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Spherical inversion on Sl$(n,\mathbb{R})$, by Jay Jorgenson and Serge Lang, Springer-Verlag, New York, 2001, xx+426 pp., \$79.95, ISBN 0-387-95115-6

Three topics in the harmonic analysis on Euclidean spaces $\mathbb{R}^n$ are:

- Fourier transform and Plancherel Theorem
- Paley-Wiener Theorem
- Heat kernel techniques and Segal-Bargmann transform

When we try to transfer these three concepts to a "curved manifold" M, the situation becomes instantly very complicated. Let us look for an appropriate class of M's for which we can expect satisfactory answers.

We need the requirement to measure distances, so M should be Riemannian. In order to have a reasonable theory of selfadjoint operators on M we also need completeness. But still the class of complete Riemannian manifolds is far too big for our purpose. Let us assume that M "looks everywhere the same"; i.e., M allows a transitive action by a Lie group G of isometries. Hence

$$M \simeq G/K$$

with K the stabilizer of a fixed point $m_0 \in M$. Notice that K is a compact subgroup of G. With the additional requirement that K is a symmetric subgroup, i.e., the fixed point set of an involution θ on G, we finally arrive where we want to be: $M = G/K$ is a *Riemannian symmetric space.*

Every Riemannian symmetric space M allows a unique decomposition

$$M = M_e \times M_c \times M_n$$

with M_e, M_c and M_n Riemannian symmetric spaces of the following types: M_e is a flat Euclidean space, M_c is compact and M_n is non-compact (and negatively curved). Our concern here is with Riemannian symmetric spaces of the non-compact type. Let us mention some examples:

Example 1. The upper half plane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ equipped with the Poincaré metric $\frac{1}{y^2}(dx^2 + dy^2)$. The special linear group $G = \mathrm{Sl}(2,\mathbb{R})$ acts, via Möbius transformations, as the group of orientation preserving isometries of $\mathcal{H}$. The stabilizer of $i \in \mathcal{H}$ is $K = \mathrm{SO}(2,\mathbb{R})$, and K is the fixed point group of the Cartan involution $\theta(g) = (g^t)^{-1}$ on G. Thus $\mathcal{H} \simeq G/K$ is a Riemannian symmetric space of the non-compact type. Nowadays one considers Riemannian symmetric spaces of the non-compact type as appropriate generalizations of the upper half plane: For example the theory of automorphic functions on $\mathcal{H}$ was extended to all Riemannian symmetric spaces of the non-compact type.

Example 2. Let $M' = \mathrm{Symm}(n,\mathbb{R})^+$ be the convex cone of all positive definite symmetric matrices. Write $V = \mathrm{Symm}(n,\mathbb{R})$ for the surrounding vector space of M' and identify all tangent spaces $T_m M'$ with V. Then the prescription

$$g_x(u,v) = \mathrm{tr}(x^{-1}ux^{-1}v) \qquad (x \in M')(u,v \in T_xM' = V)$$

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defines a complete Riemannian metric on M'. The group $G' = \mathrm{Gl}(n,\mathbb{R})$ acts transitively and isometrically on M' via $M' \ni x \mapsto gxg^t \in M'$, $g \in G'$. The isotropy subgroup of the identity matrix $\mathbf{1}$ is $K' = \mathrm{O}(n,\mathbb{R})$. Thus the mapping

$$G'/K' \to M', \quad gK' \mapsto gg^t$$

identifies $M' \simeq G'/K'$ as a Riemannian symmetric space. Finally M' decomposes as $M' = M \times \mathbb{R}^+$ with M being the section of M' consisting of all matrices with determinant one. With $G = \mathrm{Sl}(n,\mathbb{R})$ and $K = \mathrm{SO}(n,\mathbb{R})$ it is now clear that $M \simeq G/K$ is a Riemannian symmetric space of the non-compact type. Let us mention that M is universal in the sense that every Riemannian symmetric space of the non-compact type can be embedded isometrically into M.

The harmonic analysis on this particular M is the subject proper of the book of Jorgenson and Lang and will follow us throughout the review. Let us finish the discussion of this example by mentioning various choices of coordinates on M. Define subgroups A and N of G by

$$A = \{\mathrm{diag}(a_1, \ldots, a_n) : a_i > 0,\ \prod_{i=1}^n a_i = 1\}, \qquad N = \{\begin{pmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{pmatrix}\} .$$

By a result of Gauss every positive definite matrix X can be written uniquely as $X = UU^t$ with U an upper triangular matrix with positive entries on the diagonal. Thus the following map, called the *Iwasawa decomposition*, is a diffeomorphism:

$$N \times A \to M, \quad (n,a) \mapsto na(na)^t. \tag{1}$$

By the spectral theorem every positive definite matrix can be diagonalized by an orthogonal transformation. Hence the following mapping, called the *polar decomposition*, is onto:

$$K \times A \to M, \quad (k,a) \mapsto ka^2k^t. \tag{2}$$

This choice of coordinates deviates only little from being injective as the generic fiber is isomorphic to a finite group. Write $\mathcal{W}$ for the *Weyl group* of M, which in this case is S_n, the permutation group of n letters. The Weyl group $\mathcal{W}$ acts on A by permuting the diagonal entries. Then, if we write $C^\infty(M)^K$ for the space of smooth K-invariant functions on M and $C^\infty(A)^{\mathcal{W}}$ for the smooth $\mathcal{W}$-invariant functions on A, the polar decomposition (2) gives us an identification

$$C^\infty(M)^K \simeq C^\infty(A)^{\mathcal{W}} . \tag{3}$$

For the rest of this article we let $M = G/K$ be a Riemannian symmetric space of the non-compact type. When necessary we will restrict ourselves to the specific example $\mathrm{Sl}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$.

1. Fourier-transform and Plancherel Theorem

Let us first recall the theory on the real line $\mathbb{R}$. Write $\mathcal{S}(\mathbb{R})$ for the *Schwartz space* on $\mathbb{R}$, i.e., the space of all rapidly decaying smooth functions. Then the *Fourier-transform*

$$\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R});\ \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{2\pi i \xi x} f(x)\ dx$$

is an isomorphism, and we have

$$\|f\|_2 = \|\mathcal{F}(f)\|_2 \qquad (f \in \mathcal{S}(\mathbb{R}))\ .$$

The latter means that $\mathcal{F}$ extends to an isometry on the Hilbert space $L^2(\mathbb{R})$. Finally, we have the *Inversion Formula*

$$\mathcal{F}\mathcal{F}(f)(-x) = f(x)\ .$$

One way of transferring the theory of Fourier transformation to Riemannian symmetric spaces M goes through answering the following two questions: 1. How can we define an appropriate Schwartz space $\mathcal{S}(M)$ on M? 2. What is the appropriate replacement of the exponentials $e_\xi(x) = e^{2\pi i\xi x}$ which are used to define the Fourier transform?

We do not want to go into the technical aspects of the definition of $\mathcal{S}(M)$ and focus instead on the generalization of the exponentials $e_\xi(x)$ and the definition of the Fourier transform on M.

The exponentials are eigenfunctions of the Laplace operator $\frac{d^2}{dx^2}$. So it is reasonable to replace the exponentials by eigenfunctions of the Laplace-Beltrami operator Δ on M. Here one encounters the first difficulty: The eigenspaces of Δ are always infinite dimensional, whereas for a generalization in the flavour of the real line we need one-dimensional eigenspaces. The first step to overcome this difficulty is to consider only K-invariant eigenfunctions of Δ. These eigenspaces are considerably smaller—for $G = \mathrm{Sl}(2,\mathbb{R})$ they are even one dimensional—but generically they are still infinite dimensional. How can we solve this problem?

Notice that Δ commutes with the action of G on M as the Riemannian structure of M is G-invariant. Thus $\mathbb{C}[\Delta]$ is a subalgebra of the algebra $\mathbb{D}(M)$ of G-invariant differential operators on M. Let us mention that only for $\dim \mathfrak{a} = 1$ for $\mathfrak{a}$ the Lie algebra of A we have $\mathbb{D}(M) = \mathbb{C}[\Delta]$ (for example, this is the case when $G = \mathrm{Sl}(2,\mathbb{R})$). The algebra $\mathbb{D}(M)$ is commutative, as it was realized by Harish-Chandra that the Iwasawa decomposition (1) gives rise to an isomorphism

$$\mathbb{D}(M) \simeq S(\mathfrak{a})^{\mathcal{W}} \tag{4}$$

with $S(\mathfrak{a})^{\mathcal{W}}$ the $\mathcal{W}$-invariants in the symmetric algebra of $\mathfrak{a}$. One shows that the K-invariants in a common eigenspace of $\mathbb{D}(M)$ form a one dimensional subspace.

As suggested by (4) we can parametrize common eigenspaces of $\mathbb{D}(M)$ by elements of $\mathfrak{a}_{\mathbb{C}}^*$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we write φ_λ for the unique normalized (i.e. $\varphi_\lambda(\mathbf{1}) = 1$) K-fixed function in the eigenspace associated with λ. One calls φ_λ the *spherical function* with parameter λ.

An aesthetic drawback of the theory is that except for some few cases we will never be able to find nice explicit formulas for spherical functions. However, Harish-Chandra discovered an integral representation which is described next. According to the Iwasawa decomposition (1) every $m \in M$ corresponds to a tuple $(n(m), a(m)) \in N \times A$. Then the integral formula for φ_λ reads

$$\varphi_\lambda(m) = \int_K e^{(\lambda+\rho)\log a(km)}\, dk \qquad (m \in M) \tag{5}$$

where $\rho \in \mathfrak{a}^*$ is a certain spectral shift caused by the negative curvature of M.

Now we are ready to define the *spherical transform* on M, i.e., the Fourier-transform on the K-invariant functions $C_c^\infty(M)^K \simeq C_c^\infty(A)^{\mathcal{W}}$ on M (cf. (3)). The exponentials e_ξ on the real line get replaced by the φ_λ, and we define the spherical transform by

$$\mathcal{F} : C_c^\infty(M)^K \to C^\infty(i\mathfrak{a}^*)^{\mathcal{W}}; \quad \mathcal{F}(f)(\lambda) = \int_M \varphi_\lambda(m) f(m)\, dm \ .$$

In order to discuss the L^2-aspects let us introduce *Harish-Chandra's c-function*

$$c(\lambda) = \int_{\overline{N}} e^{-(\lambda+\rho)\log a(\overline{n})}\, d\overline{n} \ ,$$

where $\overline{N}$ denotes the group of lower unipotent matrices. Using the integral representation (5), it is not too hard to show that $c(\lambda)$ is the coefficient of the leading term in the asymptotic expansion of φ_λ. Surprisingly, $c(\lambda)$ can be explicitly computed, as was discovered by Karpelevic for $G = \mathrm{Sl}(3, \mathbb{R})$ and established in full generality by Gindikin and Karpelevic. The Plancherel Theorem for the spherical transform (proved by Harish Chandra) then states that $\mathcal{F}$ establishes an isomorphism

$$L^2(M)^K \simeq L^2\Big(i\mathfrak{a}^*, \frac{d\lambda}{|c(\lambda)|^2}\Big)^{\mathcal{W}} \ .$$

The main objective of the book of Jorgenson and Lang is to develop the theory of spherical transform on M and give a proof of the Plancherel Theorem in the spirit outlined above.

However, there is an alternative way to the Plancherel Theorem through representation theory. The advantage of this method is that it applies to various kinds of homogeneous spaces G/Γ where Γ might be a symmetric or discrete subgroup of G. This point of view is not the subject of the book, but I think it might be useful for the reader to see another perspective.

The representation theoretic approach. The principle can be already seen on the real line $\mathbb{R}$. We consider the regular representation of the additive group $\mathbb{R}$ on $L^2(\mathbb{R})$, i.e., the unitary representation L of $\mathbb{R}$ defined by translations in the arguments of the functions:

$$L : \mathbb{R} \to \mathcal{U}(L^2(\mathbb{R})), \quad (L(x)f)(y) = f(x+y) \qquad (f \in L^2(\mathbb{R}), x, y \in \mathbb{R}) \ .$$

Notice, L is unitary as the Lebesgue measure is translation invariant. Now, how does this representation decompose into irreducibles? Recall that the continuous irreducible representations of $\mathbb{R}$ are the characters $\pi_\xi(x) = e^{2\pi x \xi}$ parmametrized through $\xi \in \mathbb{C}$. Using the language of direct integrals of Hilbert spaces, the Plancherel Theorem on $L^2(\mathbb{R})$ can be expressed as

$$(L, L^2(\mathbb{R})) \simeq \Big(\int_{i\mathbb{R}}^{\oplus} \pi_\xi \ d\xi, \int_{i\mathbb{R}}^{\oplus} \mathbb{C}_\xi \ d\xi \Big) \ .$$

Here $\mathbb{C}_\xi \simeq \mathbb{C}$ for all $\xi \in i\mathbb{R}$. The important feature is that only unitary π_ξ, i.e. $\xi \in i\mathbb{R}$, appear in the decomposition of L into irreducibles. One calls those π_ξ *tempered*, motivated by the fact that $x \mapsto e^{2\pi x\xi}$ defines a tempered distribution if and only if $\xi \in i\mathbb{R}$.

Let us now switch to $L^2(M)$ with $M = G/K$ a Riemannian symmetric space. The group G acts unitarily on $L^2(M)$ via left translation in the arguments. Then the left regular representation $(L, L^2(M))$ decomposes as a direct integral of irreducible unitary representations of G. Based on an idea of Gelfand and Kostyuchenko, Bernstein (cf. [Be88]) has shown that only the so-called *tempered representations* of G can occur in the Plancherel-decomposition of $L^2(M)$. This together with some simple facts on the asymptotics of the spherical functions then equally leads to a proof of the Plancherel Theorem (as was outlined by Bernstein in a seminar talk at the MSRI in 2001).

2. Paley-Wiener Theorem

Again, let us first recall the result on the real line. For $R > 0$ write $C_R^\infty(\mathbb{R})$ for the space of smooth functions with support in $[-R, R]$. Further we associate to R the *Paley-Wiener space*

$$PW_R(\mathbb{R}) := \{f \in \mathcal{O}(\mathbb{C}) : \sup_{z\in\mathbb{C}} \left|z^N f(z) e^{-R|\operatorname{Im} z|}\right| < \infty, N \in \mathbb{N}\} .$$

Then the *Paley-Wiener Theorem* asserts that

$$\mathcal{F}\big(C_R^\infty(\mathbb{R})\big) = PW_R(\mathbb{R}) .$$

Let us now look for the generalization to M. For $R > 0$ we write $C_R^\infty(M)$ for the space of smooth functions supported in the ball $B_R := \{x \in M : d(x, \mathbf{1}) \le R\}$ where d denotes the Riemannian distance function. Then the Paley-Wiener Theorem for M (established by Helgason) states that

$$\mathcal{F}(C_R^\infty(M)^K) = PW_R(i\mathfrak{a}^*)^{\mathcal{W}} .$$

It is interesting to observe that the Paley-Wiener Theorem on M implies the Plancherel Theorem, an observation made by Rosenberg. The book of Jorgenson and Lang follows this strategy by first proving the Paley-Wiener Theorem and then the Plancherel Theorem.

3. Heat kernel techniques

The heat kernel on the real line is given by

$$\rho_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{4t}} \quad (t > 0, x \in \mathbb{R}) .$$

It has the property that $(\rho_t)_{t>0}$ constitutes a Dirac-sequence; in particular

$$L^p - \lim_{t\downarrow 0} \rho_t * f = f \qquad (f \in L^p(\mathbb{R})) . \tag{6}$$

Another important feature is that $\rho_t(x)$ extends holomorphically to an entire function $\rho_t(z)$ on the complex plane. This is relevant in the context of heat-kernel transforms, which we summarize next.

For $t > 0$ let us define the *Fock-space*

$$\mathcal{F}_t^2(\mathbb{C}) = \{f \in \mathcal{O}(\mathbb{C}) : \|f\|^2 = \frac{1}{2\pi t} \int_{\mathbb{C}} |f(z)|^2 e^{\frac{-|\operatorname{Im} z|^2}{2t}} \, dz < \infty\} .$$

This is a Hilbert space of holomorphic functions. The *heat kernel transform* or *Segal-Bargmann transform*

$$H_t : L^2(\mathbb{R}) \to \mathcal{F}_t^2(\mathbb{C}), \quad f \mapsto \text{analytic continuation of } (\rho_t * f)$$

is an isomorphism of Hilbert spaces.

While there is a concrete formula for the heat kernel on $\mathbb{R}$, this is no longer the case for Riemannian symmetric spaces (except for the complex ones). Using the spectral resolution of the heat kernel ρ_t on $M = G/K$ and various estimates on spherical functions, the authors establish (6) for $p = 1, 2$. In the last chapter Jorgenson and Lang point out (without proofs) how to obtain (6) in full generality using a concept of Flensted-Jensen which relates harmonic analysis on $\mathrm{Sl}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ with harmonic analysis on the much simpler complex space $\mathrm{Sl}(n,\mathbb{C})/U(n)$.

Finally, let us mention that recently the domain of analytic continuation of ρ_t was discovered and the first progress made towards the isometry of the heat kernel transform on $M = G/K$ (cf. [KS01]).

The standard reference book for the foundations of harmonic analysis on Riemannian symmetric spaces is Helgason's *Groups and geometric analysis* (cf. [Hel84]). The book of Jorgenson and Lang makes a useful contribution by exhibiting the theory on the example $\mathrm{Sl}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. Restricting to this specific example allows the authors to verify certain facts directly, which otherwise would have required some knowledge of semisimple Lie algebras and groups. For my taste the introduction could have been more focused on the contents than on certain political issues related with the people in the "field". Also one realizes an obsession of the authors to attribute a name to almost anything. The authors make an effort in axiomatizing certain parts of the theory and give detailed proofs. This makes the book readable and accessible for graduate students.

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