# BOOK REVIEWS

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*The concise handbook of algebra*, Alexander V. Mikhalev and Günter F. Pilz (Editors), Kluwer, Dordrecht, 2002, xvi + 618 pp., \$116.00, ISBN 0-7923-7072-4

Although it would be impossible to give here a substantial history of algebra, we would like to start with some brief (if rambling) historical comments, in particular about the interplay between algebra and geometry over the centuries, which continues today.

The word "algebra" comes from the Arabic "al-jabr", and the study of algebra as a mathematical discipline is generally attributed to the Arab world in the 9th century A.D. There were earlier activities, e.g., in ancient Egypt, Babylonia, India, China, and in classical Greece and Rome. Much of this earlier activity was discussed in ordinary language without symbols, but some of the problems were discussed in geometric terms with diagrams. The Greeks essentially translated algebraic equations into geometric problems and solved them geometrically. Thus the interplay between algebra and geometry has its roots in antiquity. Later Diophantus (3rd century) treated several equations in several variables more algebraically, and Hindu writers, e.g., Brahmagupta (7th century) solved linear and quadratic equations in an algebraic manner. The 9th century Arabs treated linear and quadratic equations written in words with only positive coefficients. Solutions were obtained by manipulating geometric squares. The methods were applied to problems of daily life such as inheritance laws and surveying. Although they did treat cubic equations by the 11th century, they did not succeed in developing an algorithm for their solution. This was done in the 16th century in Italy, as was the quartic equation. Failure to solve the quintic equation led to Lagrange's theory of equations in the 18th century, which brought out the idea of permutations of the roots and early notions about groups. The idea of a field started to emerge in the attempt to prove the unsolvability of the general quintic, culminating in Galois' work in the the 19th century about groups and fields. The ties of algebra to geometry then reappeared with invariant theory, and with topology via the fundamental group and with the Lie theory of continuous groups.

Backing up to the 17th century, Fermat stated results about quadratic forms, leading others, e.g., Euler and Lagrange, to introduce algebraic integers, and Kummer in the 19th century to consider ideal numbers. Gauss' composition of forms brought abelian groups into number theory. The axiomatic structure approach started in the later 19th century with Kronecker and Frobenius. Another direction

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from Gauss went earlier in the 19th century through Dirichlet to Weber, leading to class field theory. All this culminated in Dedekind's general theory of algebraic numbers and in Dedekind and Weber's development of algebraic functions, reconnecting with geometry via Riemann surfaces. Hilbert also participated in this, e.g., his Zahlbericht of 1897, and encouraged a very general set-theoretic and axiomatic approach. In the 20th century we mention Hensel's p-adic numbers and Steinitz' presentation of the theory of fields.

The notion of abstract algebra that we often think of today is usually attributed to Emmy Noether and Emil Artin. Kummer's ideal numbers (never clearly understood) led to the definition of an ideal in a ring. Noncommutative algebra methods got applied to number theory by Noether, Brauer and Hasse, e.g., central simple algebras and local-global principles, which related to geometry through cohomology. The abstract algebra approach was popularized in the book *Moderne Algebra* by van der Waerden in 1930. Noether also noted that Betti and torsion numbers of manifolds were better discussed by a Betti group. This led to a development of algebraic topology by algebraic methods.

Some of the preceding remarks indicate the interplay between algebra and geometry over the centuries. We should also mention Descartes' classification of curves and "permissible" geometry in the 17th century. The idea of mathematics as a collection of subdisciplines can be found in 5th century B.C. Greece with the Pythagoreans and in 5th century Rome with Pappus and Proclus. The question of whether algebra serves geometry or vice-versa has been a continuous issue. The late 19th century and early 20th century Italian school of algebraic geometry concentrated on specific constructions, and the general methods used are sometimes said not to be completely rigorous. Zariski and Weil put algebraic geometry on a firmer foundation, e.g., Zariski's arithmetization via ideals and algebraic function fields. Thus the stress on abstract algebra ideas starting with Emmy Noether also had its effect on algebraic geometry as it did on number theory. The group cohomology which arose in both settings (and in topology) led to the idea of a functor and to homological algebra (exposed in the book of Cartan and Eilenberg) and category theory. Thus algebraic topology invaded the domain of abstract algebra, somewhat reversing a previous invasion. Then Grothendieck "rewrote" Cartan and Eilenberg via categories of sheaves, unifying the concepts of abelian categories and spectral sequences. Grothendieck's theory of schemes revolutionized the study of algebraic geometry as well as other areas of algebra, e.g., the prime spectrum of a commutative ring. It may be ironic that today the Italian school's emphasis on specific examples has somewhat returned.

Besides the interplay of algebra and geometry, there is also the relation of algebra and analysis. We only mention here the 19th century (mostly British) manipulation of mathematical symbols (for $n^{\text{th}}$ derivatives and $n^{\text{th}}$ finite differences) to simplify analytic arguments, and algebraic elimination modeled on systems of solutions of linear differential equations. The legitimacy of the methods was not considered as important as its applications. An algebraically based development of calculus was done by Lagrange. In this direction of algebra as notation, there was a related direction (mostly in France in 1780-1815) of algebra as language, grammar (as used in analytic arguments) and logic (as a general form of ideas). This movement was more philosophical than mathematical. Monge discussed the language of descriptive geometry and tried to apply it to the language of machines. This was picked up by Babbage in connection with solving functional equations.

Before turning to the book under review, we wish to mention two other items. One is what is sometimes thought of as an attempt by Bourbaki to unify mathematics, which included abstract algebra. However important and useful this may have been, it did not effectively give a general structure in which all algebraic results would be put. Many results were repeated and re-proved in each context (group, ring, vector space, etc.). So in this sense it was not really unifying. The other item is the now mostly forgotten attempt of Ore to unify algebra by lattice theory. The idea of a lattice is implicit in Dedekind's work, and Ore was influenced by Noether's ideas about abstraction in algebras, e.g., ideas about chain conditions. Although Ore's program did not catch on (one factor may have been the advent of World War II and the turning of interests elsewhere), his ideas of studying algebraic structures by their lattices of substructures did have an important influence in group theory and gave rise to important developments in universal algebra, model theory and Boolean algebra. We will not conclude these historical remarks with remarks about more recent algebraic developments, but will mention some of these in connection with the book under review.

The book compiled by Mikhalev and Pilz consists of 146 short articles written by leading algebraists in these areas. The Preface indicates that the book is aimed at anyone interested in algebra, from graduate students to established researchers, including those who want to obtain a quick overview of or a better understanding of the selected topics. The prerequisites would be standard textbooks on higher algebra. The book would be a preparation to read the series in progress *Handbook of Algebra* edited by Hazewinkel, which was started in 1995 by Elsevier. The latter *Handbook of Algebra* consists of long articles going into considerable depth in each topic.

As might be expected of such a large collection as the book under review, the nature of the articles is uneven, although an attempt was made to use common notation and some articles refer to other ones. Some articles have no references, and some settle for names attached to main concepts and theorems. Many do give general references but not identification of where one can find specific results. So the extensive list of references is of somewhat limited value, mainly to direct the reader interested in pursuing a particular topic to a general source.

Many of the articles are written in a rather cursory manner – a short sequence of definitions and main theorems. Most tend to be four or five pages. Some give the flavor of a topic, but don't really explain its basic language. So it is difficult to judge whom this book would benefit. An expert in a certain area will know the material presented. A complete novice, even if having had a basic introduction to algebra, is not likely to get much out of an attempt to learn a particular topic, except perhaps to be guided to a general source. Personally, I felt that my main use of the book would be to satisfy my curiosity about certain terms and areas in algebra about which I had heard but did not have a clear idea as to what they were about. So to this extent, I feel that the book is of some general, but limited, value.

We list the headings of the nine sections, A – I, and the number of articles in each section in parentheses:

| | | |
|---|---|---|
| A | (17) | Semigroups |
| B | (23) | Groups |
| C | (55) | Rings, Modules, Algebras |
| D | (7) | Fields |

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|---|---|---|
| E | (10) | Representation Theory |
| F | (10) | Lattices |
| G | (10) | Universal Algebra |
| H | (9) | Homological Algebra |
| I | (5) | Miscellaneous |

Clearly there is some emphasis on Section C, and one might have expected more in Section D. The distribution was probably affected by the willingness of the authors who contributed to do so. Of course the choice of topics is determined by the editors and their success in getting authors to write articles. One can always quibble about topics omitted or barely mentioned. Adding such could lead to a book of overly large size. With this caveat, we mention that there are no specific articles in number theory or algebraic geometry. There are no articles on vertex operator algebras, quantum groups or category theory. Also there is little or no indication of certain areas in which algebra has been applied, e.g., biology, computer science, mathematical physics, combinatorics and topology. There is an occasional mention of a few of these inside an article on a relevant topic.

The book is attractively packaged on a paper of substantial quality. The writing and editing are generally quite good. Since some of the articles are by Russian authors, there are occasional omissions of "a" and "the", which causes no problem with understanding. Section I.1 ends with a promise to take up several topics in the next section, but I.2 does not mention these topics. But such lapses are few.

The reviewer had the opportunity to attend a portion of the Workshop on the History of Algebra in the 19th and 20th Centuries held at the Mathematical Sciences Research Institute in April 2003. I thank the participants for their assistance, in particular the organizer, Karen Parshall, and also Leo Corry for making available his book *Modern Algebra and the Rise of Mathematical Structures* (Birkhäuser Verlag, 1996) and for his comments. Corry's book is along more philosophical lines than the book under review, but is a good source on the development of algebra since the 18th century. We also thank Michiel Hazewinkel. His *Handbook*(s) mentioned earlier is more suited to learning a particular area in depth. The only other comparison which might be made for the Mikhalev-Pilz volume is to encyclopedias. The latter would generally have shorter descriptions of each concept or topic, although it might be easier to pinpoint a particular item there. So the book under review would be preferable to encyclopedias for getting ideas about trends in algebra and some of its specific subdisciplines.

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