Pr3] (they are implicit in a 1958 paper of B. Kostant [Ko]), and the solution of Specht's problem (the finite generation of T-ideals over a field of characteristic zero) by A. R. Kemer in 1987 [Ke1], [Ke2].

Although none of these breakthroughs were due to Amitsur, he remained the acknowledged leader of the field and continued to produce important results in both the structural and combinatorial sides of PI-theory for the rest of his life. For example, he and Regev [AR] proved in 1982 that the Young diagrams associated with the *cocharacter series* of a PI-algebra belong to a so-called "hook". This theorem underlies a great deal of subsequent work in combinatorial PI-theory.

IV. Division Algebras - commentary by David J. Saltman.

Central simple algebras were the subject of Amitsur's thesis in 1949, and they remained a favorite topic of his for his entire career. He did some of the earliest and most basic work on generic splitting fields [A6] and cohomology of rings [A13], [A16], and his determination of the finite subgroups of division rings [A9] is a decisive answer to a natural question.

His single most important article, however, is his noncrossed product paper [A21]. It not only answers a fundamental open question where little had been known, but it introduces a strategy which has since been exploited in all sorts of similar situations. E. Noether had discovered the construction of central simple algebras as *crossed products*, and the *Brauer group* classified them, up to a natural equivalence. Every division algebra is Brauer equivalent to a crossed product (namely, matrices of a suitable size over itself), but a fundamental question remained unanswered: Is every finite-dimensional division algebra itself a crossed product?

In 1966 Procesi [Pr1] first defined the *ring of generic matrices* and the *generic division ring* as concrete objects. (As noted above, Amitsur had already studied the former as the relatively free ring $K\langle \mathbf{X}\rangle/T(n)$, where $T(n)$ is the T-ideal of identities satisfied by $M_n(K)$, and the latter as its Öre ring of quotients.) Procesi suggested that if there were any noncrossed products, then the generic division ring should be a noncrossed product. Amitsur proves this as follows: First, he shows that if the $n \times n$ generic division algebra is a crossed product with respect to a group G, then every central simple algebra of dimension n^2 over its center is a crossed product with respect to the same group G. (A priori, a division algebra might be a crossed product with respect to none, or one, or several different G.) Then he constructs, for n divisible by 8 or the square of an odd prime, division algebras D_1 and D_2 such that D_i is a crossed product, but only with respect to a group G_i, where G_1 and G_2 are nonisomorphic groups of order n. The two steps combine to show that the generic division algebra is not a crossed product for such n.

This "Amitsur strategy" of defining an object with a suitable universal property has now been employed for thirty years and shows no sign of outliving its usefulness. It does not apply to division algebras of dimension p^2 over their centers, where $p \geq 5$ is a prime, and whether or not they are crossed products remains an open question.

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