BOOK REVIEWS

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Analytical mechanics: A comprehensive treatise on the dynamics of constrained systems; for engineers, physicists and mathematicians, by John G. Papastavridis, Oxford University Press, 2002, xxii + 1392 pp., \$295.00, ISBN 0-19-512697-1

This survey has the bias of the "Geometric Mechanics" school: Mechanics and Differential Geometry are jealous sisters, but are inherently devoted to each other. Lie groups play a fundamental role and symmetries of mechanical systems can be either external (usually left invariant) or internal (material, usually right invariant).

We limited our bibliography, a blend of pure and applied mathematics, with some biology (!) interspersed, to the magic number one hundred. Geometric mechanics is epitomized by the textbooks of Abraham and Marsden [1] and Arnold [6].

Our main focus is how geometry impacts on *applications* of Mechanics, in particular to nonholonomically constrained systems, one of the main topics of Papastavridis' book. However, it is very likely that two themes will become the most important sources of problems for Mechanics in the XXIst century: nonlinear geometric control of large scale systems, and biological motion, from cells to complex organisms.

For mathematicians and theoretical physicists, Analytical Mechanics centers around "Poisson brackets", introduced by Siméon-Denis Poisson (1781-1840) in his work on the three-body problem in celestial mechanics.

In the nineteenth century, Lagrange, Jacobi and Lie elucidated the main properties of the Poisson brackets and began the study of their geometry.

After a long period of dormancy, it was realized in the 80's (see Weinstein [96] for an online exposition) that general Poisson brackets can have a much richer geometry than the standard bracket

$$\{f(p,q), g(p,q)\} = \sum \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \ .$$

Although Poisson geometry has its roots in Mechanics, its branches now reach out to touch various areas of mathematics and mathematical physics, Lie groups and representation theory, integrable systems, singularity theory, noncommutative geometry, and quantum field theory, to mention a few.

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The last section, which I owe to Henrique Bursztyn and Rui L. Fernandes,[1] presents a survey of *recent mathematical developments in Poisson geometry.*

Alan Weinstein, who played a major role in these developments, advised two of us (JK and HB) almost 30 years apart and has contacts with several young Portuguese mathematicians. We take the opportunity to congratulate Alan, in anticipation, on the occasion of his 60th birthday festshriften.

I. New opportunities

The geometric approach will be an important ingredient in studying *multiscale* processes. It is not merely the computational size, but integrating complexities in many orders of magnitude.

i) Modeling cellular mechanical processes. This is a new and very exciting arena for Mechanics, which started around 10 years ago. With new instruments such as the optical tweezer, the cellular clockwork can now be observed *in vivo.* The excitement is comparable to Leeuwenhoek's, who in 1676 looked at microorganisms for the first time under his microscope.

Most of the biology one needs can be learned in "the Cell" [2]. We recommend two recent books [12], [49] for subcellular motion, and in particular, the fascinating theme of "molecular motors". An example of a very interesting work is G. Huber et al. [50], modeling flagellar motion reversal with an integrable infinite dimensional Hamiltonian system. Our geometric mechanics microswimming project is outlined in [56]; see also our web page, www.impa.br/~jair.

ii) Applications of control theory, in particular nonlinear geometric control, especially to biology [30] and quantum systems in the mesoscopic scale [74], [75], [67]. New experiments (and technologies) are melting the boundary between classical and quantum mechanics. In the last section, we discuss the new mathematical developments in Poisson geometry that may help bridge the gap between them.

II. Geometry of Analytical Mechanics with an eye to applications

Manifolds, bundles, differential forms, Lie groups and Riemann surfaces are basic tools of modern mathematical methods for Mechanics. Topological invariants can be used to predict qualitative features; most of the interesting dynamics follows from understanding the behavior near singularities.

Geometric mechanics "for the ordinary folk". Consider this simple (not so simple!) example. The configuration space of a double pendulum is the torus $S^1 \times S^1$. The homology group $H_1 = Z \times Z$ gives the types (m, n) of periodic orbits. Not much mathematical sophistication is needed to guess that m and n are the number of turns in each circle, but the existence of a periodic orbit on every homology class follows from a deep result (one learns in differential geometry the existence of a minimizing geodesic in every homology class).

In the last 30 years novel and exciting *integrable systems* have been found, with important applications ranging from solid state physics to numerical analysis; see e.g. [8] if we are allowed to choose a single reference. The study of integrable

[1] I consider them morally co-authors. Addresses: *henrique@math.toronto.edu*, Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 3G3 ; *rfern@math.ist.utl.pt*, Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal.

systems requires Lie groups and special functions on Riemann surfaces. That symmetries and integrability are closely related results from Noether's theorem. But why do Riemann surfaces appear naturally in Mechanics? This can be seen in another "simple" example. Consider a one degree of freedom system on a polynomial potential. From the energy integral, one can write time $t = t(x)$ as a hyperelliptic integral on the configuration variable x. If the degree of the polynomial is $2(k+1)$ or $2(k+2)$, then $x = x(t)$, obtained by functional inversion, is a complex meromorphic function on a genus k Riemann surface.

Engineers and physicists can benefit by taking advantage of modern mathematical techniques. But can the king, or even ordinary folks, learn the necessary geometrical tools? Well, there is no royal road to geometry - practice makes perfect. Around 20 years ago, Spivak [86] wrote a great expository treatise in Differential Geometry, so there is no excuse.

Besides the "classics" Abraham/Marsden and Arnold, we suggest the following references on geometric mechanics: [71], [69], [31], [15].

A Modern Dynamics sequence: Advanced Classical Mechanics+Introduction to Dynamical Systems. This is a teaching experience I was involved with at Caltech in the Control and Dynamical Systems Department during the 2001-2002 academic year. *This ongoing experiment is evolving as a partnership between the USA and Brazil* [76]. The course is a combined 2 - 3 quarters sequence in Dynamics and Control for beginning graduate students and senior undergraduates (Physics, Mathematics, Engineering, Chemistry and Biology majors interested in mathematical modeling).

The two quarter sequence "Introduction to Modern Dynamics" is Caltech's CDS140ab (www.cds.caltech.edu), which attracted many students in the 2002-2003 academic year (see also the 2001-2002 class web page, for which I was the instructor). Students can also enroll at the same time in Caltech's 101ab, "Principles of Feedback and Control", which also attracted a wide audience (ranging from Aeronautics to Zoology). Students interested in obtaining a more thorough intrinsic background can also enroll in CDS 202, "Geometry of Nonlinear Systems", which includes manifolds and mappings, transversality, vectorfields and flows, distributions and Frobenius' theorem, matrix Lie groups and Lie algebras, exterior differential forms and Stokes' theorem.

The fall quarter has a $50\%-50\%$ mix of basic Particle Mechanics and Dynamical Systems. For the latter, I recommend sources such as Strogatz [87], Smale/Hirsch [84], Verhulst [90], Perko [80], Guckenheimer/Holmes [48], covering the basic tools and concepts of ODEs (both qualitative and quantitative, including computer simulations) up to, say, Lyapunov stability theory. Strogatz can be covered entirely in a quarter by a diligent student (it is wonderful when some real lab demonstrations proposed there are performed). For basic Particle Mechanics, there is still good old Sommerfeld [85]: terse, intelligent exercises, interesting for all publics - a masterpiece (written after about 50 years of teaching in Munich).

In the winter quarter, again a $50\%-50\%$ mix. One would cover more groundwork material of Dynamical Systems (starting, say, with Floquet theory and including an introduction to homoclinic phenomena and other chaos concepts and diagnostics); regarding Mechanics, we propose following a little more mathematically sophisticated avenue, using Arnold [6] or Marsden/Ratiu [71]. Basic concepts and notations about manifolds and Lie groups, and some invariant language can be introduced without difficulty taking the rigid body as a concrete realization. Using Papastravidis' book as a dictionary, the Mechanical Engineering student will not feel he has strayed away.

The spring quarter could have a more open agenda, according to the instructor's preferences, focusing on his/her personal research interests; mine have been divided between Geometric Mechanics applications such as geometric phases, nonholonomic and subriemannian systems, and biological motion.

Papastavridis' treatise, which we will review now, fits well for supplementary material and historical sources, especially regarding nonholonomic systems. But it can hardly be used as a textbook, due to the very large size (and price).

III. Papastavridis' book

Analytical Mechanics has a long and honorable tradition from the perspective of engineering and applied physics, which is aptly maintained by the treatise being reviewed.

A quick search in Amazon.com, under Analytic Mechanics (Mathematical Aspects) gave 522 results. MathSciNet, for the last 20 years, yielded 206 items under 70-01 (Mechanics of particles and systems/Instructional exposition: textbooks, tutorial papers, etc.), 114 items under 70-02 (Research exposition: monographs, survey articles, etc.), and 510 under 70-03 (Historical). The more general heading 71.0X (Mechanics) gave 1372 entries. Even eliminating surveys and keeping only books, doing a thorough comparison is overwhelming.

Nonetheless, I recommend without hesitation Prof. Papastravidis' treatise as a reference source to be acquired by every library of Mathematics, Physics, or Mechanical/Aeronautical/Electrical Engineering department. It is a *different* book, especially in our Internet era where instant satisfaction is often the primary (sometimes sole) goal of the student or researcher. Putting together 1392 (!!) pages of carefully prepared text and 172 figures (which then become somehow sparse) represents a major effort, to say the least.

The book is a sound scholarly effort, remarkably even in the historical account - tribute is given to all schools of thought: French, German, British, Russian, Italian and others. *Papastravidis' treatise gives an important contribution to the historical record, from the golden age of Euler and Lagrange until WWII.*

Having said that, it may be (perhaps) also fair to say that, for a mathematician, most themes and techniques presented seem a little old fashioned. This is not necessarily a bad thing: one feels the flavor of an Indiana Jones movie, a fascinating and thrilling plot with treasures to be dug out and turned into topics for current research.

In terms of notation, moderation is advocated even to classic style tensorial indices and matrices. "Things" like TQ, T^*Q, $dH = -\omega(X_H, \bullet)$ are out of the question. Papastravidis expresses strong views and openly takes sides on several (old and new) battles for priority, correctness, and relevance (more on that later in this review).

An impressive list of references runs from pages 1323 to 1370. But a finger could be pointed to a heresy: just one reference for Poincaré? This list could easily be doubled if more connections between theory and applied research were included. Papastravidis explains, in the preface, why he did not intend to emphasize applications: on p. xii he opens his heart regarding "computerization, applications, and ultimate goals of research."

There is actually a lot of Geometry, even though it is not written in modern differential geometric language. But precisely *because* it is not, Papastavridis' treatise can surely help in the study of the literature until WWII.

What is in the book? The main subject, as indicated clearly in the title, is the theoretical study of *constrained* mechanical systems. This means, roughly speaking, that there are relations among the velocities that must be satisfied at all times. Most often in practical applications, constraints are *linear* in the velocities, but Papastravidis also describes the literature on nonlinear constraints.

Chapter 1 covers basic undergraduate mechanics from a mature viewpoint, and Chapter 2 gives a classification of the main types of constraints encountered in Engineering. A nice discussion about Frobenius' theorem on distributions is given, and it is interesting for a mathematician to see it from a different standpoint.

Chapter 3 gives the core material of the book: Lagrange-D'Alembert's principle and some variants (connected to control, section 3.17) are discussed in detail. Chapter 4 deals with impulsive motion and Chapter 5 with nonlinear contraints.

In Chapters 6 and 7, Papastravidis follows the historical path to describe the roots and connections between several of the *differential and integral variational principles* of Mechanics. I think these chapters are quite valuable, both for the instructor and for the researcher.

The last chapter, Chapter 8, is an introduction to the Hamiltonian formalism. In my view it is "squarely and unabashedly traditional" (in Papastravidis' own words) but very useful for checking the original sources and learning about some of the priority wars.

The presentation stops short of the mathematical developments from the last 30 years, many of which have percolated to wonderful applications. Important themes such as reduction and momentum maps [97], and new methods for integrable systems are left for supplementary references.

What is not in the book? In fact, Papastravidis makes a caveat in the Preface: "Since this is not an encyclopaedia of theoretical and applied dynamics, an inescapable and necessary selection has operated, and so, the following important topics are *not* covered: applications of differential forms/exterior calculus (of Cartan, Galissot et al.) and symplectic geometry to Lagrangian and Hamiltonian mechanics; group theoretical applications; stability of motion (nonlinear theory) and theory of orbits; and computational/numerical techniques. For all these, there already exists an enormous and competent literature."

This is indeed a very honest disclaimer. Is the author diminishing the merits of his own work? After all, these *mainstream* themes and directions have been left out; some researchers/instructors could get the impression that the baby was thrown out with the bath water.

We think that the author was wise to confine himself to a focused range of themes and to discuss them carefully.

The treatise is primarily geared to those interested in applications to multibody dynamics and robotics, so Papastavridis gives a short but lucid list of some other areas associated to Analytical Mechanics (p. 8): "in addition to ... the classical tradition of Whittacker, Hamel, Luré, Pars, Gantmacher et al. followed here ... the following *complementary* formulations of Classical Mechanics also exist:[2]

Variational, Vakonomic, Algebraic, Nonlinear Dynamics,
Geometrical, Statistical, Celestial

All these, and other, formulations testify once more to the vitality and importance of CM for the entire natural science, even today" (italics added by the reviewer).

We agree entirely with this appraisal. Mechanics is alive and well, and since it is not a mausoleum nor a pantheon, there is a permanent fight for "hearts and minds", as we now discuss.

There's Something About Mary, or who will marry Mechanics? Papastravidis classifies contemporary expositions on advanced dynamics in three very distinct groups:

i) formalistic/abstract, ii) applied, and iii) mainstream/traditionalist.

One could suggest moving the world "mainstream" to the other groups, as we did in the previous section, but let us take this classification at face value. Mechanics would be like the beautiful woman (Cameron Diaz) of a popular movie, with whom all the men fall in love. The author places himself "squarely and unabashedly" in the last group and states his mission as: "... to help counter the very real and disturbing trend, brought about by the proponents of the first two groups, toward a dynamical tower of Babel." This is a challenge! Can a Mechanics lover from the second group make an effort, of comparable breath, to rival Prof. Papastravidis' impressive work? Time is also ripe for "our" side to do it too!

It is curious that a fierce debate within a culture may pass unnoticed in others. From the mathematician's perspective, Poincaré's generalization of Euler-Lagrange equations ([81], 1901) suffices to resolve many controversies (which still persist) within the engineering community about quasi-velocities. For instance, Newton's equations in "quasi-coordinates" (known also as Kane's equations) are of great importance for multibody systems and robotics, and spurred a long quarrel among mechanical engineers (see pp. 713-717). To a mathematician, these disputes may look somewhat curious if not a little bizarre. The whole business would be interpreted by a geometric mechanician (see [58]) as an interesting application of Cartan's moving frames [21].

On mathematical theories, or abstract nonsense. Contributions of mathematicians were lumped generally speaking inside the first category. Papastavridis states (Preface, p. x): "*Formalistic/Abstract*, of the by-and-for-mathematicians variety, and, as such, of next to zero relevance and/of usefulness to most nonmathematicians. Considering the high mental effort and time that must be expended toward their mastery vis-à-vis their meager results in understanding mechanics better and/or solving new and nontrivial problems, these works represent a pretty poor investment of ever scarce intellectual resources; that is, they are not worth their 'money'... I categorically reject soothing apologies of the type 'Oh, well, that is a book for mathematicians;' that is, the book has little or no consideration for ordinary folk."

[2]Papastravidis gives an excellent list of main sources for these branches; however, noteworthy is the absence of *Fluid, Geophysical, ...* and other areas of applied dynamics.

As Mark Twain would say, this opinion is perhaps a little exaggerated; we mathematicians should not resent it[3] but, rather, recognize that there may be some reason in it. History shows that solid mathematical work will, sooner rather than later, bring about important applications. Therefore we mathematicians should try harder, making a stronger outreach effort to the other sciences.

On applications, or concrete nonsense. I wish to make a slightly dissenting comment though about the "ultimate goal of research" that Papastravidis alludes to in the last section of his preface. Many examples from history of science demonstrate the inherent BUT healthy tension between the "Platonic" point of view vs. the "Archimedian" one. Mathematicians (especially those living in developing countries) feel this tension frequently. Gauss relaxed from his work as a geodesist by creating differential geometry. On the other hand, he should have felt wonderful every time his theoretical work connected with applications that had the potential of increasing mankind's well-being.

IV. Some new and old opportunities in applications

Geometry organizes the Analysis of Mechanical Systems

There is a good way to maintain the very rich genetic heritage of Mechanics. We give a few examples of recent research, in which there is contact between different "tribes and cultures". But first, the usual disclaimer: as it is impossible to touch all areas of applications, a personal choice of themes was made. Even in a single area, I will certainly forget many relevant works. I apologize in advance for the omissions.

D'Alembert's principle. We (mathematicians) regard it as so obvious that we tend to ignore its importance for theory and practical examples as well. Sommerfeld [85] emphasized that d'Alembert's principle should be taken as *THE* foundation of the mechanics of constrained systems.

Mathematicians can take advantage of the fact that *the principle of virtual work matches very nicely with sub-differential analysis.* For impulsive motion, (Dirac)-distributions also come into the picture. On the other hand, engineers and physicists should take note that *virtual work is simply the natural duality between cotangent and tangent bundles.*[4] Vectors belong to the latter and forces to the former. As Prof. S. S. Chern teaches, vectors are masculine and forms, much more sophisticated, are of feminine gender (I was fortunate to be in his elementary differential geometry class in 1971).

For an example of beautiful work, using the principle of virtual work together with sub-differentials, see M. Frémond [42], [41] and references therein. Frémond's work (exemplifying others) does not appear in Papastavridis' list of references; he would be classified most likely as a member of the "second group" of *Mechaniciéns.*

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[3]Even if it is the party line of the Theoretical and Applied Mechanics community (let's say the American Society of Mechanical Engineers), but we hope not.

[4]Prof. Laurant Schwartz used to say that life for applied people would be much easier if they could use dual spaces. We learned this quote from Elon Lima, who attended his *Cours d'Analyse* in Paris.

Cartan at NASA. A wonderful viewpoint that apparently has been going unnoticed, even by experts, is "Cartan's mechanics" ([17], 1928). Consider a mechanical system on a configuration space Q^n with kinetic energy T and subject to constraints (holonomic or nonholonomic) defined by a distribution E of dimension $m \leq n$ and an external force F (written in contravariant form, so that $F \in TQ$, instead of T^*Q). D'Alembert's principle translates into the equation $D_{\dot{c}}\dot{c} = F^{\|}$ where the right hand side is the orthogonal projection of F over E, and D is Cartan's "projected connection" (think of the Levi-Civita connection, but with E instead of a submanifold). The trajectory to be found in Q is $c(t)$.

By parallel transporting backwards we would get the "hodograph of c", a curve γ on fixed euclidian space $E_{q_o} \equiv \Re^m$. Taking a *constant* orthonormal frame $f_1, \cdots, f_m$ on $\Re^m$ we could write $\gamma(t) = \sum_{i=1}^m \gamma_i(t) f_i$. Let e_i be the parallel transported frame along $c(t)$ and decompose $F^{\|} = \sum_{i=1}^m f_i e_i$. Here's the punch line:

The hodograph curve obeys Newton's law $\ddot{\gamma}_i(t) = f_i(c(t))$ $(1 \leq i \leq m)$.

Hence, if one couples this equation with those of parallel transport, the whole dynamics can be depicted on a "computer screen at Houston" as a curve $\gamma(t)$. Then $c(t)$ can be constructed in configuration space Q by parallel transporting forward. We observe that F can represent *control forces* actuating over the system.

Very recently K. Ehlers [36] made good progress towards a classification programme for nonholonomic systems using Cartan's equivalence method [18], [43].

Discrete mechanics. This is an example of cross fertilization between the mathematical side and the applied side. It is an impressive work on computational algorithms which arose from *discrete variational principles*, being developed as a collaboration between engineers (Michael Ortiz and students) and mathematicians (Jerrold Marsden and students) at Caltech; see [72], [54]. We should pay tribute to Juan Simo, whose untimely death a few years ago saddened both the Engineering and Mathematical communities. Juan pioneered the use of geometrical approaches for computational codes.

Making a long story short or, rather, straight: nonholonomic systems. It has been a dogma in Theoretical Physics that every fundamental phenomenon has a Lagrangian functional, and one must find its stationary curves. Everything should have a global variational principle.

Leibnitz claimed that "ours is the best of all possible worlds." Voltaire at that time (and engineers now) knew better. On Earth, one *does not* live (unfortunately) in the best of all worlds.

The best one can do is follow the middle road, advocated by Confucius: the straightest path (which is not necessarily the shortest). This is Hertz' "principle of least curvature", which among several foundations for Mechanics (Papastavridis, Chapters 6 and 7) is my favorite one.

Linear homogeneous nonholonomic constraints are easily defined geometrically, as a nonintegrable subbundle of the tangent bundle.[5] Hertz was the first one to

[5]The usual terminology *holonomic* vs. *nonholonomic* actually is due to Hertz. Alan Weinstein introduced me to Hertz's book, *Foundations of Mechanics*, back in 1982. When I told him, in Berkeley's coffee room, that I was interested in studying symmetries of nonholonomic systems, Alan immediately replied, "Well, then take a connection on a principal bundle." This was the beginning of my most cited research [55].

notice that *in general*, D'Alembert's principle (which is a local, differential procedure) is not equivalent to any integral (i.e., global, calculus of variations) procedure. In this sense, Euler-Lagrange or Hamilton's principle are just wonderful shortcuts, true only for holonomic systems as a mysterious miracle.

A prototype example of a nonholonomic system is the rattleback (known also as the Celtic stone), which the unaware reader is encouraged to watch. A basic reference, updated until the 60's, is Neimark and Fufaev [77]. This tradition is still going strong in the former Soviet Union, centered around PMM (*J. Applied Math. Mech.*) and in China. In the Western Hemisphere Engineering community, it is centered around ASME's *J. Appl. Mech.* All these sources are very well referenced in Papastavridis.

The realization of nonholonomic constraints as limit processes was discussed mathematically by Kozlov awhile ago [60]. We outline, very briefly, some recent work. Ruina [82] has demonstrated how it may happen, both computationally and in the laboratory, as a result of dissipative collisions. Stability properties of nonholonomic systems (which differ from those of holonomic systems) have been studied by Zenkov et al. [99], among others. *Control* of nonholonomic systems is an important issue for robotics, and one representative work is [79]. A very nice toy model (literally) is Caltech's snakeboard [70].

The *mathematical structure* of genuinely nonholonomic systems deserves attention for its own sake. However, one should not risk being taken as a crackpot, believing that fundamental physical forces can be modeled by nonholonomic systems. The truth is that its mathematical theory still lags a long way behind the superb structure created for holonomic (Hamiltonian) systems. Recently, a concentrated effort has been attempted to understand *nonholonomic systems from the geometric viewpoint*, and the subject is rapidly evolving.

Because we would make serious omissions, we just indicate a collection of papers which appeared in *Reports of Mathematical Physics* [32] and an excellent recent book by Monforte [73]. In the West researchers working in the area include L. Bates, A. Bloch, R. Cushman, W. S. Koon, P. S. Krishnaprasad, M. de Leon, A. D. Lewis, C. Marle, J. E. Marsden, B. M. Mashke, A. van der Schaft, D. Schneider, J. S̀niatyki, D. Zenkov, and their coworkers.

To finish this part we present a small list of works, related in fact to the last section of this review. *The programme is to study nonholonomic systems via a (pseudo)-Poisson bracket.* Although the Jacobi identity fails, [53], [89], [22], [57], hopefully a very nice geometry will eventually unravel.

V. Mathematical developments in Poisson geometry

Poisson geometry (see e.g. [95], [88]) is the geometry of manifolds equipped with a Poisson bracket. Basic examples of spaces carrying canonical Poisson brackets are phase-spaces in mechanics, as well as their quotients by groups of symmetry.

What is a Poisson manifold? A Poisson algebra A is a commutative associative algebra together with a Lie bracket $\{\ ,\ \}$ for which each adjoint operator $X_h = \{\ , h\}$ is a derivation. When A is the algebra of smooth functions on a manifold M, we say that M is a Poisson manifold and $\{\ ,\ \}$ is a Poisson bracket. In this case, X_h is called the Hamiltonian vector field corresponding to h. It is simple to check that a linear Poisson structure on a vector space is equivalent to a Lie algebra structure on the dual space.

Poisson manifolds occur as phase-spaces for classical particles (or, in the infinite-dimensional case, fields). However, Poisson geometry is also relevant to Quantum Mechanics, playing an important role in "quantization" problems, as briefly discussed below.

On a Poisson manifold M, the collection of Hamiltonian vector fields X_h forms an integrable distribution with varying rank. Each leaf of the corresponding singular foliation carries a natural symplectic form, so Poisson manifolds are foliated by symplectic manifolds of varying dimensions. For duals of Lie algebras, the symplectic leaves coincides with the coadjoint orbits.

Recent history and progress. The modern period of Poisson geometry started with the study of local geometric properties of Poisson manifolds (see [64], [92]). More recently, Poisson geometry has expanded in many directions, including such topics as

- singularities and linearization of Poisson structures [23], [24], [100];
- equivariant Poisson geometry: Poisson-Lie groups and Poisson actions, reduction of Hamiltonian systems and their generalizations, group-valued moment maps [3], [4], [5], [78], [66];
- Poisson invariants, such as Poisson cohomology and homology [45], [51], [98];
- integrability: symplectic realizations and symplectic groupoids [26], [28]

to mention a few. In the last 5 years, much progress has been achieved in the understanding of global properties of Poisson manifolds. This progress is illustrated by the discovery of new Poisson invariants (such as modular classes, secondary characteristic classes, and Poisson K-groups [27], [38], [39], [46], [94]) and by new approaches to the study of the topology and geometry of the symplectic foliations (such as the development of the idea of Poisson holonomy and applications of modern ideas of symplectic topology [9], [39]).

On top of the development of new tools to study the geometry of Poisson manifolds, Poisson geometry has also had interactions with other fields of mathematics and mathematical physics where significant progress has been achieved. We comment on a few of them below.

Poisson geometry and quantization. In physical terms, the process of *quantization* describes the transition from Classical to Quantum Mechanics. Mathematically, quantization connects Poisson geometry to the world of noncommutative algebras and to the geometry of "noncommutative spaces" [25].

A powerful way to produce noncommutative algebras out of a Poisson manifold M is by associative deformations [44] of the commutative algebra $C^\infty(M)$, a process known as deformation quantization [7]. In this context, Poisson structures can be seen as "first-order approximations" to noncommutative algebras (or "spaces") [16], motivating the idea that the "Poisson category" should occupy an intermediate place between ordinary differential geometry and noncommutative geometry. In fact, this point of view has proven very fruitful, and several important results suggest that the Poisson geometry of a manifold M is intimately related to the noncommutative geometry of the "quantum" algebras associated to it. This powerful principle is illustrated by the connections between quantum groups [33], [34] and their semi-classical limits, Poisson-Lie groups. Another example is Kontsevich's proof of the formality conjecture [59], which solves the long-standing problem of existence and classification of deformation quantizations of arbitrary Poisson manifolds and has impact on many other areas of mathematics.

Lie algebroids, groupoids and sigma models. Another field of research that is closely related to Poisson geometry is that of Lie algebroids [16], [61]. The corresponding global objects, Lie groupoids, are a very rich source of noncommutative algebras [25], providing another link between Poisson and noncommutative geometries [93].

A groupoid is a small category whose morphisms are all isomorphisms, and a Lie groupoid is a groupoid with a smooth structure compatible with its algebraic structure. The corresponding infinitesimal objects are called Lie algebroids. These concepts extend those of Lie groups and Lie algebras and play a prominent role in the study of nonhomogeneous geometric structures, such as a Lie group action, a foliation or a Poisson manifold.

The classical Lie's third theorem, establishing the correspondence of Lie groups and Lie algebras, does not extend to the realm of Lie algebroids and groupoids, and a difficult problem that remained open for many years was to understand the precise obstructions. A solution to this general problem, found in [29], was inspired by techniques used in the construction of symplectic groupoids for Poisson manifolds [20]. This construction involves Poisson-sigma models [83], which are recognized as instrumental in gauge field theory and have strong connections with Kontsevich's formality theorem [19].

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