BOOK REVIEWS

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The geometry and cohomology of some simple Shimura varieties, by Michael Harris and Richard Taylor, with an appendix by Vladimir G. Berkovich, Annals of Mathematics Studies, Number 151, Princeton University Press, Princeton, NJ, 2001, viii + 276 pp., \$35.00 (paperback), ISBN 0-691-09092-0; \$65.00 (cloth), ISBN 0-691-09090-4

In the late sixties, Langlands proposed some problems on automorphic forms. Elaborated upon and tested in special cases, these have grown into a web of far-reaching conjectures linking representation theory, number theory and arithmetic geometry, now known as the Langlands program. In 1998, Harris and Taylor announced a proof of an essential part of the program, the local Langlands conjecture for GL_n of a p-adic field. This book gives a full account of their proof. It is based on a careful study of the bad reduction of certain Shimura varieties, itself an achievement of independent interest. My first and principal goal here is to indicate the statement and, in part, context of the conjecture. We will then turn to Harris and Taylor's work.

1. Motivation

We begin with the motivating case of class field theory. Let F be a number field, that is, a finite extension of $\mathbb{Q}$, also called a global field of characteristic zero. A finite extension of F is called abelian if it is Galois with abelian Galois group. Class field theory describes such extensions and their Galois groups directly in terms of the arithmetic of F. It can be elegantly formulated using the ring of adeles $\mathbb{A}_F$ of F.

To describe this ring, suppose first that $F = \mathbb{Q}$. Then for each prime p, we have the p-adic absolute value $|\ \ |_p$ on $\mathbb{Q}$. Thus if $x = p^n r/s \in \mathbb{Q}^\times$ with $r, s, n \in \mathbb{Z}$ such that $p \nmid rs$, then

$$|x|_p = p^{-n}.$$

The completion of $\mathbb{Q}$ with respect to $|\ \ |_p$ is the field $\mathbb{Q}_p$ of p-adic numbers. The absolute value $|\ \ |_p$ extends to an absolute value on $\mathbb{Q}_p$ which we again denote by $|\ \ |_p$. The elements x of $\mathbb{Q}_p$ with $|x|_p \leq 1$ form the unique maximal compact subring $\mathbb{Z}_p$. This is a principal ideal domain with unique nonzero prime ideal $p\mathbb{Z}_p$. There is also the prime at infinity for which $|\ \ |_\infty$ is the standard absolute value so that the completion $\mathbb{Q}_\infty = \mathbb{R}$. The fields $\mathbb{Q}_p$ for $p \leq \infty$ are exactly the locally compact non-discrete fields (up to isomorphism) that contain $\mathbb{Q}$ as a dense subfield.

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Then $\mathbb{A}_{\mathbb{Q}}$ is the subring of $\prod_{p\leq\infty}\mathbb{Q}_p$ of all (x_p) such that $x_p \in \mathbb{Z}_p$ for all but finitely many p. It carries a natural locally compact topology, and $\mathbb{Q}$ embeds diagonally as a discrete subgroup.

In the general case, $\mathbb{A}_F$ is again defined in terms of locally compact completions F_v of F indexed by the places v of F. There are a finite number of archimedean completions $F_v = \mathbb{R}$ or $\mathbb{C}$ corresponding to the infinite places given by embeddings of F in $\mathbb{R}$ or $\mathbb{C}$ (up to complex conjugation). Then there are an infinite number of non-archimedean completions. These correspond to the finite places v of F, one for each prime ideal in the ring of integers of F as for $F = \mathbb{Q}$. If the prime ideal given by v contains $p\mathbb{Z}$, then F_v is a p-adic field, that is, a finite extension of $\mathbb{Q}_p$. The integral closure of $\mathbb{Z}_p$ in F_v is then the unique maximal compact subring $\mathcal{O}_v$ of F_v. It is again a principal ideal domain with a unique nonzero prime ideal $\mathcal{P}_v$. Any generator of $\mathcal{P}_v$ is called a uniformizer.

Then $\mathbb{A}_F$ consists of all (x_v) in $\prod_v F_v$ such that $x_v \in \mathcal{O}_v$ for all but finitely many v. Again it carries a natural locally compact topology, and F embeds diagonally as a discrete subgroup. More generally, if $\underline{G}$ is a linear algebraic group over F, then the topology on $\mathbb{A}_F$ induces a locally compact topology on $\underline{G}(\mathbb{A}_F)$ and $\underline{G}(F)$ embeds as a discrete subgroup. In particular, if $\underline{G} = GL_1$, we obtain the locally compact groups $GL_1(\mathbb{A}_F) = \mathbb{A}_F^\times$ and $GL_1(\mathbb{A}_F)/GL_1(F) = \mathbb{A}_F^\times/F^\times$.

Class field theory gives an explicit bijection between the finite abelian extensions of F and the open subgroups of $\mathbb{A}_F^\times/F^\times$ of finite index. The Galois group of a given finite extension of F is then canonically isomorphic to the quotient of $\mathbb{A}_F^\times/F^\times$ by the corresponding open subgroup. This, in explicit form, is Artin's famous reciprocity law. We can express these isomorphisms all at once in terms of the Galois group of an algebraic closure $\overline{F}$ of F, called the absolute Galois group of F. Note first that $\overline{F}$ is a union of finite Galois extensions E of F and that each F-automorphism of E extends to $\overline{F}$. It follows that

$$\mathrm{Gal}(\overline{F}/F) = \lim_{\leftarrow} \mathrm{Gal}(E/F)$$

where the inverse limit is over all finite Galois extensions E of F in $\overline{F}$ (with respect to the restriction maps $\mathrm{Gal}(E_1/F) \to \mathrm{Gal}(E_2/F)$ for $E_1 \supset E_2$). If we give each $\mathrm{Gal}(E/F)$ the discrete topology, then $\lim_{\leftarrow} \mathrm{Gal}(E/F)$ becomes a compact topological group (as a closed subgroup of $\prod \mathrm{Gal}(E/F)$ with the product topology). This is the Krull topology on $\mathrm{Gal}(\overline{F}/F)$. Write $\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$ for the quotient of $\mathrm{Gal}(\overline{F}/F)$ by the closure of its commutator subgroup. Then Artin reciprocity yields an explicit continuous surjective homomorphism

$$r_F : \mathbb{A}_F^\times/F^\times \to \mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}.$$

There is a similar description of the abelian extensions of the completions F_v of F. Of course, for the archimedean completions this is trivial. The non-archimedean case constitutes local class field theory. A local field is a non-discrete locally compact field. Any such non-archimedean field of characteristic zero is isomorphic to a finite extension of some $\mathbb{Q}_p$ and arises as F_v for appropriate F and v, in general in several different ways.

Suppose then that K is a p-adic field. Let $\mathcal{O}$ denote the maximal compact subring of K and $\mathcal{P}$ the unique nonzero prime ideal in $\mathcal{O}$. Fix an algebraic closure $\overline{K}$ of K. The integral closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ in $\overline{K}$ has a unique nonzero prime ideal $\overline{\mathcal{P}}$, and $\overline{k} = \overline{\mathcal{O}}/\overline{\mathcal{P}}$ is an algebraic closure of the finite residue field $k = \mathcal{O}/\mathcal{P}$ of K. There is therefore a natural homomorphism

$$\mathrm{Gal}(\overline{K}/K) \to \mathrm{Gal}(\overline{k}/k).$$

This map is continuous and surjective. Its kernel is called the inertia subgroup I_K. Let $q = |k|$. Then $\mathrm{Gal}(\overline{k}/k)$ is topologically generated by the Frobenius element $x \mapsto x^q$. Any element of $\mathrm{Gal}(\overline{K}/K)$ which maps onto $x \mapsto x^q$ is called a Frobenius element, denoted Fr, of $\mathrm{Gal}(\overline{K}/K)$. There is a local Artin reciprocity law which provides a continuous, now injective, homomorphism

$$r_K : K^\times \to \mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$$

with dense image. A key property of r_K is that it maps uniformizers to (the images of) Frobenius elements.

It is often more convenient to rephrase this in terms of the Weil group W_K of K. By definition, W_K consists of those elements of $\mathrm{Gal}(\overline{K}/K)$ that act by an integer power of Frobenius on $\overline{k}$. It carries a unique locally compact topology for which I_K is an open subgroup (with the induced topology from $\mathrm{Gal}(\overline{K}/K)$). Then r_K induces an isomorphism of topological groups $K^\times \cong W_K^{\mathrm{ab}}$.

The two class field theories, local and global, are closely related. Indeed, for any finite place v of F, let G_v denote the absolute Galois group of F_v. Then there is a family of conjugate embeddings $G_v \hookrightarrow \mathrm{Gal}(\overline{F}/F)$ and hence a well-defined homomorphism $G_v^{\mathrm{ab}} \to \mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$. Precomposing with the local Artin maps r_{F_v}, we obtain maps $F_v^\times \to \mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$. The global Artin map r_K is then built from these local maps, via the obvious embeddings $F_v^\times \hookrightarrow \mathbb{A}_F$. (Of course, we have ignored the archimedean places, but these are easily treated.) In particular, r_F takes uniformizers in any $F_v^\times$ (for v finite) to the images of the Frobenius elements of G_v in $\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$. It is determined by this property and its triviality on $F^\times$. There are now several purely local proofs of local class field theory. Once the local theory is intrinsically established, the global theory follows quickly. However, historically, the global theory was proved first and the local theory obtained as a consequence. The local Langlands conjecture for GL_n can be viewed as a generalization of local class field theory. Harris and Taylor's work and the alternative approach due to Henniart ([5], [7]) rely crucially on global results, in the spirit of the initial proofs of local class field theory.

2. Conjectures

A direct description of $\mathrm{Gal}(\overline{F}/F)$ in the manner of global class field theory's description of $\mathrm{Gal}(\overline{F}/F)^{\mathrm{ab}}$, if such exists, appears completely out of reach. What emerges from Langlands' work is a dual viewpoint which links representations of $\mathrm{Gal}(\overline{F}/F)$ and representations of general linear groups over $\mathbb{A}_F$. From this perspective, global class field theory is presented as a bijective correspondence $\chi \leftrightarrow \chi \circ r_F$ between continuous characters $\chi : \mathrm{Gal}(\overline{F}/F) \to \mathbb{C}^\times$ and finite order complex characters of $\mathbb{A}_F^\times/F^\times$. In place of characters or one-dimensional representations, one considers automorphic representations of $GL_n(\mathbb{A}_F)$. In rough terms, these are essentially the irreducible constituents of the regular representation of $GL_n(\mathbb{A}_F)$ on $L^2(GL_n(\mathbb{A}_F)/GL_n(F))$. For $n = 2$ and $F = \mathbb{Q}$, they are closely related to classical modular forms and Maass forms. The class of cuspidal automorphic representations, analagous to classical cusp forms, plays a special role in that a general automorphic representation of $GL_n(\mathbb{A}_F)$ is built from cuspidal automorphic representations of $GL_m(\mathbb{A}_F)$ for $m \leq n$.

Since $GL_n(\mathbb{A}_F)$ is assembled from the groups $GL_n(F_v)$, as v varies through the places of F, an irreducible representation Π of $GL_n(\mathbb{A}_F)$ decomposes as a suitably interpreted infinite product $\bigotimes_v \Pi_v$ where each Π_v is an irreducible representation of $GL_n(F_v)$. For all but finitely many v, each local component Π_v must be unramified. This means that it admits a nonzero fixed vector for the maximal compact subgroup $GL_n(\mathcal{O}_v)$ of $GL_n(F_v)$. Representations of this sort are classified by the Satake isomorphism which establishes a bijection between isomorphism classes of unramified representations of $GL_n(F_v)$ and elements of $(\mathbb{C}^\times)^n$ modulo permutations. The unordered n-tuple corresponding to an unramified representation is called its Satake parameter. A cuspidal automorphic representation Π is rigid in that it is determined by its local unramified components Π_v, for any collection of v that includes all but finitely many places.

On the Galois side, one considers complex representations, that is, continuous homomorphisms

$$\Sigma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}_{\mathbb{C}} V$$

with V finite-dimensional over $\mathbb{C}$. Such representations are unramified almost everywhere, in the sense that, for all but finitely many v, the restriction $\Sigma_v = \Sigma \mid G_v$ is trivial on the inertia subgroup of G_v. Let Fr_v denote a Frobenius element of G_v. Then the irreducible constituents of Σ are determined by the family $\{\Sigma_v(\mathrm{Fr}_v)\}$, again for any collection of v that includes all but finitely many places.

Langlands has conjectured that if Σ is an irreducible complex representation of $\mathrm{Gal}(\overline{F}/F)$ of dimension n, then there is an automorphic cuspidal representation $\Pi = \Pi(\Sigma)$ of $GL_n(\mathbb{A}_F)$ such that, at those finite places v at which Σ_v and Π_v are unramified, the Satake parameter of Π_v is given by the eigenvalues of $\Sigma_v(\mathrm{Fr}_v)$. Of course, the representation $\Pi(\Sigma)$ is uniquely determined, if it exists, by the rigidity property of cuspidal automorphic representations stated above. The equality between Satake parameters and eigenvalues of Frobenius elements generalizes the relation between uniformizers and Frobenius elements in class field theory. The conjecture includes, as a special case, the famous conjecture of Artin on the holomorphicity of L-functions of (non-trivial) complex irreducible Galois representations. However, the cuspidal automorphic representations that can occur in this conjectural correspondence are quite restricted.

Arithmetic geometry is a rich source of further Galois representations. In particular, for a prime l, the l-adic cohomology groups of suitable varieties over F give rise to l-adic Galois representations, that is, suitably continuous homomorphisms

$$\Sigma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{Aut}_{\overline{\mathbb{Q}}_l}(V)$$

where V is now finite-dimensional over an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$. Such representations, coming from geometry, are again unramified almost everywhere. In contrast to the complex case, they do not in general have finite image.

There is again a similar conjecture relating irreducible l-adic Galois representations Σ and cuspidal automorphic representations Π, and all cuspidal automorphic representations in a wide class should occur in these correspondences. To convey something of its depth, note that the conjecture includes, as a very special case, the statement that every elliptic curve over $\mathbb{Q}$ is modular, proved recently by Breuil-Conrad-Diamond-Taylor following the path-breaking work of Wiles and Taylor-Wiles.

These global correspondences $\Sigma \leftrightarrow \Pi$ should give rise to, and arise from, similar local correspondences $\Sigma_v \leftrightarrow \Pi_v$. For a p-adic field K, the desired local correspondence is most easily stated in terms of the Weil group W_K of K. For this one needs the notion of a Weil-Deligne representation of W_K. This is a pair $\sigma = (\rho, N)$ with $\rho : W_K \to \mathrm{Aut}_{\mathbb{C}} V$ a continuous homomorphism (for V finite-dimensional) and $N : V \to V$ a nilpotent endomorphism such that

$$\rho(w)N\rho(w)^{-1} = q^{-\nu(w)}N, \qquad w \in W_K.$$

Here $\nu(w)$ is the power of Frobenius by which w acts on an algebraic closure of the residue field of K. For $l \neq p$ continuous l-adic representations of W_K or $\mathrm{Gal}(\overline{K}/K)$ can be understood in these terms [2]. Of course, if $N = 0$, then σ is just an ordinary continuous complex representation of W_K. We say that σ is semisimple if ρ is semisimple. The dimension of σ is simply the dimension of ρ.

On the GL_n side, we need the notion of an irreducible smooth representation $\pi = (\pi, \mathcal{V})$ of $GL_n(K)$. This is a complex vector space $\mathcal{V}$ and a homomorphism $\pi : GL_n(K) \to \mathrm{Aut}_{\mathbb{C}}\mathcal{V}$ such that 1) the stabilizer of each vector in $\mathcal{V}$ is open for the natural topology on $GL_n(K)$ induced by the p-adic topology on K (smoothness) and 2) $\mathcal{V}$ has no non-trivial π-stable subspaces (irreducibility). Such representations are closely related to the local components of automorphic representations of $GL_n(\mathbb{A}_F)$ at finite places v with $F_v = K$. The space $\mathcal{V}$ is almost always infinite-dimensional. For any smooth irreducible π, the center of $GL_n(K) \cong K^\times$ acts by a smooth homomorphism $\omega_\pi : K^\times \to \mathbb{C}^\times$. Of course, for any smooth homomorphism $\chi : K^\times \to \mathbb{C}^\times$, the representation $\pi\chi$ given by $g \mapsto \pi(g)\chi(\det g)$ is again smooth and irreducible. Similarly, one can also twist a Weil-Deligne representation by a smooth, or equivalently continuous, character of W_K.

The local Langlands conjecture for GL_n roughly asserts that the irreducible smooth representations of the groups $GL_n(K)$ mirror the arithmetic of the algebraic extensions of K. To formulate it more precisely, let $\mathcal{A}_n(K)$ denote the set of isomorphism classes of irreducible smooth representations of $GL_n(K)$ and $\mathcal{G}_n(K)$ the set of isomorphism classes of semisimple Weil-Deligne representations of dimension n. Then the conjecture asserts that there is a family of bijections $\sigma_n = \sigma_{n,K}$, for $n \geq 1$,

$$\pi \stackrel{\sigma_n}{\longmapsto} \sigma(\pi) : \mathcal{A}_n(K) \to \mathcal{G}_n(K)$$

satisfying the following properties:

I. Under the isomorphism $K^\times \cong W_K^{\mathrm{ab}}$ of local class field theory, the determinant of $\sigma(\pi)$ corresponds to the central character ω_π of π. In particular, σ_1 is induced by local class field theory.

II. For each χ in $\mathcal{A}_1(K)$ and $\pi \in \mathcal{A}_n(K)$,

$$\sigma_n(\pi\chi) = \sigma_n(\pi)\sigma_1(\chi).$$

III. The dual $\pi^\vee$ of an irreducible smooth representation π of $GL_n(K)$ and the dual $\sigma^\vee$ of a Weil-Deligne representation σ of W_K are defined in the obvious way. The bijections σ_n preserve duals: for $\pi \in \mathcal{A}_n(K)$,

$$\sigma_n(\pi^\vee) = \sigma_n(\pi)^\vee.$$

IV. Finally and most crucially the bijections preserve certain arithmetic invariants. On the GL_n side, these are the L-factors and ε-factors of pairs defined in [8]. They also arise from a wider construction due to Shahidi. These factors contribute to corresponding objects attached to pairs of automorphic representations which play a key role in the global theory. On the Galois side, if (ρ, V) is a representation of W_K and V^{I_K} denotes its space of I_K-fixed vectors, then the Artin L-factor $L(s,\rho)$ is defined by

$$L(s,\rho) = \det(1 - \rho(\mathrm{Fr})q^{-s} : V^{I_K})^{-1},$$

where s is a complex variable. Thus if ρ is unramified, then $L(s,\rho)$ simply records the eigenvalues of $\rho(\mathrm{Fr})$. The definition extends naturally to Weil-Deligne representations. There is also an ε-factor for a Weil-Deligne representation. Roughly speaking, it reflects the complexity of the restriction of the representation to I_K. There is an obvious notion of tensor product for Weil-Deligne representations. Now let $\pi \in \mathcal{A}_n(K)$, $\pi' \in \mathcal{A}_{n'}(K)$. The final requirement then is that the L-factor and ε-factor for $\sigma(\pi) \otimes \sigma(\pi')$ should coincide with the L-factor and ε-factor of the pair (π, π'). The L-factor condition implies that σ_n induces the bijection between unramified representations of $GL_n(K)$ and unramified semisimple representations of W_K of dimension n for which Satake parameters correspond to eigenvalues of Frobenius, as demanded by compatibility with the global conjectures.

Langlands has deemed the emphasis on ε-factors in this current formulation unsatisfactory. With a much deeper understanding of the representation theory of p-adic groups, it may be possible to express and prove the conjecture in a more natural way, but at present this seems a long way off.

The semisimple Weil-Deligne representations of W_K of dimension n are constructed in a canonical way from the m-dimensional irreducible representations of W_K for $m \leq n$. On the GL_n side, there is a parallel, though much deeper, procedure—the Zelevinsky classification [10]—that assembles the irreducible smooth representations of $GL_n(K)$ from certain building blocks, the irreducible supercuspidal representations of $GL_m(K)$ for $m \leq n$. These constructions are compatible with the local Langlands conjecture. More precisely, let $\mathcal{A}_n^0(K)$ denote the set of isomorphism classes of irreducible supercuspidal representations of $GL_n(K)$ and $\mathcal{G}_n^0(K)$ the set of isomorphism classes of irreducible representations of dimension n of W_K. Suppose that $\sigma_n^0 : \mathcal{A}_n^0(K) \to \mathcal{G}_n^0(K)$ is a family of bijections satisfying I-IV. Then the maps σ_n^0 extend to bijections $\sigma_n : \mathcal{A}_n(K) \to \mathcal{G}_n(K)$ that again satisfy I-IV. Moreover, by work of Henniart announced in [7], there is at most one set of bijections σ_n satisfying these requirements. In particular, the certainly natural procedure of extending the maps σ_n^0 to the maps σ_n is necessarily unique. In earlier work, Henniart had shown that the bijections σ_n^0 are uniquely determined by preservation of ε-factors [4].

3. Theorems

The problem then was to prove that the maps σ_n^0 actually exist. Harris and Taylor accomplish this by generalizing an approach of Deligne in the case $n = 2$ and $K = \mathbb{Q}_p$. Deligne used the geometry of modular curves to produce a correspondence between certain cuspidal automorphic representations of $GL_2(\mathbb{A}_\mathbb{Q})$ and certain global Galois representations. From this, with much work, one obtains a well-defined local correspondence that meets the necessary requirements.

Shimura varieties are a generalization of modular curves with a similar rich arithmetic. Examples arise naturally in the study of moduli problems for abelian varieties. From the initial definition they appear as complex manifolds, but they are known to have a canonical variety structure, by work of Baily and Borel. Further they admit canonical models over number fields, and Langlands has proposed that their Hasse-Weil zeta functions can be understood in terms of automorphic representations. However, there are substantial technical obstacles to realizing this proposal. Kottwitz isolated a class of Shimura varieties, attached to certain twisted unitary groups over suitable number fields F, where most of these obstacles could be overcome. These are the varieties used by Harris and Taylor, following a suggestion of Carayol.

They realize K as F_v for a finite place v of F. The l-adic cohomology of the Shimura varieties gives rise to a global correspondence between certain cuspidal automorphic representations Π and certain global Galois representations Σ. From a careful study of the bad reduction of these varieties, they show that the semisimplification of Σ_v depends only on Π_v and arises from a geometric construction involving purely local objects. In particular, this gives a geometric model of the local Langlands correspondence (for supercuspidals) in spaces of vanishing cycles, in the spirit of earlier predictions of Carayol. This requires a result of Berkovich which is included in an appendix. The strategy of obtaining the local correspondence by global means is analogous to the initial proofs of local class field theory. While they modestly describe their work as simply a natural generalization of Deligne's, the technical complications are formidable and require an array of sophisticated techniques, many beyond the ken of this reviewer.

A few months after Harris and Taylor circulated a preliminary version of their work, Henniart found a clever and considerably more direct proof of the existence of the correspondence. It again relies in an essential way on the geometry of Shimura varieties but avoids consideration of places of bad reduction. However, it does not give a geometric model of the local correspondence that is compatible with many instances of the global Langlands correspondence. Neither proof gives an explicit description of the maps σ_n^0 which remains as a fundamental open problem. Bushnell and Henniart have made much progress on this using the classification of $\mathcal{A}_n^0(K)$ via restriction to compact open subgroups, due to Bushnell and Kutzko.

The Langlands conjectures also make sense for local and global fields of positive characteristic. Here the local conjecture for GL_n was proved in the early nineties by Laumon, Rapoport and Stuhler. More recently Lafforgue, in a celebrated tour de force, has established the global correspondence, building on earlier ideas of Drinfeld. However, the tools he uses are unavailable in the number field case where a general solution of the corresponding problem is likely a long way off.

There are now several good expository accounts of the proofs of Harris and Taylor and of Henniart. In particular, Carayol's Bourbaki report [1] gives a detailed overview of the geometric aspects of Harris and Taylor's work. Henniart's paper [6] stresses the local-global principles underlying both approaches. In addition, Taylor's account of his Beijing ICM talk [9], written expressly for nonspecialists, discusses the broader global conjectures hinted at above. Harris' ICM talk [3] reports on some recent developments and highlights several natural open problems.

Finally, the book itself is clearly and carefully written. In sum, it represents an awe-inspiring achievement and is a model of good exposition.

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