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Abstract root subgroups and simple groups of Lie-type, by Franz Georg Timmesfeld, Birkhäuser-Verlag, Basel, Boston, Berlin, 2001, xiii + 389 pp., $125.00, ISBN 3-7643-6532-3

In the course of the 1950's Jacques Tits developed the notion of a building with the aim, at first, of providing a systematic geometric interpretation of the exceptional complex Lie groups. The theory turned out to apply to arbitrary simple algebraic groups over an arbitrary field. More precisely, to each absolutely simple algebraic group of positive rank (i.e. isotropic), there is a canonically associated building (which is thick irreducible and spherical) of the same rank defined in terms of parabolic subgroups. These are the spherical buildings of algebraic type. Spherical buildings can also be defined in terms of the geometry of a classical group (e.g. unitary, orthogonal, etc., of finite Witt index or linear of finite dimension) over an arbitrary field or skew-field. These are the spherical buildings of classical type. (A spherical building of classical type is not of algebraic type if, for example, the skew-field is not finite dimensional over its center.) A third much smaller family of spherical buildings consists of those associated with certain "mixed" groups – essentially algebraic groups in some sense defined over a *pair* of fields k and K of characteristic p, where $K^p \subset k \subset K$ and p is two or three. These are the spherical buildings of mixed type. In 1974, Tits proved that every thick irreducible spherical building of rank at least three is of algebraic, classical or mixed type [7]. The spherical building of classical type associated with a linear group over a field or skew-field is essentially the corresponding projective space. Tits' result is thus, in some sense, a grand generalization of the Veblen-Young characterization of projective spaces [9].

Around the same time, Bernd Fischer was developing the theory of 3-transpositions. A conjugacy class D of a group G is called a class of 3-transpositions if each element of D is of order two and any two elements of D either commute or generate a subgroup isomorphic to the symmetric group S_3 (equivalently, $|d| = 2$ for all $d \in D$ and $|de| \leq 3$ for all $d, e \in D$). In the basic example, $G = S_n$ for n arbitrary and D is the class of transpositions in G, i.e. the permutations which just interchange two letters and leave the others fixed. This work culminated (in 1971) in the result that if G is a *finite* group generated by a class of 3-transpositions which does not have any solvable normal subgroups, then $|G/G'| \leq 2$ (where G' denotes the commutator subgroup of G) and G' is either an alternating group or one of a few classical groups defined over the field with two or three elements or one of three previously unknown simple groups subsequently denoted Fi_{22}, Fi_{23} and Fi_{24}' [4]. Although this seems to be a much more special and almost whimsical result compared to Tits' classification of spherical buildings, Fischer's result caused a comparable stir, at least among finite group theorists. This stir was due, of course, to the discovery of the three sporadic simple groups but even more to the sheer beauty and originality of Fischer's arguments. Generalizations followed quickly. His classification of groups generated by 3-transpositions was generalized to $\{3, 4\}^+$-transpositions by Franz Timmesfeld (a student of Fischer), to odd-transpositions by Michael Aschbacher and then to

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root involutions, again by Timmesfeld. (Fischer himself was led through the consideration of $\{3,4\}$-transpositions to the discovery of the Monster and Baby Monster sporadic groups. The Monster was discovered simultaneously and constructed by Robert Griess [5]. The classification of groups generated by 3-transpositions was extended to the infinite case by Hans Cuypers and Jon Hall [3].) This work and subsequent developments had an enormous influence on the eventual classification of finite simple groups. In the present book, Timmesfeld presents the most far-reaching advances in these directions – the theory and classification of what he calls "abstract root groups" – from which one can see that Fischer's work was a lot closer to Tits' theory of buildings than anyone suspected at the time.

In the theory of abstract root groups, Fischer's 3-transpositions (certain elements of order two) are replaced by subgroups (the abstract root groups), and his condition that two non-commuting 3-transpositions generate a symmetric group S_3 is replaced by the condition that two non-commuting abstract root groups generate a subgroup which either fulfills a certain condition implying it is nilpotent or is "a group of rank one." Let X be the subgroup generated by two groups A and B and let Ω be the conjugacy class of A in X. Then X is a group of rank one (equivalently: X has a split BN-pair of rank one) if $B \in \Omega$, X acts 2-transitively on Ω and A is nilpotent and acts regularly on $\Omega \backslash \{A\}$.

This book consists of five chapters. Chapter I is devoted to a thorough study of groups of rank one. Of course, S_3 is such a group (with A and B any two subgroups of order two), but there are many known families of such groups defined over arbitrary fields, skew-fields, etc., some of them involving pseudo-quadratic forms, Cayley-Dickson division algebras, etc. (Notice that the assumption of finiteness is now gone.) This chapter describes important examples (and serves as an introduction to the algebraic structures needed to describe them) and gives basic results on the structure of groups of rank one. In fact, this is the best collection of such results I know. Classification, however, seems to be out of the question, at least for the forseeable future.

Chapter II introduces groups generated by abstract root groups, again presents many families of examples, introduces the labeled graph on the set of abstract root groups which is the principal organizing tool of the whole theory (and Fischer's legacy) and begins to use it to obtain basic structural results about groups generated by abstract root groups. In this chapter, it is also shown how to associate a group generated by abstract root groups with a building. More precisely, let Δ be a thick irreducible spherical building of rank n. Suppose that Δ is Moufang. (We will say something about this important property below; here let us just note that this condition is automatically fulfilled if $n \geq 3$.) Then associated with each root α (certain substructures of other important substructures called apartments) of Δ is a certain subgroup U_α of $\mathrm{Aut}(\Delta)$. The roots lying in a given apartment of Δ can be canonically identified with the roots of a certain (not necessarily reduced) root system Φ. In §5, it is shown that if α is a highest root of Φ, then the set of subgroups of $\mathrm{Aut}(\Delta)$ conjugate to U_α forms a class of abstract root groups (except in the case that $n = 2$ and Δ is a generalized octagon) in the subgroup of $\mathrm{Aut}(\Delta)$ they generate. These are the "groups of Lie type" in the title; in almost all cases, they are simple. In the classical case, Δ is the building associated with a projective space or a polar space of finite rank. Projective spaces and polar spaces of infinite rank give rise to further groups generated by abstract root groups.

Chapter III is the most important one. It contains the complete proof that the only groups generated by a class of abstract root groups having no non-trivial solvable subgroup (a condition which can be made much more precise), with at least two commuting root groups and not involving groups of rank one defined over a field with less than four elements (this keeps the three sporadic groups generated by 3-transpositions from reappearing) are those coming from a Moufang spherical building of rank at least two or from a projective space or polar space of infinite rank. This is a difficult and deep result which builds on a whole tradition of characterizations involving different families of simple groups and the various kinds of geometries on which they act as well as the whole machinery of abstract root groups developed by Timmesfeld. Generically, Timmesfeld's strategy is to construct (in each of many cases) a point-line geometry associated with the target building (or polar space, etc.), thereby reducing the problem to one of several known classification results, most of which either depend on[1] or are a part of[2] or grew out of[3] or were subsumed in[4] Tits' classification of spherical buildings of rank at least three. For the geometries of rank two which arise, this means appealing to the classification of Moufang polygons [8].

A (thick irreducible) spherical building of rank two is also called a generalized polygon. Generalized polygons are too numerous to classify (generalized triangles and projective planes are equivalent notions, for instance), but Tits observed that the generalized polygons which appear as the rank two residues of a spherical building of rank at least three as well as all those which arise as the spherical building associated with an algebraic, classical or mixed group of rank two all exhibit a symmetry property he called Moufang (in honor of Ruth Moufang, a pioneer in the characterization of projective planes). In fact, the Moufang condition can be formulated for spherical buildings of arbitrary rank, and the main result of the famous Chapter 4 of Tits' Lecture Notes says (in other language) that a thick irreducible spherical building of rank at least three automatically has this property. This is the result which makes the classification of these buildings possible. Tits' proof of it is an astonishing tour de force.

Chapters IV and V (the last two) are devoted to applications of the classification of abstract root groups, Chapter IV to a new proof of the classification of finite groups generated by root involutions and Chapter V to results on quadratic pairs and subgroups of Lie type groups generated by long root elements.

The theory of abstract root groups and the theory of spherical buildings both have as their theme the mysterious and deep connection between group theory and geometry. Although in the end (not to imply that either theory is at its end!) the two theories result in characterizations of (almost) the same families of groups, they offer several interesting points of contrast. The notion of a spherical building was derived from a consideration of simple algebraic groups over an arbitrary field k. It resulted in a theory which almost as a footnote applies also to finite groups (i.e. to the case when k is finite). The notion of abstract root groups, on the other hand, grew from Fischer's work on 3-transpositions. This work was thoroughly finite in

[1] e.g. the characterization of parapolar spaces by Arjeh Cohen and Bruce Cooperstein [2]

[2] e.g. the characterizations of metasymplectic spaces, of buildings of type D_4 and of the exceptional polar spaces of rank three

[3] e.g. the characterization of polar spaces by Francis Buekenhout and Ernie Shult [1] and the characterization of polar spaces of infinite rank by Peter Johnson [6]

[4] e.g. early work on classical "polar geometries" by F. D. Veldkamp [10]

nature (or at least it certainly seemed so at the time), with much of its focus on the three new sporadic groups to which it gave rise. In a spherical building, groups of rank one appear as groups generated by pairs of "opposite" root groups, i.e. root groups U_α and U_β such that the union of α and β is an apartment. As noted above, we are far from a classification of groups of rank one. In the classification of Moufang buildings, in fact, these subgroups are avoided to the maximal extent possible. The philosophy of abstract root groups is just the opposite – groups of rank one are enshrined in the hypothesis themselves and play a central role in the whole theory.

Timmesfeld's classification of abstract root groups is a major achievement and the culmination of three decades worth of remarkable developments. This book is a valuable document for all those interested in simple groups and the geometries on which they act.

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