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Essays in the history of Lie groups and algebraic groups, by Armand Borel, History of Mathematics, vol. **21**, American Mathematical Society and London Mathematical Society, Providence, RI, and London, 2001, xiii+184 pp., \$39.00, ISBN 0-8218-0288-7

Recently, two books have appeared on the history of Lie theory. The first, written by Thomas Hawkins [1] presents a highly detailed study up to 1926, concentrating on the development of Lie groups and algebras. The second, and the book under review, written by Armand Borel, comprises essays on topics ranging in time from the 1890s through about 1970. In addition to discussing Lie groups and Lie algebras, Borel pays considerable attention to algebraic groups, beginning with the little known work of Ludwig Maurer in the 1890s up through the work of Claude Chevalley, Ellis Kolchin, Borel and others in the post-World War II era.

In the beginning, Sophus Lie hoped to develop a Galois theory for differential equations. Felix Klein's famous "Erlangen Program," which aimed to connect firmly geometry with group theory, also influenced Lie's thinking. By 1893, Lie (together with Friedrich Engel) had completed the final, third volume of the massive treatise *Theorie der Transformationsgruppen.* Also by 1890, Wilhelm Killing had succeeded in classifying (modulo a few gaps in his arguments) the complex simple Lie algebras. Killing's work was rigorously treated and extended in Élie Cartan's 1894 thesis. In Chapter I, Borel gives a brief overview of Lie's early work as well as the related work of Killing and Cartan. Ironically, Lie algebras was an area Lie had little interest in pursuing. From Lie's point of view, a Lie group was a transformation group, i.e., there was some geometric space present upon which the Lie group acted. The notion of an abstract group with the compatible structure of a complex manifold came later; mathematicians such as Klein resisted this notion, feeling that it would distance the subject too far from its applications. In addition, Lie's theory was largely *local*. Early workers were aware of global examples (such as $\mathbf{SL}_n(\mathbb{C})$), but the theory was mostly developed for what we today would call "germs" of Lie groups. A global theory emerged only with the work of Weyl and Cartan in the twentieth century.

After the introductory chapter, the book divides into two (related) parts. The first part (Chapters II, III, and IV) focuses on Lie groups, Lie algebras, and related topics (e.g., symmetric spaces). The second part (Chapters V, VI, VII, and VIII) deals with algebraic groups.

Lie groups and Lie algebras. The chronological setting for Chapter II, entitled "Full Reducibility and Invariants for $\mathbf{SL}_2(\mathbb{C})$", is the 1890s, the era in which Heinrich Maschke, using E. H. Moore's idea of averaging over a finite group, proved that any complex representation of a finite group was fully reducible. According to Lie in his *Transformationsgruppen*, Eduard Study had claimed (in an unpublished manuscript) full reducibility for finite dimensional, holomorphic representations of the complex Lie group $\mathbf{SL}_2(\mathbb{C})$. Study also conjectured a similar result for $\mathbf{SL}_n(\mathbb{C})$. A related issue concerned the finite generation for the ring of invariant polynomial

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functions on a finite dimensional representation. Borel gives a detailed mathematical discussion of three different approaches to these questions: an algebraic one (due to É. Cartan in his thesis), an analytic one (due to Adolf Hurwitz) and an algebraic-geometric one (due to Gino Fano). Moore's averaging process could not be directly applied to $\mathbf{SL}_2(\mathbb{C})$, so to "surmount that difficulty, Hurwitz used a procedure which turned out later to be far-reaching, namely to integrate over a compact subgroup G_u, which insures convergence, but choosing it big enough so that invariance under G_u implies invariance under the whole group, an argument later called the 'unitarian trick' by Weyl" (p. 12). A similar method would have shown full reducibility for holomorphic representations of $\mathbf{SL}_2(\mathbb{C})$ (and more generally, $\mathbf{SL_n}(\mathbb{C})$ $(n \geq 3)$ and $\mathbf{SO_n}(\mathbb{C})$ $(n \geq 4)$). Calling this work a "landmark paper," Borel goes on to say that "[t]his is only hindsight, because the paper was completely forgotten for about 25 years and, apparently, no specialist in Lie groups or Lie algebras was aware of it and had realized that a proof of Study's conjecture for $\mathbf{SL_n}(\mathbb{C})$ was at hand" (p. 15).

Hermann Weyl, whose work is surveyed in Chapter III from the early 1920s to the late 1930s, became interested in Lie theory from problems in the theory of general relativity. Almost immediately, he was drawn to Cartan's work, especially that on the classification and representation theory of simple Lie algebras. Also, by this time, Issai Schur had used Hurwitz's method to attack problems for certain of the classical groups. Seeing this work, "Weyl was now ready to strike. He first extended Schur's method to $\mathbf{SL_n}(\mathbb{C})$ and the symplectic group $\mathbf{Sp}_{2n}(\mathbb{C})$ ... and then almost immediately afterwards combined the Hurwitz-Schur and Cartan approaches in an extraordinary synthesis .... Until Weyl came on the scene, Cartan did not know about the work of Hurwitz and Schur, nor did Schur know about Cartan's ... " (p. 33). Among the results obtained are Weyl's famous character and degree formulas for irreducible representations. In addition, "these papers mark the birthdate of the systematic global theory of Lie groups. The original Lie theory, created in 1873, was in principle local, but during the first fifty years, global considerations were not ruled out, although the main theorems were local in character. However, a striking feature was that algebraic statements were proved by global arguments, which, moreover, seemed unavoidable at the time" (p. 35).

Chapter IV, entitled "Élie Cartan, Symmetric Spaces, and Lie Groups", roughly mirrors Chapter III chronologically and focuses in considerable detail on Cartan's major papers. The chapter is divided into two parts. Part A takes up the development of the theory of symmetric spaces. In particular, in his development of a global theory of symmetric spaces, Cartan was heavily influenced by the global point of view of Weyl. Throughout the chapter, Borel does a wonderful job of describing the impact of Cartan and Weyl on each other. By 1930, "Cartan felt it was time to outline a theory of Lie groups stressing global aspects" (p. 78). He did this in his book *La théorie des groupes finis et continus et l'analysis situs*. Part B ("Further Developments") first takes up Cartan's efforts to extend the Peter-Weyl theorem to the homogeneous spaces of compact groups, providing a detailed mathematical discussion of his 1929 paper in that direction. Borel then describes Cartan's work on the topology of Lie groups and their homogeneous spaces. These results would influence mathematics for many years after, e.g., algebraic topology, the cohomology of Lie algebras, etc.

Algebraic groups. Chapter V is devoted to the 19th century and Chapter VI to the 20th century. The concluding chapters, VII and VIII, reproduce with some modifications two previously published articles, one on Chevalley and one on Kolchin, providing more details on their work.

As for the 19th century, I will confine my remarks to Maurer, who published four papers on algebraic groups during the years 1888–1899. Borel provides a thorough exposition of this work, using modern terminology, but also paying attention to Maurer's own formulations. His results included a careful investigation of the Lie algebra $\mathfrak{g}$ of a linear algebraic group G (i.e., an algebraic subgroup of $\mathbf{GL_n}(\mathbb{C})$), including the Jordan decomposition in $\mathfrak{g}$, the algebraic hull of a one-parameter subgroup of G (see comments on Chevalley's work below), a proof of the rationality of G, main properties of unipotent groups, construction of maximal tori, Cartan subgroups, etc.

Maurer's last paper in this period (which Borel calls "unfortunate" (p. 111)) attempted to prove the false fact that the invariants for a connected Lie group are finitely generated. Maurer did, however, prove this result correctly for a one-dimensional unipotent group (or, equivalently, a complex nilpotent matrix). Twenty-four years later and with Maurer's work largely forgotten, the incorrect proof was essentially reproduced by Roland Weitzenböck, together with the correct one-dimensional case, now known as Weitzenböck's theorem.

Speaking about Maurer's work on linear algebraic groups, Borel writes (p. 112):

> His results resurfaced fifty or sixty years later, with at best perfunctory acknowledgements, quite inadequate, even though the more recent results are usually proved in greater generality. It is true that there are some errors, and the presentation, in particular the notation, is often awkward, but this should not hide the fact that he had developed a considerable insight into this topic.

During the 1940s, the theory of algebraic groups attracted the attention first of Chevalley and then of Kolchin. Motivated by his interest in Lie groups, Chevalley, whose 1946 classic, *Theory of Lie Groups, I*, was the first textbook to adopt the global point of view, initially worked in characteristic zero, relying heavily on Lie algebra methods. For example, influenced by Maurer's work on the Lie algebra of a complex group and thus extending it to general fields k of characteristic zero, Chevalley introduced the notion of a *replica* of a linear operator X on a finite dimensional vector space V. For example, a replica Y of a semisimple operator X is a polynomial, without constant term, in X whose eigenvalues satisfy the integer linear relations satisfied by the eigenvalues for X. "Chevalley calls a linear Lie algebra *algebraic* if it contains the replicas of all its elements. In that terminology, the main result of Maurer's 1888 paper ... asserts that the Lie algebra of a complex linear algebraic group is algebraic" (p. 119).

In 1948, Kolchin, motivated like Lie by differential equations, published several landmark papers which "constitute in fact the birth certificate of the theory of linear algebraic groups over algebraically closed fields of *arbitrary* characteristic" (p. 169). Actually, Kolchin was concerned with generalizing the Picard-Vessiot theory (discussed in Chapter V) to a Galois theory of differential fields. As Borel states (p. 165):

> From the beginning, it appeared that these Galois groups would be algebraic groups, or, rather, naturally isomorphic to such. However, the theory of algebraic groups was not suitably developed in Kolchin's view for his purpose when he first needed it, so that a minor, but persistent and essential, theme in his work is the theory of algebraic groups.

Unlike the work of this period by Chevalley, Lie algebras played no role in Kolchin's approach, but rather his methods came from algebraic geometry. The famous Lie-Kolchin theorem, that any connected and solvable algebraic group of linear transformations has a common eigenvector, derives from this work.

The methods of algebraic geometry, in fact, soon played a dominant role in the theory of algebraic groups. By the mid-1950s, Chevalley was able to classify completely the semisimple algebraic groups over an arbitrary algebraically closed field k. This was presented in his famous Paris notes and his equally famous "Tôhoku" paper. The main Paris breakthrough was made possible by Borel's landmark 1956 *Annals* paper in which the central importance of the maximal connected solvable subgroups (later called Borel subgroups by Chevalley) came to the fore.

The above brief narrative omits many interesting topics discussed in the book, including the theory of algebraic groups over non-algebraically closed fields, the work of various mathematicians on "abstract" homomorphisms between algebraic groups, the Tits theory of buildings, etc. In many of these areas, Borel himself was a key player, and he describes the mathematics with great authority.

To end this part of the review, I quote from Chevalley (as quoted by Borel, from an unpublished source): "The principal interest of the algebraic groups seems to me to be that they establish a synthesis, at least partial, between the two main parts of group theory, namely the theory of Lie groups and the theory of finite groups" (p. 157). Probably written in 1953–1954, these words seem highly prescient a half-century later, given the explosion of work on the representations of finite groups of Lie type and such matters as the Alperin Conjecture, which tries to establish a kind of "weight theory" for general finite groups.

Final remarks. Borel's essays tell much of the story of a great mathematical theory over an 80 year span. The mathematics is beautiful and deep. The story has twists and turns, injustices and triumphs. Mathematicians have their results forgotten or ignored, then rediscovered many years later. Lack of interaction between researchers even in adjacent countries hampers progress. Opportunities are missed. Points of view shift dramatically over the years.[1] Borel takes a very internalistic approach in this book, never straying far from the mathematics. Although he does provide photographs of the principals, we learn essentially nothing about their personal lives. Other approaches are possible and valuable, but I personally found this to be a fascinating book, and I highly recommend it.

[1] As for Lie's original idea of developing a Galois theory for differential equations, one expert recently wrote [2] that "the remarkable range of applications of Lie groups to differential equations in geometry, in analysis, in physics, and in engineering over the past 40 years has resurrected Lie's original vision into one of the most active and rewarding fields of contemporary research."

References

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