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RIEMANN'S ZETA FUNCTION AND BEYOND

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Dedicated to Ilya Piatetski-Shapiro, with admiration

Abstract. In recent years L-functions and their analytic properties have assumed a central role in number theory and automorphic forms. In this expository article, we describe the two major methods for proving the analytic continuation and functional equations of L-functions: the method of integral representations, and the method of Fourier expansions of Eisenstein series. Special attention is paid to technical properties, such as boundedness in vertical strips; these are essential in applying the converse theorem, a powerful tool that uses analytic properties of L-functions to establish cases of Langlands functoriality conjectures. We conclude by describing striking recent results which rest upon the analytic properties of L-functions.

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1. Introduction

In 1859 Riemann published his only paper[1] in number theory, a short ten-page note which dramatically introduced the use of complex analysis into the subject. Riemann's main goal was to outline the eventual proof of the Prime Number Theorem

$$\pi(x) \;=\; \#\{\text{primes } p \leq x\} \;\sim\; \frac{x}{\log x}\,, \qquad x \to \infty\,,$$

$$i.e. \qquad \lim_{x\to\infty} \pi(x)\,\frac{\log x}{x} \;=\; 1\,,$$

by counting the primes using complex integration (the proof was completed half a century later by Hadamard and de la Vallee Poussin). Along this path he first shows that his ζ-function, initially defined in the half-plane Re $(s) > 1$ by

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p\ (\text{prime})} \frac{1}{1-\frac{1}{p^s}}\,,$$

has a meromorphic continuation to $\mathbb{C}$. Secondly, he proposes what has remained as perhaps the most-famous unsolved problem of our day:

The Riemann Hypothesis: $\zeta(s) \neq 0$ *for* $\operatorname{Re} s > 1/2$.

For more on the history of ζ and Riemann's work, the reader may consult [16], [30], [36], [180]. Our role here is not so much to focus on the *zeroes* of $\zeta(s)$, but in some sense rather on its *poles*. In particular, our emphasis will be on explaining how we know that $\zeta(s)$ extends meromorphically to the entire complex plane and satisfies the functional equation

$$\xi(s) \;:=\; \pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) \;=\; \xi(1-s)\,.$$

It is one purpose of this paper to give two separate treatments of this assertion. We want also to characterize the ζ-function as satisfying the following three classical properties (which are simpler to state in terms of $\xi(s)$, the *completed* ζ-function).

- **E**ntirety (**E**): $\xi(s)$ has a meromorphic continuation to the entire complex plane, with simple poles at $s = 0$ and 1.
- **F**unctional **E**quation (**FE**): $\xi(s) \;=\; \xi(1-s)$.

[1]See [145], and [36], for translations.

- **B**oundedness in **V**ertical strips (**BV**): $\xi(s) + \frac{1}{s} + \frac{1}{1-s}$ is bounded in any strip of the form $-\infty < a < \text{Re}\,(s) < b < \infty$ (i.e. $\xi(s)$ is bounded in vertical strips away from its two poles).

A second purpose is to overview how these treatments and properties extend to L-functions assigned to more general groups such as $GL(n)$, the group of invertible $n \times n$ matrices (the function $\zeta(s)$ is attached, we shall see, to $GL(1)$). A major motivating factor for studying these analytic conditions (especially the technical **BV**) is that they have become crucial in applications to the Langlands Functoriality Conjectures, where they are precisely needed in the "Converse Theorem", which relates L-functions to automorphic forms (see Theorems 3.1, 3.2, and Section 7.3). More to the point, the study and usefulness of L-functions has pervaded many branches of number theory, wherein complex analysis has become an unexpectedly powerful tool. In Section 9 we discuss the connections with some of the most dramatic recent developments, including the modularity of elliptic curves, progress towards the Ramanujan conjectures, and the results of Kim and Shahidi. The two treatments we describe are, in fact, the major methods used for deriving the analytic properties of L-functions.

The Two Methods. A first method (Section 2) of analytic continuation is Riemann's, initiated in 1859. In fact, it was one of several different, though similar, proofs known to Riemann; Hamburger, and later Hecke, moved the theory along remarkably following this line of attack. Almost a century later, Tate (Section 6) recast this method in the modern language of adeles in his celebrated 1950 Ph.D. thesis [174], another famous and important treatment of ζ-functions. The second – and lesser known – method is via Selberg's "constant term" in the theory of Eisenstein series (Section 4.1). This theory, too, has an important expansion: the Langlands-Shahidi method (Section 8).

As we shall see, both methods take advantage of various (and sometimes hidden) group structures related to the ζ-function. They also suggest a wide generalization of the methods: first to handle Dirichlet L-functions, ζ-functions of number fields, and then quite general L-functions on a wide variety of groups. In Section 3 we begin by explaining this through the connection between modular forms and L-functions. In fact, this nexus has been fundamentally important in resolving many classical problems in number theory. After surveying the classical theory of Hecke, we turn to the modern innovations of Langlands. We have in mind ideas of Weil, Langlands, Jacquet, Godement, Piatetski-Shapiro, Shalika, Shahidi, and others. For broader and deeper recent reports on the nature of L-functions and the application of their analytic properties, see, for example, [70], [154].

The reader will notice that we have left out many important properties of the Riemann ζ-function, some related to the most famous question of all, the Riemann Hypothesis, which can be naturally restated in terms of ξ as

$$\text{All } \rho \text{ such that } \xi(\rho) = 0 \text{ have Re } \rho = 1/2.$$

This is because we are primarily interested in results related to the three properties **E, BV**, and **FE**. As indicated above, we are also following the development of only a few approaches (see [176] for many more, though which mainly follow Riemann). Also, to lessen the burden on the reader unfamiliar with adeles, we will more or less describe the historical development in chronological order, first treating the classical results of Riemann, Hecke, Selberg, and Weil before their respective generalizations to adele groups.

To wit, the paper is organized in three parts. The first, Sections 2, 3, and 4, gives the background on the classical theory: Section 2 discusses Riemann's theory of the ζ-function and its analytic properties; Section 3 focuses on Hecke's theory of modular forms and L-functions; and Section 4 centers on Selberg's theory of non-holomorphic Eisenstein series. The second part of the paper redescribes these topics in more modern, adelic terms. Section 5 leads off with a short introduction to the adeles. Sections 6, 7, and 8 then give a parallel discussion of the respective topics of Sections 2, 3, and 4, but in a much more general context. Finally, the last part of the paper is Section 9, where we recount some recent results and applications of the analytic properties of L-functions. Sections 8 and 9 are quite linked, in that many of the recent developments and analytic properties used in Section 9 come from the Langlands-Shahidi method, the topic of Section 8. However, the latter is quite technical, and we have made an attempt to make Section 9 nonetheless accessible without it.

A word is in order about what we *don't* cover. Because our theme is the analytic properties of L-functions, we have left out a couple of important and timely topics that lie somewhat outside our focus. Chief among these are some developments towards the Langlands conjectures, for example the work of Lafforgue [92] over function fields. This is mainly because the analytic properties of L-functions in the function field setting were long ago established by Grothendieck (see [79]) and are of a significantly different nature. Some resources to learn more about these additional topics include [1], [2], [4], [37], [42], [88], [102], [103], [105], [106], [146].

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2. Riemann's integral representation (1859)

As we mentioned in the introduction, Riemann wrote only a single, ten-page paper in number theory [145]. In it he not only initiated the study of $\zeta(s)$ as a function of a complex variable, but also introduced the Riemann Hypothesis and outlined the eventual proof of the Prime Number Theorem! At the core of Riemann's paper is the *Poisson summation formula*

$$\sum_{n\,\in\,\mathbb{Z}} f(n) \;=\; \sum_{n\,\in\,\mathbb{Z}} \widehat{f}(n)\ , \tag{2.1}$$

which relates the sum over the integers of a function f and its Fourier transform

$$\widehat{f}(r) \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\,\pi\, i\, r\, x}\, dx\,. \tag{2.2}$$

The Poisson summation formula is valid for functions f with suitable regularity properties, such as Schwartz functions: smooth functions which, along with all their derivatives, decay faster than any power of $\frac{1}{|x|}$ as $|x| \to \infty$. However, by temporarily neglecting such details, one can in fact quickly see why the Poisson summation formula implies the functional equation for $\zeta(s)$, at least on a formal level. Indeed, let $f(x) = |x|^{-s}$, so that

$$\widehat{f}(r) := \int_{\mathbb{R}} |x|^{-s} e^{-2\pi i r x} dx = |r|^{s-1} G(s), \tag{2.3}$$

where

$$G(s) = \int_{\mathbb{R}} |x|^{-s} e^{-2\pi i x} dx. \tag{2.4}$$

Using the convention that $|0|^s = 0$, we can already see from the Poisson Summation Formula that

$$2\zeta(s) = 2G(s)\zeta(1-s), \tag{2.5}$$

a functional equation relating $\zeta(s)$ to $\zeta(1-s)$. In fact the integral (2.4) is a variant of the classical Γ-integral

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \qquad \mathrm{Re}\ s > 0 \tag{2.6}$$

and can be shown to equal

$$G(s) = \frac{\pi^{(s-1)/2}\,\Gamma(\frac{1-s}{2})}{\pi^{-s/2}\,\Gamma(\frac{s}{2})}, \tag{2.7}$$

at least in the range $0 < \mathrm{Re}\ s < 1$ (see [51] or [30, p. 73]). Thus the functional equation (2.5) is formally identical to Riemann's functional equation

$$\xi(s) = \pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s) = \xi(1-s). \tag{2.8}$$

Of course, neither sum defining $\zeta(s)$ in (2.5) converges when the other does, much less in the range $0 < \mathrm{Re}\ s < 1$ where we computed $G(s)$. Indeed, the functional equation cannot be proven in the absence of some form of analytic continuation beyond the region where $\sum_{n=1}^\infty n^{-s}$ converges. The argument sketched here for the functional equation seems to have been first considered by Eisenstein, who succeeded in proving the functional equation not for $\zeta(s)$ itself but for a closely related Dirichlet L-function (for these, see (2.12) and [30]). André Weil has written historical accounts [179], [180] which suggest that Riemann was himself motivated by Eisenstein's papers to analyze $\zeta(s)$ by Poisson summation. The rigorous details omitted from the above formal summation argument can be found in [117, §5].

2.1. **Mellin Transforms of Theta Functions.** Riemann's own, rigorous argument proceeds by applying the Poisson summation formula (2.1) to the Gaussian $f(x) = e^{-\pi x^2 t}$, $t > 0$, whose Fourier transform is

$$\widehat{f}(r) = \frac{1}{\sqrt{t}}\, e^{-\pi r^2/t}.$$

The Gaussian is a Schwartz function and can be legitimately inserted in the Poisson summation formula. Its specific choice is not absolutely essential, but rather a matter of convenience, as we will see in Section 6. However, it was an inspired selection by Riemann, in that it is connected to the theory of modular forms (see Section 3.1). Thus Riemann's contribution to the functional equation went far beyond simply making a formal argument rigorous – it launched the link between modular forms and L-functions that remains at the forefront of much mathematical activity a century and a half later.

By applying Poisson summation to $f(x) = e^{-\pi x^2 t}$, one thus obtains Jacobi's transformation identity

$$\theta(i\,t) \quad = \quad \frac{1}{\sqrt{t}}\,\theta(\tfrac{i}{t})\,, \tag{2.9}$$

where

$$\theta(\tau) \quad = \quad \frac{1}{2}\sum_{n\,\in\,\mathbb{Z}} e^{\pi\,i\,n^2\,\tau} \quad = \quad \frac{1}{2} \;+\; \sum_{n\,=\,1}^{\infty} e^{\pi\,i\,n^2\,\tau}$$

(more later in Section 3.1 on θ as a function of a complex variable for Im $\tau > 0$). Riemann then obtained an integral representation for $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ as follows:

$$\begin{aligned}
\Gamma(s) \quad &= \quad \int_0^\infty t^{s-1}\,e^{-t}\,dt\,, && \text{Re } s > 0\\
\pi^{-s}\,\Gamma(s)\,\zeta(2s) \quad &= \quad \sum_{n=1}^{\infty}\int_0^\infty (\pi\,n^2)^{-s}\,t^{s-1}\,e^{-t}\,dt\,, && \text{Re } s > 1/2\\
&= \quad \int_0^\infty t^{s-1}\,(\theta(it) - \tfrac{1}{2})\;dt
\end{aligned}$$

$$\begin{aligned}
&= \int_1^\infty t^{\,s-1}\,(\theta(it)-\tfrac{1}{2})\;dt \;+\; \int_0^1 t^{s-1}\,\theta(it)\,dt \;-\; \frac{t^s}{2s}\bigg|_0^1\\
&= \int_1^\infty t^{s-1}\,(\theta(it)-\tfrac{1}{2})\,dt \;+\; \int_1^\infty t^{-s-1}\,\theta(\tfrac{i}{t})\;dt \;-\; \tfrac{1}{2s}\\
&= \int_1^\infty (t^{s-1} \,+\, t^{1/2-s-1})\,(\theta(it)-\tfrac{1}{2})\;dt \;-\; \frac{1}{2s} \;-\; \frac{1}{1-2s}\,.
\end{aligned}$$

Indeed, replacing s by $s/2$, the above expression reads

$$\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \;=\; \int_1^\infty (t^{s/2-1}+t^{(1-s)/2-1})(\theta(it)-\frac{1}{2})\;dt \;-\; \frac{1}{s} \;-\; \frac{1}{1-s}\,. \tag{2.10}$$

The integral representation (2.10) allows us to conclude the main analytic properties mentioned in the introduction:

Theorem 2.1. *The function*

$$\xi(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,\zeta(s)$$

satisfies properties **E, BV**, *and* **FE** *of Section 1.*

Proof: We first note that

$$\theta(it) - \frac{1}{2} \;=\; \sum_{n=1}^{\infty} e^{-\pi\,n^2\,t} \;\leq\; \sum_{n=1}^{\infty} e^{-\pi\,n\,t} \;=\; \frac{e^{-\pi\,t}}{1-e^{-\pi\,t}} \;=\; O(e^{-\pi\,t}) \tag{2.11}$$

for $t \geq 1$.[2] Since

$$\int_1^\infty |\,t^{\,s}\,e^{-\pi\,t}\,|\;dt \quad\leq\quad \int_1^\infty t^{\,b}\,e^{-\pi\,t}\,dt \;<\; \infty$$

for Re $s \leq b$, the integral in (2.10) converges – for *any* value of s – to an entire function which is bounded for s in vertical strips. Thus $\xi(s)$ is meromorphic with

[2]The notation $A = O(B)$ indicates that there exists some absolute constant $C > 0$ such that $|A| \leq C \cdot B$.

only simple poles at $s = 0$ and 1, demonstrating properties **E** and **BV**. Having established that (2.10) gives an analytic continuation, we may conclude that $\xi(s) = \xi(1-s)$ (property **FE**) because of the symmetry present in (2.10). □

2.2. **Hecke's Treatment of Number Fields (1916).** In this section we shall briefly describe Hecke's generalization [61] of Riemann's work to certain zeta functions associated to number fields (that is, finite extensions of $\mathbb{Q}$). These subsume Riemann's ζ-function, as well as the related Dirichlet L-functions. The latter are simply Dirichlet series

$$L(s,\chi) \quad = \quad \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}\ . \tag{2.12}$$

Here χ is a "Dirichlet character", meaning a non-trivial function $\chi : \mathbb{Z} \to \mathbb{C}$ which: (i) is periodic modulo some integer N; (ii) obeys $\chi(nm) = \chi(n)\chi(m)$ ("complete multiplicativity"); and (iii) vanishes on integers sharing a common factor with N. A Dirichlet character can equally be thought of as a homomorphism from $(\mathbb{Z}/N\mathbb{Z})^*$ to $\mathbb{C}^*$, extended to $\mathbb{Z}$ as a periodic function that vanishes on $\{n \mid (n,N) > 1\}$. The Dirichlet L-functions $L(s,\chi)$ satisfy the properties **E, BV,** and **FE** analogous to those of $\zeta(s)$ (which corresponds to the trivial character); for a complete discussion and precise analog of Theorem 2.1, see [30].

Our goal here is to describe the generalizations of $\zeta(s)$ and $L(s,\chi)$ that are the objects of Hecke's work, in some sense following the earlier exposition in [41]. This will necessitate some algebraic background; accordingly this section requires some familiarity with the concepts involved. However, it is not essential to the rest of the paper, and readers may wish to skip directly to Section 2.3, or instead to consult [93], [142] for definitions and examples.

To describe Hecke's accomplishment, we need to recall some of the local and global terminology involved. Let F be a number field, and $\mathcal{O}_F$ its ring of integers. We will refer to a non-archimedean place v of F as a prime ideal $\mathfrak{P} \subset \mathcal{O}_F$. A fractional ideal of $\mathcal{O}_F$ is an $\mathcal{O}_F$-submodule $\mathfrak{U}$ such that $x\mathfrak{U} \subset \mathcal{O}_F$ for some $x \in F^*$. All fractional ideals are invertible (i.e. there exists a fractional ideal $\mathfrak{U}^{-1}$ such that $\mathfrak{U}\mathfrak{U}^{-1} = \mathcal{O}_F$), and all fractional ideals factor uniquely into products of positive and negative powers of prime ideals. We let $ord_{\mathfrak{P}}(x)$ denote the exponent of $\mathfrak{P}$ occurring in the unique factorization of the principal ideal $x\mathcal{O}_F$, and set

$$|x|_v \quad = \quad |x|_{\mathfrak{P}} \quad = \quad (N\mathfrak{P})^{-ord_{\mathfrak{P}}(x)}\,,$$

where $N\mathfrak{P}$ is the number of elements in the finite field $\mathcal{O}_F/\mathfrak{P}$. Any real embedding $\sigma : F \to \mathbb{R}$ of F gives rise to a "real" infinite place via the norm $|x|_v = |\sigma(x)|$; complex places are defined analogously, and the real and complex places together comprise the archimedean places of F. For each place of F, the norm $|\cdot|_v$ gives a different completion F_v of F. For example, when $F = \mathbb{Q}$, $F_\infty = \mathbb{R}$ and $F_p = \mathbb{Q}_p$, the p-adic numbers (see Section 5 for much more on this theme).

We now come to the generalization of a Dirichlet character to the number field setting: a Hecke character (also known as a *Grössencharacter*). We shall think of one as the product of a family of homomorphisms $\chi_v : F_v^* \to \mathbb{C}^*$, one for each place of F:

$$\chi(x) \quad = \quad \prod_v \chi_v(x)\,.$$

Two constraints must be made on the family: firstly that χ be trivial on F^*, i.e. for any $x \in F \subset F_v^*$

$$\chi(x) \;=\; \prod_v \chi_v(x) \;=\; 1;$$

and secondly that all but a finite number of the χ_v be *unramified*, i.e. trivial on $\{x \in F_v^* \mid |x|_v = 1\}$. If v is such an unramified place, corresponding to a prime ideal $\mathfrak{P}$, $\chi(\mathfrak{P})$ is defined as $\chi_v(\varpi_v)$, where ϖ_v is an element of F_v such that $|\varpi_v|_v = N\mathfrak{P}^{-1}$ (a "uniformizing parameter" for F_v). This definition can of course be extended to ordinary ideals $\mathfrak{U}$ of $\mathcal{O}_F$, provided they are products of prime ideals corresponding to places where χ_v is unramified. Hecke's (abelian) L-series for the character χ is then defined as the Dirichlet series

$$L(s,\chi) \;=\; \sum \frac{\chi(\mathfrak{U})}{(N\mathfrak{U})^s} \;=\; \prod_{\mathfrak{P}} \big(1 \;-\; \chi(\mathfrak{P})\,(N\mathfrak{P})^{-s}\big)^{-1}. \tag{2.13}$$

Here $\mathfrak{U}$ is summed over these ordinary ideals of $\mathcal{O}_F$ just mentioned, and the product is only over the prime ideals corresponding to these unramified places.

When χ is the trivial character, i.e., $\chi_v \equiv 1$ for all v, then $L(s,\chi)$ specializes to be the *Dedekind zeta-function* $\sum (N\mathfrak{U})^{-s}$ of F. For $F = \mathbb{Q}$ this reduces to $\zeta(s)$, and if χ is instead of finite order, $L(s,\chi)$ becomes the Dirichlet L-function (2.12). Using very clever and intricate arguments, Hecke was able to express his L-series in terms of generalized "θ-functions" and to then derive their analytic continuation, functional equation, and boundedness in vertical strips, *à la* Riemann.

2.3. **Hamburger's Converse Theorem (1921).** Now let us return to the Riemann ζ-function. The next point of the theory is that the **F**unctional **E**quation for $\zeta(s)$ nearly characterizes it. Indeed, Hamburger [55] showed in 1921 that any Dirichlet series satisfying ζ's functional equation *and suitable regularity conditions* is necessarily a constant multiple of $\zeta(s)$. We state Hamburger's Theorem 2.4 at the end of this section, but first begin by describing these conditions, which are closely related to **B**oundedness in **V**ertical strips. In fact, our main motivation in describing Hamburger's theorem here is to explain the role of **BV** and the related "finite order" conditions; in the modern picture, these are crucial for applications involving the Converse Theorem (see Sections 3.1, 3.2, and 7.3).

Recall that property **BV** was stated earlier in terms of the function $\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. One may ask how the individual factors themselves behave as $|\mathrm{Im}\ s| \to \infty$. Clearly

$$|\pi^{-s/2}| \;=\; \pi^{-\mathrm{Re}(s)/2}\,, \tag{2.14}$$

while Stirling's formula states that

$$\begin{aligned} &|\Gamma(\sigma+it)| \;\sim\; \sqrt{2\pi}\,|t|^{\sigma-1/2}\,e^{-\pi\,|t|/2}\,, \\ &\text{uniformly for } a \,\le\, \sigma \,\le\, b, \quad |t| \to \infty. \end{aligned} \tag{2.15}$$

Yet the size of $|\zeta(s)|$ in the critical strip is quite difficult to pin down. In fact, one of the central unsolved problems in analytic number theory is the following and its generalizations.

The Lindelöf Hypothesis: *For any fixed* $\varepsilon > 0$ *and* $\sigma \ge 1/2$,

$$\zeta(\sigma \,+\, i\,t) \;=\; O(|t|^{\varepsilon}) \qquad \text{as} \qquad |t| \to \infty. \tag{2.16}$$

The implied constant in the O-notation here depends implicitly on the value of ε. In particular, $|\zeta(1/2+it)| = O(|t|^{\varepsilon})$ for $|t|$ large (this case turns out to be equivalent to (2.16) via the Phragmen-Lindelöf Principle, Proposition 2.5). The Lindelöf Hypothesis is implied by the Riemann Hypothesis and conversely implies that very few zeros disobey it (see [176, §13]).

Note that by (2.14), (2.15), and the **F**unctional **E**quation, the behavior for Re $(s) \leq 1/2$ is given by

$$(2.17) \qquad |\zeta(\sigma+it)| \;\sim\; |\zeta(1-\sigma-it)| \left|\frac{t}{2\pi}\right|^{1/2-\sigma}, \quad \sigma \text{ fixed, } |t| \text{ large.}$$

The Lindelöf conjecture is far out of reach, but we can easily prove (weaker) polynomial bounds.

Proposition 2.2. $\zeta(s) - \frac{1}{s-1} = O(|s|)$ *for Re* $s \geq 1/2$.

Proof:
For Re $s > 1$,

$$(2.18) \qquad \zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} n^{-s} - \int_1^{\infty} x^{-s} dx$$

$$(2.19) \qquad = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s})\, dx\,.$$

The integrand in (2.19) is bounded by

$$|n^{-s} - x^{-s}| = \left| \int_n^x s\, t^{-s-1}\, dt \right| \leq |s|\, n^{-\operatorname{Re} s - 1}\,.$$

We conclude that

$$\left| \zeta(s) - \frac{1}{s-1} \right| \leq |s|\, \zeta(\operatorname{Re}\, s+1)\,,$$

and so (2.19) gives an analytic continuation of $\zeta(s) - \frac{1}{s-1}$ to the region Re $s > 0$. In particular, $|\zeta(s) - \frac{1}{s-1}| \leq |s|\,\zeta(3/2)$ for Re $s \geq 1/2$. □

Definition: An entire function $f(s)$ is *of order* ρ if

$$(2.20) \qquad f(s) = O(e^{|s|^{\rho+\epsilon}}) \qquad \text{for any } \epsilon > 0.$$

It will turn out that the ζ-function and (conjecturally) all L-functions connected to automorphic forms have order 1. However, many other generalizations of zeta functions (such as Selberg's Zeta functions) in fact have order greater than 1.

Proposition 2.3. *The function*

$$s\,(s-1)\,\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,\zeta(s)$$

is (entire and) of order 1.

Proof: By the **F**unctional **E**quation, it suffices to consider Re $s \geq 1/2$. We have already seen this function is **E**ntire in Theorem 2.1. Another form of Stirling's Formula gives that

$$(2.21) \quad \Gamma(s) \;\sim\; \sqrt{2\pi}\, e^{-s}\, s^{s-\frac{1}{2}} = \sqrt{2\pi}\, e^{-s+(s-\frac{1}{2})\log s}, \quad \text{Re } s \geq 1/2,\ |s| \to \infty,$$

and hence $\Gamma(s) = O(e^{M|s|\log|s|})$ for some $M > 0$. Thus by (2.14), (2.21), and Proposition 2.2

$$s\,(s-1)\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \;=\; O(|s|^3 e^{\frac{M}{2}|s|\log|s|}).$$

Since for any $\varepsilon > 0$, $\frac{\log|s|}{|s|^\varepsilon} \to 0$ as $|s| \to \infty$, we conclude

$$s\,(s-1)\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \;=\; O(e^{|s|^{1+\varepsilon}}). \quad \square$$

We note that $s\,(s-1)\,\xi(s)$ is not of any order $\rho < 1$, as can be seen from (2.14) and (2.21) as $s \to \infty$ along the real numbers, where $\zeta(s)$ is always greater than 1.

Theorem 2.4. *(Hamburger's Converse Theorem)*[3]

Let $h(s) = \sum_{n=1}^{\infty} a_n\, n^{-s}$ *and* $g(s) = \sum_{n=1}^{\infty} b_n\, n^{-s}$ *be absolutely convergent for Re* $s > 1$, *and suppose that both* $(s-1)h(s)$ *and* $(s-1)g(s)$ *are entire functions of finite order. Assume the functional equation*

$$\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,h(s) \;=\; \pi^{-\frac{1-s}{2}}\,\Gamma(\tfrac{1-s}{2})\,g(1-s)\,. \tag{2.22}$$

Then in fact $h(s) = g(s) = a_1\,\zeta(s)$.

This is the theorem which says that $\zeta(s)$ is *uniquely* determined by its functional equation (subject to certain regularity conditions). Hamburger's theorem was greatly generalized and enlightened by Hecke approximately 15 years later. We will in fact later show how to derive Theorem 2.4 from Hecke's method (see the discussion after Theorem 3.1.) See also [132], [138].

The original proof of Hamburger's Theorem relies on the *Mellin transform* and *inversion formulas*; that is, if

$$\pi^{-s}\,\Gamma(s)\,\zeta(2s) \;=\; \int_0^\infty t^{s-1}\,(\theta(it) - \frac{1}{2})\,dt\,, \tag{2.23}$$

then

$$\theta(it) - \frac{1}{2} \;=\; \frac{1}{2\pi i}\int_{Re(s)\,=\,c} t^{-s}\,\left(\pi^{-s}\,\Gamma(s)\,\zeta(2s)\right)\,ds \tag{2.24}$$

for sufficiently large $c > 0$. Using the Phragmen-Lindelöf principle (Proposition 2.5), *plus* the regularity conditions of $g(s)$ and $h(s)$, one can fairly directly show that every a_k is equal to a_1; that is, $h(s) = a_1\zeta(s)$. By the way, it is of course known that $(s-1)\zeta(s)$ is entire and of order 1. This is because both $s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\frac{1}{s\Gamma(s/2)}$ are entire and of order 1 (Theorem 2.1).

2.4. **The Phragmen-Lindelöf Principle and Convexity Bounds.** A standard fact from complex analysis, the *Phragmen-Lindelöf Principle*, can be used to obtain estimates on $\zeta(s)$ in vertical strips from ones on their edges:

Proposition 2.5. *(Phragmen-Lindelöf). Let* $f(s)$ *be meromorphic in the strip* $U = \{s \,|\, a \le Re\ (s) \le b\}$, $a, b \in \mathbb{R}$, *with at most finitely many poles. Suppose that* $f(s)$ *satisfies the finite order inequality*

$$f(s) \;=\; O\left(e^{|s|^A}\right)\,, \quad \textit{for some } A > 0\,,$$

on U *for* $|Im\ s|$ *large and obeys the estimate*

$$f(\sigma + it) \;=\; O(|t|^M) \quad \textit{for}\ \ Re\ s = a, b, \quad |t| \to \infty.$$

[3]Actually Hamburger proved a more general statement, allowing for an arbitrary, finite number of poles (see [55], [176, p. 31]).

Then

$$f(\sigma+it) \quad = \quad O(|t|^M) \qquad \textit{for} \;\; a \le \textit{Re } s \le b, \quad |t| \to \infty$$

as well.

See [94] for a detailed exposition and proof of Proposition 2.5. An immediate application of the Phragmen-Lindelöf Principle is to provide a standard bound for $\zeta(s)$ and other L-functions in the critical strip. As an example, let us note the following bound towards the Lindelöf conjecture:

Lemma 2.6. *For any* $\varepsilon > 0$*,*

$$\zeta(1/2 + i\,t) \quad = \quad O_\varepsilon(t^{1/4+\varepsilon}) \;, \qquad |t| \to \infty \tag{2.25}$$

where the implied constant depends on ε*.*

Note that this is a sizeable improvement on the trivial bound in Proposition 2.2 towards (2.16).

Proof of Lemma 2.6: First we observe that

$$|\,\zeta(1+\varepsilon+it)\,| \quad \le \quad \sum_{n=1}^{\infty} |\,n^{-1-\varepsilon-it}\,| \quad = \quad \zeta(1+\varepsilon)\,,$$

which is a positive constant. By (2.17), which comes from the functional equation,

$$|\,\zeta(-\varepsilon-it)\,| \quad = \quad O_\varepsilon(|t|^{1/2+\varepsilon})\,, \qquad |t| \to \infty\,.$$

Now, set $f(s) = \zeta(s)\zeta(1-s)$, $a = -\varepsilon$, $b = 1+\varepsilon$, and $M = 1/2+\varepsilon$. Because of the discussion at the very end of Section 2.3, the conditions of Proposition 2.5 are met; we conclude $|\zeta(1/2+it)\zeta(1/2-it)| = O_\varepsilon(|t|^{1/2+\varepsilon})$ as $|t| \to \infty$. To finish the proof we replace ε by 2ε and observe that $\zeta(\bar{s}) = \overline{\zeta(s)}$ because of the Schwartz reflection principle ($\zeta(s) = \sum n^{-s}$ is real for $s > 1$). □

The estimate (2.25) for $\zeta(s)$ has been improved many times over; however, for general L-functions, the bounds given by the above argument are usually the best known. Because (2.25) interpolates between the bounds at Re $s = -1-\varepsilon$ and ε, results given by this argument are known as the *convexity bounds* for L-functions. A very important problem is to improve these by *breaking convexity*; even slight improvements to the convexity bounds for more general L-functions – still falling far short of Lindelöf's conjecture – have had many profound applications. Let's consider, for example, the possible ways of writing a positive integer as the sum of three squares. Gauss' famous condition asserts that the equation

$$x^2 \;+\; y^2 \;+\; z^2 \quad = \quad n \tag{2.26}$$

is solvable by some $(x,y,z) \in \mathbb{Z}^3$ if and only if n is not of the form $4^a(8b+7)$ for some integers $a, b \ge 0$ (see, for example, [160]). Linnik conjectured that the solutions to (2.26) are randomly distributed in the sense that the sets

$$\mathcal{D}_n \quad = \quad \left\{ \frac{(x,y,z)}{\sqrt{n}} \;\middle|\; x^2+y^2+z^2 \;=\; n\;, \quad x,\,y,\,z \,\in\, \mathbb{Z} \right\} \tag{2.27}$$

become equidistributed in the sphere $S^2 \subset \mathbb{R}^3$ as $n \neq 4^a(8b+7)$ increases. This was in fact proven by W. Duke (see [32], [33], [67]) and can be shown to follow quite directly from subconvexity estimates on automorphic L-functions ([34]), although this was not Duke's original argument. For a survey of recent results on subconvexity bounds, see [70].

The proof of Lemma 2.6 shows the strength of the finite-order condition, for it allows us to conclude that $\xi(s)$ decays rapidly as $|\mathrm{Im}\ s| \to \infty$ (and uniformly so in vertical strips), given only the functional equation and the absolute convergence of $\zeta(s)$ for $\mathrm{Re}\,(s)$ large. This will be useful in the proofs of Theorems 2.4 and 3.1. To wrap up this section, let's formally state this for future use.

Lemma 2.7. *Assume the conditions of Theorem 2.4 (notably* **E***ntirety,* **F***unctional* **E***quation, and the finite order hypothesis). Then both sides of* (2.22) *are* **B***ounded in* **V***ertical strips.*

We remark that the conclusion of the lemma does not depend particularly on the exact form of the **F**unctional **E**quation (2.22); similar conclusions follow when the functional equation involves different configurations of Γ-functions and powers of π.

Proof: The assumption of absolute convergence implies that

$$|h(\sigma + it)| \ \ \le \ \ \sum_{n=1}^{\infty} |a_n|\, n^{-\sigma} \ \ < \ \ \infty\ , \qquad \sigma > 1.$$

Then for any $\varepsilon > 0$, $|h(s)|$ is uniformly bounded in the range $\mathrm{Re}\ s \ge 1 + \varepsilon$, as is $|g(s)|$ by symmetry. Using the **F**unctional **E**quation, we see that both

$$|h(\sigma + it)|\ ,\ \ |g(\sigma + it)| \ \ = \ \ O(|t|^{1/2-\sigma})\ , \qquad |t| \to \infty$$

for $\sigma < -\varepsilon$, and uniformly so in vertical strips (see (2.17)).

We are assuming that $(s-1)g(s)$ and $(s-1)h(s)$ are of finite order, so the Phragmen-Lindelöf Principle (Proposition 2.5) applies. This shows that

$$|g(\sigma + it)|\ ,\ \ |h(\sigma + it)| \ \ = \ \ O(|t|^{1/2+\varepsilon})\,, \qquad \text{for } -\varepsilon < \sigma < 1 + \varepsilon.$$

Thus we have shown that in any vertical strip $a \le \mathrm{Re}\ s \le b$, both $g(s)$ and $h(s)$ are bounded by $|\mathrm{Im}\ s|^M$ for some $M > 0$, as $|\mathrm{Im}\ s| \to \infty$. Stirling's estimate (2.15) shows that (2.22) decays rapidly as $|\mathrm{Im}\ s| \to \infty$ in the strip $a \le \mathrm{Re}\ s \le b$, and hence is bounded there. □

3. Modular forms and the Converse Theorem

3.1. **Hecke (1936).** As already suggested, Hamburger's Converse Theorem did not become completely understood until greatly generalized by Hecke in 1936 ([60], [61, paper #33]); to describe it, we thus encounter the notion of the space of modular forms to which functions like θ belong. Note that

$$\theta(\tau) \ \ = \ \ \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{\pi\, i\, n^2\, \tau}$$

is holomorphic in the upper half plane $\mathrm{Im}\,(\tau) > 0$; moreover, because it satisfies (2.9) (when $\mathrm{Re}\ \tau = 0$), clearly

$$\theta\left(\frac{-1}{\tau}\right) \ \ = \ \ \left(\frac{\tau}{i}\right)^{1/2} \theta(\tau)\ , \qquad \theta(\tau + 2) \ \ = \ \ \theta(\tau)\,. \tag{3.1}$$

These two equations say that $\theta(\tau)$ is a *modular form of weight* $\frac{1}{2}$ for the group generated by $\tau \mapsto \tau + 2$ and $\tau \mapsto -\frac{1}{\tau}$. More generally, a *modular form of weight*

$k > 0$ and *multiplier condition* C for the group of substitutions generated by $\tau \mapsto \tau + \lambda$ and $\tau \mapsto -\frac{1}{\tau}$ is a holomorphic function $f(\tau)$ on the upper half plane satisfying

- (i) $f(\tau + \lambda) \;=\; f(\tau)$,
- (ii) $f(-\frac{1}{\tau}) \;=\; C\,(\frac{\tau}{i})^k\, f(\tau)$, and
- (iii) $f(\tau)$ has a Taylor expansion in $e^{\frac{2\pi i \tau}{\lambda}}$ (cf. (i)): $f(\tau) = \sum_{n=0}^{\infty} a_n\, e^{\frac{2\pi i n \tau}{\lambda}}$; i.e., f is "holomorphic at ∞".

We denote the space of such f by $M(\lambda, k, C)$; f is a *cusp form* if $a_0 = 0$. For example, the space $M(2, \frac{1}{2}, 1)$ is one dimensional and consists of multiples of the θ-function.

Now, given a sequence of complex numbers $a_0, a_1, a_2, \ldots$ with $a_n = O(n^d)$ for some $d > 0$, and given $\lambda > 0, k > 0, C = \pm 1$, set

$$\phi(s) \;=\; \sum_{n=1}^{\infty} \frac{a_n}{n^s}\,, \tag{3.2}$$

$$\Phi(s) \;=\; \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\,\phi(s)\,, \tag{3.3}$$

and

$$f(\tau) \;=\; \sum_{n=0}^{\infty} a_n\, e^{\frac{2\pi i n \tau}{\lambda}}\,. \tag{3.4}$$

(The O-condition on the a_n ensures that $\phi(s)$ converges for Re $s > d+1$ and that $f(s)$ is holomorphic in the upper half plane. In fact, $f(\tau) - a_0 = O(e^{-\frac{2\pi}{\lambda}\,\mathrm{Im}\,\tau})$; for example, see (2.11).)

Theorem 3.1. *(Hecke's Converse Theorem) The following two conditions are equivalent:*

(A) $\Phi(s) + \frac{a_0}{s} + \frac{C a_0}{k-s}$ *is an entire function which is bounded in vertical strips* **(EBV)** *and satisfies* $\Phi(s) \;=\; C\,\Phi(k-s)$ **(FE)***;*

(B) f *belongs to* $M(\lambda, k, C)$*.*

We will come to the proof of Theorem 3.1 shortly, but first wish to explain the connection to the results of Riemann and Hamburger. Riemann's Theorem 2.1 is an example of the direction (B)⇒(A). In the other direction, set

$$\phi(s) \;=\; \zeta(2s) \;=\; \sum_{n\geq 1} (n^2)^{-s}\,,$$
$$C = 1\,,\;\; k = 1/2\,,\;\; a_0 = 1/2\,,\;\; \text{and} \;\; \lambda = 2\,.$$

In this special case, the direction (A)⇒(B) of Theorem 3.1 asserts that $\theta(\tau)$ obeys the modular relations (3.1). Similarly, Theorem 2.4 can be derived from this direction of Theorem 3.1 as well. For simplicity, suppose that the coefficients a_n and b_n in the statement are equal (these are not the same a_n involved here). Then assumptions of Theorem 2.4 actually match the properties of ζ and ξ needed in (A). They guarantee that $\Phi(\frac{s}{2}) = \pi^{-s/2}\Gamma(\frac{s}{2})h(s)$ is holomorphic in Re $s > 0$, except perhaps for a simple pole at $s = 1$. By the functional equation (2.22), $\Phi(s)$ has an analytic continuation to $\mathbb{C}$ except for potential simple poles at $s = 0$ and $1/2$. Because of (2.22) the residues of $\Phi(s)$ at those points are negatives of each other, and thus $\Phi(s) + \frac{a_0}{s} + \frac{a_0}{k-s}$ is **E**ntire, where a_0 is the residue of $\Psi(s)$ at $s = k$. Lemma 2.7 shows the finite order assumption implies that $\Phi(s)$ satisfies the **BV** condition of

(A) as well. Theorem 3.1 therefore produces a modular form f in $M(2,1/2,1)$, which is a one-dimensional space spanned by $\theta(\tau)$. So f must in fact be a multiple of the θ-function, and we conclude that the original Dirichlet series in Theorem 2.4 are multiples of ζ.

Proof of Theorem 3.1: As in Hamburger's proof of Theorem 2.4, the proof begins by using Mellin inversion (see (2.24)):

$$f(ix) \; - \; a_0 \quad = \quad \frac{1}{2\pi i}\int_{\sigma\,=\,c} x^{-s}\,\Phi(s)\,ds\,, \tag{3.5}$$

for $x > 0$, where $\sigma = \text{Re }(s)$, and c is chosen large enough to be in the domain of absolute convergence of $\phi(s)$ (since we are assuming that $a_n = O(n^d)$, we may take any $c > d+1$).

Now assume (A). We first want to argue that we can push the line of integration to the left, past $\sigma = 0$, picking up residues of $C\,a_0\,x^{-k}$ at $s = k \le c$ and $-a_0$ at $s = 0$:

$$f(ix) \; - \; C\,a_0\,x^{-k} \quad = \quad \frac{1}{2\,\pi\,i}\int_{\sigma\,=\,k\,-\,c\,<\,0} x^{-s}\,\Phi(s)\,ds\,. \tag{3.6}$$

To see this, we need to show that the integrals of $\Phi(s)$ over the horizontal paths $[k-c\pm ir, c\pm ir]$ tend to zero as $r \to \infty$. We shall use the **B**oundedness in **V**ertical strips assumption to prove the integrand decays rapidly there; in fact, the contour shift here is the primary importance of the **BV** property. The assumption that $a_n = O(n^d)$ implies that $\phi(s)$ converges absolutely for Re $s \ge c > d+1$, where

$$|\phi(s)| \quad \le \quad \sum_{n=1}^{\infty} |a_n|\,n^{-c} \quad = \quad O(1)\,. \tag{3.7}$$

Stirling's asymptotics (2.21) show that $\Phi(s)$ satisfies the order-one estimate $O(e^{|s|^{1+\varepsilon}})$ in the region Re $s \ge c$. By the functional equation, $\Phi(s)$ does as well in the reflected region Re $s \le k-c$, and the **BV** assumption from (A) handles the missing strip; therefore $\Phi(s) + \frac{a_0}{s} + \frac{Ca_0}{k-s}$ is of order one on $\mathbb{C}$. Since $\frac{1}{s\Gamma(s)}$ is entire and of order 1, $(s-k)\phi(s) = (s-k)(\frac{2\pi}{\lambda})^s\Gamma(s)^{-1}\Phi(s)$ is also entire and of order 1 (cf. the end of Section 2.3). The functional equation

$$\phi(s) \quad = \quad C\left(\frac{2\,\pi}{\lambda}\right)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(s)}\,\phi(k-s)$$

shows that

$$|\phi(\sigma+it)| \quad = \quad O(t^{2c-k})\,, \quad \sigma \,=\, k\,-\,c\,<\,0 \tag{3.8}$$

just as in the proof of Lemma 2.6. We conclude from the Phragmen-Lindelöf Principle (Proposition 2.5) that $\phi(s)$ is $O(|\text{Im } s|^K)$ for some K, uniformly as $|\text{Im } s| \to \infty$ in the strip $k-c \le \text{Re } s \le c$. Since this growth is at most polynomial, the exponential decay from Stirling's formula (2.15) gives us that $\Phi(\sigma+it)$ decays faster than any polynomial in $|t|$ as $|t| \to \infty$, uniformly for σ in the interval $[k-c, c]$. Thus the integrals

$$\int_{k-c+ir}^{c+ir} x^{-s}\,\Phi(s)\,ds\;,\;\int_{k-c-ir}^{c-ir} x^{-s}\,\Phi(s)\,ds \quad \longrightarrow\; 0\;,\;\text{as } r\to\infty\,,$$

and the contour shift between (3.5) and (3.6) is valid.

Now, let us resume from (3.6) and apply the functional equation from (A):

$$\begin{aligned} f(ix) \;-\; C\, a_o\, x^{-k} \;&=\; \frac{C}{2\pi i}\int_{\sigma = k-c<0} x^{-s}\,\Phi(k-s)\,ds \\ &=\; \frac{C}{2\pi i}\int_{\sigma = c>k} x^{s-k}\,\Phi(s)\,ds \qquad \text{(upon } s\mapsto k-s\text{)} \\ &=\; C\,x^{-k}\,(f(\frac{i}{x}) \;-\; a_0) \qquad \text{by (3.5)}\,, \end{aligned}$$

or

$$f(ix) \;=\; C\,x^{-k}\,f(\frac{i}{x})\,,$$

which is property (ii) of the definition of $M(\lambda, k, C)$. Properties (i) and (iii) are immediate from the definition of $f(\tau)$ in (3.4), and we conclude the proof that (B) follows from (A).

Now suppose (B). We will essentially follow Riemann's original argument from Section 2.1, using the integral representation (cf. (2.23))

$$\Phi(s) \;=\; \int_0^\infty t^{s-1}\,(f(it)\;-\;a_0)\,dt\,.$$

Then

$$\begin{aligned} \int_0^1 t^{s-1}\,(f(it)\;-\;a_0)\,dt = &\int_1^\infty t^{-s-1}\,f(\frac{i}{t})\,dt \;-\; a_0\,\frac{t^s}{s}\Big|_0^1 \\ =\;& C\int_1^\infty t^{k-s-1}\,(f(it)\;-\;a_0)\,dt \;-\; \frac{a_0}{s} \;-\; \frac{C\,a_0}{k-s}\,. \end{aligned}$$

Thus

$$\Phi(s) \;+\; \frac{a_0}{s} \;+\; \frac{C\,a_0}{k-s} \;=\; \int_1^\infty [t^{s-1}\,(f(it)\;-\;a_0) \;+\; t^{k-s-1}\,C(f(it)\;-\;a_0)]\;dt\,.$$

This expression is clearly **EBV**, and $\Phi(s) \;=\; C\,\Phi(k-s)$ (just as in the proof of Theorem 2.1), whence (A). □

By reducing a question about Dirichlet series to one about modular forms, Theorem 3.1 represents a great step forward from Riemann's treatment of ζ. In particular, it puts his original argument into a very useful and fruitful context. Note that a specified type of Dirichlet series is connected to any modular form satisfying

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

for

(3.9)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\in\; SL(2,\mathbb{Z}) \;=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle|\; a,\,b,\,c,\,d\in\mathbb{Z}\,,\; a\,d\;-\;b\,c\;=\;1 \right\},$$

the group of substitutions generated by $\tau \to \tau+1$ and $\tau \to \frac{-1}{\tau}$.

3.2. **Weil's Converse Theorem (1967).** A. Weil in 1967 completed Hecke's theory by similarly characterizing modular forms for *congruence* subgroups, such as

$$\Gamma_0(N) \quad = \quad \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2,\mathbb{Z}) \middle| \; c \equiv 0 \pmod{N} \right\}. \tag{3.10}$$

(These subgroups in general have many generators, whereas Hecke's Theorem deals with modular forms only for the groups generated by $\tau \mapsto \tau + \lambda$ and $\tau \mapsto -\frac{1}{\tau}$.) Weil's breakthrough was to *twist* the series $\phi(s)$ by *Dirichlet characters.* Recall from Section 2.2 that a Dirichlet character modulo r is a periodic function $\chi : \mathbb{Z} \to \mathbb{C}$ which is completely multiplicative (i.e. $\chi(nm) = \chi(n)\chi(m)$), and satisfies

$$\chi(1) \quad = \quad 1\,, \qquad \chi(n) \quad = \quad 0, \;\; \text{if } (n,r) > 1.$$

Given a Dirichlet character χ modulo r and a proper multiple r' of r, one may form a Dirichlet character χ' modulo r' by setting

$$\chi'(n) \quad = \quad \begin{cases} \chi(n) & , \; (n,r') \,=\, 1, \\ 0 & , \; \text{otherwise.} \end{cases}$$

Such a character χ' obtained this way is termed *imprimitive*, and one which is not, *primitive*. The importance of primitive characters is that their functional equations are simpler (see [30]). Weil's converse theorem gives a condition for modularity under $\Gamma_0(N)$ in terms of the functional equations of Dirichlet series twisted by primitive characters:

Theorem 3.2. *(Weil [178]) Fix positive integers N and k, and suppose $L(s) = \sum_{n=1}^{\infty} a_n\, n^{-s}$ satisfies the following conditions:*

- *(i) $L(s)$ is absolutely convergent for Re s sufficiently large;*
- *(ii) for each primitive character χ of modulus r with $(r,N) = 1$,*

  $$\Lambda(s,\chi) \quad = \quad (2\pi)^{-s}\;\Gamma(s)\;\sum_{n=1}^{\infty} a_n\,\chi(n)\,n^{-s}$$

  continues to an **E***ntire function of s,* **B***ounded in* **V***ertical strips;*
- *(iii) Each such $\Lambda(s,\chi)$ satisfies the* **F***unctional* **E***quation*

  $$\Lambda(s,\chi) \quad = \quad w_\chi\; r^{-1}\; (r^2 N)^{\frac{k}{2}-s}\; \Lambda(k-s,\bar{\chi})\,, \tag{3.11}$$

  where

  $$w_\chi \quad = \quad i^k\,\chi(N)\,g(\chi)^2$$

  and the Gauss sum

  $$g(\chi) \quad = \quad \sum_{n \pmod{r}} \chi(n)\; e^{2\,\pi\,i\,n/r}\,.$$

Then $f(z) = \sum_{n=1}^{\infty} a_n\, e^{2\,\pi\,i\,n\,z}$ belongs to the space of modular forms for $\Gamma_0(N)$ (i.e.

$$f\left(\frac{a\,z+b}{c\,z+d}\right) \quad = \quad (c\,z+d)^k\; f(z)\;, \quad \textit{for all} \;\; \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)\;,$$

and satisfies a holomorphy condition at its "cusps" analogous to property (iii) in the definition of $M(\lambda,k,C)$ in Section 3.1).

Note that the trivial character (with $\chi(n) \equiv 1$) is primitive, so the statement includes the L-functions used in Theorem 3.1. Property (iii) is certainly satisfied if $L(s)$ is the L-function of a modular form, as can be shown using a slight variant of Hecke's argument used in proving Theorem 3.1. For the obvious reason, we refer to this theorem as "Weil's converse to Hecke Theory". For a proof, see [12], [68], or [125].

3.3. **Maass Forms (1949).** In addition to the holomorphic modular forms on the complex upper half plane $\mathbb{H}$, there are the non-holomorphic modular forms introduced by Maass [111]. These are equally important, but far more mysterious. The literature has slight differences in the terminology, but for us a Maass form will be a non-constant eigenfunction of Laplace operator $\Delta = -y^2\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)$ in $L^2(\Gamma\backslash\mathbb{H})$, where Γ is a discontinuous subgroup of $SL(2,\mathbb{R})$, e.g. a congruence subgroup. The laplacian condition replaces the holomorphy condition here. In contrast to the holomorphic modular forms, all of which have constructions and geometric interpretations, the vast majority of Maass forms lack constructions or identification. Their mere existence is so subtle that Selberg invented the trace formula [156] simply to show that they exist for $\Gamma = SL(2,\mathbb{Z})$! In fact, deformation results such as those of Phillips-Sarnak and Wolpert [128], [129], [130], [148], [182], [183] demonstrate that Maass forms are scarce for the generic discrete subgroup $\Gamma \subset SL(2,\mathbb{R})$. For this reason we shall stick to congruence subgroups Γ for the rest of this exposition.

For now, consider a Maass form ϕ for $\Gamma = \Gamma_0(N)$ (for simplicity the reader may take $N = 1$ and $\Gamma = SL(2,\mathbb{Z})$). The Fourier expansion of ϕ is given by

$$\phi(x+iy) \quad = \quad \sum_{n\neq 0} a_n \, \sqrt{y}\; K_\nu(\,2\,\pi\,|n|\,y\,)\; e^{2\,\pi\,i\,n\,x}\,, \tag{3.12}$$

where a_n are coefficients and $K_\nu(t)$ is the K-Bessel function

$$K_s(z) \quad = \quad \frac{\pi}{2}\,\frac{I_{-s}(z)-I_s(z)}{\sin \pi s} \quad , \qquad I_s(z) \quad = \quad \sum_{m=0}^{\infty} \frac{(\frac{z}{2})^{\,s+2m}}{m!\;\Gamma(s\,+\,m\,+\,1)}\,. \tag{3.13}$$

The parameter ν is related to the Laplace eigenvalue of ϕ by $\lambda = 1/4 - \nu^2$, where $\Delta\phi = \lambda\phi$. Hecke's method was extended by Maass to obtain the analytic continuation and functional equations of the L-functions $L(s,\phi) = \sum_{n=1}^{\infty} a_n n^{-s}$ of Maass forms on $\Gamma_0(N)$ through the integral $\int_0^\infty \phi(i\,y)\,y^{s-1/2}\,\frac{dy}{y}$. When $\Gamma = SL(2,\mathbb{Z})$, for example, this integral is unchanged by the substitution $s \mapsto 1-s$. Maass also proved a converse theorem for his Maass forms for $\Gamma = SL(2,\mathbb{Z})$; see the comments at the end of the section.

3.4. **Hecke Operators.** The Euler product structure of the Riemann ζ-function has an analog for modular form L-functions through Hecke operators. For any positive integer n, the Hecke operator

$$T_n(f)(z) \quad = \quad \frac{1}{n} \sum_{a\,d\,=\,n} a^k \sum_{0\,\le\,b\,<\,d} f\left(\frac{a\,z+b}{d}\right) \tag{3.14}$$

preserves the space of modular forms of weight k for $\Gamma_0(N)$, so long as n and N are relatively prime. The same formula applies to Maass forms when $k = 0$ and the prefactor $\frac{1}{n}$ is replaced by $\frac{1}{\sqrt{n}}$. A few other operators are used as well, to take into account symmetries of $\Gamma_0(N)$ by which modular forms can be "diagonalized." In addition to being eigenfunctions of a differential operator (i.e. either the Cauchy-Riemann operator $\bar{\partial}$ or the laplacian Δ), a basis of modular forms or Maass forms can be chosen among eigenfunctions of the Hecke operators as well. As a result, identities amongst the coefficients can be proven. These are nicely expressed as factorizations of the L-functions of modular forms. For example, when $\Gamma = SL(2,\mathbb{Z})$ the L-series of a holomorphic form of weight k factors as

$$L(s) \;=\; \sum_{n=1}^{\infty} a_n\, n^{-s} \;=\; \prod_p \left(1 - a_p\, p^{-s} + p^{k-1-2s}\right)^{-1}, \tag{3.15}$$

a formula which remains valid for Maass forms if k is taken to be 1.

We end this section with some remarks about the Converse Theorem 3.2. Maass observed that Hecke's argument for Theorem 3.1 applies to his Maass forms for $\Gamma = SL(2,\mathbb{Z})$ as well, but this method does *not* prove a converse theorem for $\Gamma_0(N)$ for N large. The reason for this is that the group $\Gamma_0(N)$ can have many generators, which are not accounted for by simply one functional equation alone. Interestingly, Conrey and Farmer [29] have found that by using Hecke operators, a converse theorem can be proved for a surprisingly large range of N using only a single functional equation. In another direction, Booker [6] has recently discovered that the converse theorem requires only a single functional equation when it is specialized to the L-functions coming from *Galois representations*, regardless of how large N is. It is an open question whether or not Weil's argument applies to Maass forms. A key point for Weil is that radially symmetric holomorphic functions are necessarily constant; this is not true in the non-holomorphic case because there are spherical functions (formed by radially-symmetrizing $\mathrm{Im}(z)^s$), and so Weil's argument does directly apply. However, there is nevertheless an applicable converse theorem due to Jacquet and Langlands, which we will come to in Section 7.1.

4. L-functions from Eisenstein series (1962-)

In the last section we saw the Mellin transform provided a connection between holomorphic modular forms and certain Dirichlet series generalizing $\zeta(s)$. Another quite different connection comes from a family of non-cuspidal modular forms, the Eisenstein series

$$G_k(z) \;=\; \sum_{(m,n)\,\in\,\mathbb{Z}^2-\{0,0\}} \frac{1}{(m\,z+n)^k}\,, \qquad k \text{ even}, \;\geq 2. \tag{4.1}$$

It is not difficult to show that $G_k(z)$ is a holomorphic modular form of weight k for $SL(2,\mathbb{Z})$. Via a Poisson summation argument over m, one can obtain the Fourier expansion

$$G_k(z) \;=\; 2\,\zeta(k) \;+\; \frac{2\,(\,2\,\pi\,i\,)^{\,k}}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\; e^{2\,\pi\,i\,n\,z}\,, \tag{4.2}$$

where $\sigma_{k-1}(n)$ is defined in terms of the divisors of n by

$$\sigma_t(n) \;=\; \sum_{d|n} d^{\,t}$$

(see [160, Section 7.5.5] for details). The appearance of $\zeta(k)$ here is the first example of a very general phenomena, which ultimately leads to the Langlands-Shahidi method (Section 8). In the next section, we will describe the generalized *non-holomorphic* Eisenstein series considered by Selberg and their connection to the analytic properties of the Riemann ζ-function throughout the complex plane, not just at special integral values alone.

4.1. **Selberg's Analytic Continuation.** Selberg's method [157] can be used to obtain the analytic continuation and functional equations of the L-functions that arise in the "constant terms" of Eisenstein series. We shall sketch a form of it in the classical case of the upper half plane $\mathbb{H} = \{z = x + iy \mid y > 0\}$, and the simplest possible Eisenstein series. Here we shall summarize the main steps involved; the details can be found in [7], [91]. We will turn to the general case in Section 8.

Define

$$
\begin{aligned}
E(z,s) &= \frac{1}{2} \sum_{\substack{(m,n)=\mathbb{Z}^2-\{0\} \\ gcd(m,n)=1}} \frac{y^s}{|m\,z+n|^{2s}} \\
&= \frac{1}{2}\,\frac{1}{\zeta(2\,s)} \sum_{(m,n)=\mathbb{Z}^2-\{0\}} \frac{y^s}{|m\,z+n|^{2s}}\,,
\end{aligned}
\tag{4.3}
$$

for $z \in \mathbb{H}$ and $\sigma = \text{Re } s > 1$. This series converges absolutely and uniformly in any compact subset of the region Re $s > 1$ and is the first example of a non-holomorphic Eisenstein series. Very importantly, $E(z,s)$ is unchanged by the substitutions $z \mapsto \frac{az+b}{cz+d}$ coming from any matrix in (3.9). Selberg considers the problem of analytically continuing $E(z,s)$ with respect to s to obtain another functional equation, as we shall now explain. (Actually, Selberg had several different arguments to do this, but they mainly appeal to spectral theory to obtain the important properties of analytic continuation and functional equation of Eisenstein series.)

To motivate the statement of the functional equation, let us first consider the Fourier expansion of $E(z,s)$. It is given by

$$
E(z,s) \;=\; E(x\,+\,i\,y\,,\,s) \;=\; \sum_{m\in\mathbb{Z}} a_m(y,s)\,e^{\,2\,\pi\,i\,m\,x}
\tag{4.4}
$$

where $e(x) = e^{\,2\,\pi\,i\,x}$, and

$$
a_m(y,s) \;=\; \int_0^1 E(x+iy,s)\,e^{-2\,\pi\,i\,m\,x}\,dx\,.
$$

We shall need here only the coefficients a_0 and a_1. If one computes directly for Re $s > 1$, using the "Bruhat decomposition"[4] for $SL(2,\mathbb{Z})$ and recalling (3.13), one obtains

$$
a_0(y,s) \;=\; y^s \;+\; \phi(s)\,y^{1-s}
\tag{4.5}
$$

[4]The Bruhat decomposition states that all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ with $c \neq 0$ may be written as products $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, where: r and s range over $\mathbb{Z}$; γ over $\mathbb{Z}-\{0\}$; δ over $(\mathbb{Z}/\gamma\mathbb{Z})^*$; and α and β are any two integers (which depend on γ and δ, of course) satisfying $\alpha\delta - \beta\gamma = 1$.

and

$$a_n(y,s) \;=\; 2\,\frac{\sqrt{y}\;K_{s-\frac{1}{2}}\,(2\,\pi\,|n|\,y)}{\pi^{-s}\,\Gamma(s)\,\zeta(2s)}\;|n|^{s-1}\;\sigma_{1\,-\,2\,s}(n)\,, \tag{4.6}$$

with

$$\phi(s) \;=\; \pi^{1/2}\,\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)} \;=\; \frac{\xi(2s-1)}{\xi(2s)}$$

(see [12] for details). In general, $\phi(s)$ is called the "constant term" or "scattering" matrix of $E(z,s)$.

Having described the Eisenstein series $E(z,s)$, we now state and prove Selberg's theorem:

Theorem 4.1. *(Selberg – see* [157]*) $E(z,s)$ has a meromorphic continuation to the whole complex s-plane and satisfies the functional equation*

$$E(z,s) \;=\; \phi(s)\;E(z,1-s)\,. \tag{4.7}$$

(A Misleading) Proof: Theorem 2.1 and (4.5) and (4.6) can then be applied to show that each term in the Fourier expansion

$$E(z,s) \;=\; \sum_{n\in\mathbb{Z}} a_n(y,s)\,e^{\,2\,\pi\,i\,n\,x}$$

is meromorphic and satisfies the functional equation (4.7). The sum converges rapidly because $K_s(y)$ decays exponentially as $y\to\infty$. Hence the whole sum is meromorphic on $\mathbb{C}$ and satisfies (4.7). □

We wrote that the above proof is "misleading" because, although it demonstrates a connection to Theorem 2.1, in practice it has turned out to be much more fruitful to reverse the logic – and conclude properties of L-functions from those of Eisenstein series! Indeed, Theorem 4.1 can be proven using spectral theory and even in a very non-arithmetic setting (see [7], [27], [91] for more details). The reader may already have noticed a similarity between the Fourier expansion of Eisenstein series in (4.4)-(4.6), and those of Maass forms in (3.12). In fact, the Eisenstein series $E(x+iy,s)$ is an eigenfunction of the Laplace operator $\Delta=-y^2\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)$, with eigenvalue $s(1-s)$. A main point in arguing the functional equation is *Maass' lemma*, which ultimately implies that because $E(z,s)$ and $E(z,1-s)$ share the Laplace eigenvalue $s(1-s)$, the two must be multiples of each other. The ratio can be found to be $\phi(s)$ by inspecting the constant term $a_0(y,s)$, and so the functional equation (4.7) can be proven without knowing ζ's functional equation $\xi(s)=\xi(1-s)$.

Of course, Selberg proved his Theorem 4.1 in much greater generality than we have stated. Our point is that the analytic continuation and functional equation for the Eisenstein series furnish an analytic continuation and functional equation for the Riemann ζ-function. To analytically continue $\zeta(s)$, basically "the constant term" is enough: reading through the spectral proof of the analytic continuation of $\phi(s)$ for $E(z,s)$, one demonstrates that $\xi(s)$ is holomorphic everywhere, save for simple poles at $s=0$ and 1. To get the functional equation, we need to consider

the *non-trivial* Fourier coefficient $a_1(y,s)$. Theorem 4.1 yields
(4.8)
$$\frac{2\sqrt{y}\,K_{s-1/2}(2\,\pi\,y)}{\xi(2s)} \;=\; a_1(y,s) \;=\; \frac{\xi(2s-1)}{\xi(2s)}\,a_1(y,1-s) \;=\; \frac{\xi(2s-1)}{\xi(2s)}\,\frac{2\sqrt{y}\,K_{1/2-s}(2\,\pi\,y)}{\xi(2-2s)}\,;$$
then, using $K_s = K_{-s}$ and setting $s = \frac{1+s'}{2}$, we have
$$\xi(s') \;=\; \xi(1-s')\,,$$
exactly the **F**unctional **E**quation for $\zeta(s')$. Incidentally, the same analysis applied to the general Fourier coefficient $a_n(y)$ from (4.6) does not give any additional information (this is because the extra factor $|n|^{s-1}\sigma_{1-2s}(|n|)$ already obeys the functional equation). **B**oundedness in **V**ertical strips is another matter, which we will return to in Section 8.3. Selberg's work on $GL(2)$ was extended by Langlands [95], [97] to cover Eisenstein series on general groups, where the analysis is much more difficult. This forms the basis of the Langlands-Shahidi method, the topic of Section 8.

5. Generalizations to adele groups

In the remaining sections of the paper, we will revisit the techniques and topics of the earlier sections, but in the expanded setting of automorphic forms on groups over the adeles. The adeles themselves enter as a language to keep track of the arithmetic bookkeeping needed for complicated expressions, such as the computations over general number fields in Section 2.2. They are convenient even in the simplest examples when the ground field is $\mathbb{Q}$. For instance, we shall see in the next section how Tate's thesis naturally produces the Euler product formula for the Riemann ζ-function:
$$\zeta(s) \;=\; \prod_p \left(1-p^{-s}\right)^{-1} \;=\; \prod_p \left(1+p^{-s}+p^{-2s}+\cdots\right),$$
a formula which itself is a restatement of the unique factorization theorem for integers. They will be useful in Section 8 for computations involving the Eisenstein series for $SL(2,\mathbb{Z})$ from Section 4. In general, they are extremely valuable on general groups, where they give clues for how to structure terms in large sums into an "Eulerian" form.

The adeles and their notable features are perhaps better explained later on, within the context of the arguments in which they are used. Nevertheless we give the basic definitions before proceeding. Given a rational number x, let
$$|x|_p \;:=\; p^{-\mathrm{ord}_p(x)}, \tag{5.1}$$
where $\mathrm{ord}_p(x)$ denotes the exponent of p occurring in the unique factorization of $x \in \mathbb{Q}$. This p-adic valuation defines a metric on $\mathbb{Q}$ by $d_p(x,y) = |x-y|_p$, and its completion is $\mathbb{Q}_p$, the field of p-adic numbers. More concretely, $\mathbb{Q}_p$ may be viewed as the formal Laurent series in p
$$x \;=\; c_k\,p^{\,k} \,+\, c_{k+1}\,p^{\,k+1} \,+\, \cdots\,, \qquad c_k \neq 0\,, \qquad k \;\in\; \mathbb{Z} \tag{5.2}$$
with integral coefficients $0 \le c_j < p$; alternatively it may be thought of as consisting of base-p expansions with only finitely many digits to the right of the "decimal"

point, but perhaps infinitely many to the left. Within $\mathbb{Q}_p$ lies its ring of integers, $\mathbb{Z}_p$, which is the completion of $\mathbb{Z}$ under $|\cdot|_p$. It may instead be viewed as the elements of $\mathbb{Q}_p$ as in (5.2) which have $k \geq 0$, or those with no digits to the right of the decimal point in their base-p expansion. The p-adic valuation of course extends to $\mathbb{Q}_p$: the absolute value of x given in (5.2) is p^{-k}, and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$. Similarly, the multiplicative subgroups are $\mathbb{Q}_p^* = \mathbb{Q}_p - \{0\}$ and $\mathbb{Z}_p^* = \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$.

The adeles are formed by piecing together all $\mathbb{Q}_p$ along with $\mathbb{R}$, which may be viewed as $\mathbb{Q}_\infty$, the completion of $\mathbb{Q}$ under the usual archimedean absolute value. Concretely, the adeles $\mathbb{A}$ are the restricted direct product of the $\mathbb{Q}_p$ with respect to the $\mathbb{Z}_p$; that means the adeles are infinite-tuples of the form

$$a \;=\; (a_\infty;\, a_2,\, a_3,\, a_5,\, a_7,\, a_{11},\, \ldots)\,, \quad a_p \in \mathbb{Q}_p \ \text{ for all } \ p \leq \infty \tag{5.3}$$

such that all but finitely many a_p lie in $\mathbb{Z}_p$. Similarly the ideles $\mathbb{A}^*$ are the restricted direct product of all $\mathbb{Q}_p^*$ with respect to $\mathbb{Z}_p^*$. Addition and multiplication are defined componentwise in $\mathbb{A}$ and $\mathbb{A}^*$. The rational numbers embed diagonally into the ring $\mathbb{A}$ and play a fundamental role, which will become apparent shortly when it appears in Tate's thesis. The adeles, or more properly the ideles, themselves have an absolute value; its value on a in (5.3) is

$$|a|_{\mathbb{A}} \;=\; \prod_{p \leq \infty} |a_p|_p.$$

Note that this is actually a finite product, because almost all a_p have absolute value equal to one, a theme which underlies many adelic concepts. The diagonally embedded $\mathbb{Q}^*$ consists of the ideles with $|a|_{\mathbb{A}} = 1$.

The above construction can be generalized to an arbitrary number field – or even "global field" – F to obtain its adele ring $\mathbb{A}_F$ (see [93], [142]). Most constructions involving $\mathbb{A}_{\mathbb{Q}}$ generalize to $\mathbb{A}_F$, though we will mainly focus on $F = \mathbb{Q}$ for expositional ease. Adeles are usually viewed much more algebraically and with much greater emphasis on their topology (which we have hardly touched); our intention here is rather to give enough background to illuminate their effectiveness in analysis.

6. Tate's thesis (1950)

In his celebrated 1950 Ph.D. thesis [174], J. Tate reinterpreted the methods of Riemann and Hecke in terms of harmonic analysis on the ideles $\mathbb{A}^*$ of a number field F. Tate's method succeeded in precisely isolating and identifying the contribution to the functional equation from each of the ramified prime ideals $\mathfrak{P}$ not treated in the product (2.13), a delicate problem which appeared complicated from the perspective of Hecke's classical method. At the same time, Tate's method is powerful enough to uniformly reprove the analytic continuation and functional equations of Hecke's L-functions. For this local precision, uniformity, and flexibility, Tate's method has influenced the many adelic methods at the forefront today. In this section we explain Tate's construction and the role of the devices he employs, via a comparison with Riemann's argument in Section 2.

Let us recall Riemann's integral from Section 2, after a harmless change of variables:

$$\xi(s) \;=\; \pi^{-s/2}\,\Gamma(\tfrac{s}{2})\,\zeta(s) \;=\; \int_0^\infty x^s \sum_{n \neq 0} e^{-\pi\, n^2\, x^2}\, d^*x\,. \tag{6.1}$$

Tate instead considers the sum over $\mathbb{Z} - \{0\}$ as an integral over a disconnected group. In order to keep the flexibility of treating more general sums, he instead essentially integrates the characteristic function of $\mathbb{Z}$ over a much larger set in his generalized ζ-*integral*

$$\zeta(f,c) \quad = \quad \int_{\mathbb{A}_{\mathbb{Q}}{}^*} f(a)\, c(a)\, d^*a\,. \tag{6.2}$$

Here $c(a)$ is any quasi-character of $\mathbb{A}^*$ – that is, a continuous homomorphism from $\mathbb{A}^*$ to $\mathbb{C}^*$ – which is trivial on $\mathbb{Q}^*$ (for example, we saw before that $|a|_{\mathbb{A}}$ is trivial on $\mathbb{Q}^*$); d^*a is the multiplicative Haar measure on $\mathbb{A}^*$ pieced together as a product of the local Haar measures $d^*x_\infty = \frac{dx}{|x|}$ and d^*x_p. The latter is normalized so that $\mathbb{Z}_p^*$ has measure 1. Finally, the function f is taken to be a product

$$f(a_\infty;\, a_2,\, a_3,\, a_5, \ldots) \quad = \quad \prod_{p\le\infty} f_p(a_p) \tag{6.3}$$

of functions f_p on $\mathbb{Q}_p$, which may depend on the quasi-character c. In the simplest possibility, which is that $c(a) = |a|_{\mathbb{A}}^s$, let us choose

$$f_p(x) \quad = \quad \chi_{\mathbb{Z}_p}(x) \quad = \quad \begin{cases} 1 & ,\ |x|_p \le 1\,, \\ 0 & ,\ \text{otherwise}\,, \end{cases}$$

and $f_\infty(x) = e^{-\pi\, x^2}$; then the integral $\zeta(f, |\cdot|_{\mathbb{A}}^s)$ actually recovers Riemann's integral. This can be seen as follows: first we may "fold" the integral to one over $\mathbb{Q}^*\backslash\mathbb{A}^*$:

$$\int_{\mathbb{A}^*} f(a)\, |a|_{\mathbb{A}}^s\, d^*a \quad = \quad \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} |a|_{\mathbb{A}}^s \left(\sum_{q\in\mathbb{Q}^*} f(qa) \right) d^*a\,. \tag{6.4}$$

The *strong approximation principle* states that $(0,\infty) \times \widehat{\mathbb{Z}}^*$ is a fundamental domain for $\mathbb{Q}^*\backslash\mathbb{A}^*$, where $\widehat{\mathbb{Z}}^* = \prod_{p<\infty} \mathbb{Z}_p^*$. It is easy to see that $f(qa) \equiv 0$ on this fundamental domain unless the rational q is actually an integer, for otherwise, the p-adic valuation $|qa|_p = |q|_p > 1$ for any prime p in the denominator of q. Thus the role of the f_p is to select the integers amongst $\mathbb{Q}$, and (6.4) becomes

$$\int_{(0,\infty)\times\widehat{\mathbb{Z}}^*} |a|_{\mathbb{A}}^s \left(\sum_{n\neq 0} f(na) \right) d^*a\,. \tag{6.5}$$

Now $f_p((na)_p) \equiv 1$ for all $p < \infty$, and so the integrand is independent of the $\widehat{\mathbb{Z}}^*$ factor, which has volume 1 under the Haar measure. Now (6.5) amounts to

$$\int_0^\infty |a_\infty|^{\,s} \sum_{n\neq 0} e^{-\pi\, n^2\, a_\infty^2}\ d^*a_\infty\,, \tag{6.6}$$

i.e. (6.1). Thus Tate's and Riemann's integrals match for $\zeta(s)$.

At the same time, the global integral on the lefthand side of (6.4) factors as a product

$$\prod_{p\le\infty} \int_{\mathbb{Q}_p^*} f_p(x)\ |x|_p^s\ d^*x_p \quad = \quad \left(\int_{\mathbb{R}} e^{-\pi|x|^2} |x|^s\, \frac{dx}{|x|} \right) \cdot \prod_p \int_{\mathbb{Z}_p} |x_p|_p^s\ d^*x_p\,. \tag{6.7}$$

The integral over $\mathbb{R}$ gives $\pi^{-s/2}\Gamma(\frac{s}{2})$, and the p-adic integral may actually be broken up over the "shells" $p^k\,\mathbb{Z}_p^* = \{|x_p|_p = p^{-k}\}$, $k \ge 0$, to give the geometric series

$\sum_{k=0}^{\infty} p^{-k\,s} = (1-p^{-s})^{-1}$. This gives the Euler product formula for $\zeta(s)$, along with its natural companion factor $\pi^{-s/2}\Gamma(\frac{s}{2})$ for $p = \infty$ – in other words, the completed Riemann ξ-function.

We should note that the role of the adelic absolute value (and in particular that its value is 1 on $\mathbb{Q}$) corresponds to the change of variables $x \mapsto x/n$ in the classical picture. In general for a global field F, we may write the quasi-character $c(a)$ in the form $c_0(a)|a|^s$, where $c_0 : \mathbb{A}^* \to \mathbb{C}^*$ is a character of modulus 1. Then $c_0(a)$ corresponds to χ, a "Hecke character" for F (Section 2.2), and $\zeta(f,c)$ differs from

$$L_F(s,\chi) \quad = \quad \prod_{\mathfrak{P}} \big(1 \,-\, \chi(\mathfrak{P})\,(N\mathfrak{P})^{-s}\big)^{-1}$$

(where $\mathfrak{P}$ now runs over *all* prime ideals of F) by only a finite number of factors.

In this idelic setting, Tate uses a Fourier theory and Poisson summation formula on the ring of adeles $\mathbb{A}$, and proves the elegant functional equation

$$\zeta(\,f\,,\,c\,) \quad = \quad \zeta(\,\widehat{f}\,,\,\widehat{c}\,)\,, \tag{6.8}$$

where $\widehat{f}$ is the "adelic Fourier transform" of f and $\widehat{c}(a) = \overline{c_0(a)}|a|^{1-s}$. The functional equation for $L_F(s,\chi)$ may be extracted from this. To illustrate with our example of the Riemann ζ-function, recall that we had taken c_0 to be identically equal to 1, and in fact our $f = \widehat{f}$, so that

$$\zeta(f_0, |\cdot|^s) \quad = \quad \pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})\,\prod_p\,(1-p^{-s})^{-1} \quad = \quad \xi(s)\,.$$

The functional equation **FE** is then immediate from (6.8). Tate's method of course also yields the **E**ntirety and **B**oundedness in **V**ertical strips.

7. Automorphic forms on $GL(n)$

Thus far we have seen two types of L-functions: The Riemann ζ-function and its cousins that are treated in the Riemann-Hecke-Tate theory (Section 2), and the L-functions of modular forms in Hecke's (other) theory (Section 3.1). We now understand these L-functions to be part of a family, the L-functions of automorphic forms on $GL(n,\mathbb{A})$. The integrals in Tate's thesis are over $\mathbb{A}^*$, which is just $GL(1,\mathbb{A})$, and the quasi-characters are viewed as automorphic forms on $GL(1,\mathbb{Q})\backslash GL(1,\mathbb{A})$. We shall now explain how to view the modular forms we saw in Section 3.1 as automorphic forms on $GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})$. This leads to two major generalizations: first to a general number field (or indeed even a global field) F instead of $\mathbb{Q}$, and second to an arbitrary reductive algebraic group G instead of $GL(1)$ or $GL(2)$.

To recap from Section 3.1, a holomorphic modular form of weight k for $\Gamma = SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ is a holomorphic function on the complex upper half plane $\mathbb{H}$ such that

- $f\left(\frac{az+b}{cz+d}\right) \;=\; (cz+d)^k\, f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
- $f(z)$ has a Fourier expansion $f(z) = \sum_{n\geq 0} c_n\, e^{2\,\pi\,i\,n\,z}$. In addition f is a cusp form if $c_0 = 0$.

The above definition of course extends to more general groups Γ, such as the congruence subgroups in (3.10). Before considering f as a function on $GL(2,\mathbb{A})$, we must first explain how to consider f as a function on $GL(2,\mathbb{R})$, or even $SL(2,\mathbb{R})$.

Indeed, there is a correspondence between holomorphic modular forms f of weight k for $\Gamma\backslash\mathbb{H}$, and certain functions F on $\Gamma\backslash SL(2,\mathbb{R})$ defined via the following relations:

$$(7.1)\qquad \begin{aligned} F\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= f\left(\frac{a\,i+b}{c\,i+d}\right)(c\,i+d)^{-k}\,, \\ f(x+i\,y) &= y^{-k/2}\,F\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix}\right). \end{aligned}$$

(We leave matrix entries blank if they are zero.) The key reason for this correspondence is that $\mathbb{H}$ is isomorphic to the quotient $SL(2,\mathbb{R})/SO(2,\mathbb{R})$. For more details and a precise characterization of F, see [12] or [40].

Aside from the holomorphic modular forms, the most significant automorphic forms on $\mathbb{H}$ are the non-holomorphic Maass forms: non-constant, L^2 Laplace eigenfunctions on the quotient $\Gamma\backslash\mathbb{H}$. We described these in Section 3.3. Because of the identification $\mathbb{H} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$, Maass forms can directly be viewed as functions on $\Gamma\backslash SL(2,\mathbb{R})$.

Now that we view the holomorphic and Maass modular forms on the group $G = SL(2,\mathbb{R})$, wide generalizations are possible, and techniques from representation theory may be applied. The group G acts on $L^2(\Gamma\backslash G)$ by the right regular representation, which is translation on the right:

$$(7.2)\qquad [\rho(g)f](h) \;=\; f(hg)\,.$$

The study of automorphic forms on $\Gamma\backslash G$ now becomes understanding the decomposition of the very large (and highly reducible) representation ρ into irreducible components. This is the starting point for the notion of "automorphic representation", but for that we first need to delve more into the arithmetic nature of Γ, and consider G adelically.

In addition to the action on the right, left-translation by rational matrices is very important in many constructions in automorphic forms. We have, therefore, also the left regular representation:

$$(7.3)\qquad [\lambda(g)f](h) \;=\; f(g^{-1}h)\,,$$

which maps $L^2(\Gamma\backslash G)$ to $L^2(g\Gamma g^{-1}\backslash G)$. In general this moves automorphic forms for one congruence subgroup Γ to those on a conjugate, which may be wildly different. For this reason it is natural to act on the left only by rational matrices g, so that the conjugate of Γ is still closely related to a congruence subgroup.[5] In fact, many fundamental constructions (such as Hecke operators) require action by rational matrices g which lie in $GL(2,\mathbb{Q})$, but not $SL(2,\mathbb{Q})$. It is for this reason that we will consider adelic automorphic forms on $GL(2,\mathbb{A})$, not $SL(2,\mathbb{A})$, though a theory exists for that group as well. Because $GL(2,\mathbb{R})$ is one dimension larger than $SL(2,\mathbb{R})$, we technically need to consider $L^2_\omega(Z\Gamma\backslash GL(2,\mathbb{R}))$ where Z is the center of $GL(2,\mathbb{R})$ (=scalar multiples of the identity matrix). Here ω is a central character (that is, a character of Z), and this L^2 space consists of functions on G which transform by Z according to ω, but which are otherwise square-integrable on the quotient $Z\Gamma\backslash GL(2,\mathbb{R})$. As a minor technicality, we will now consider $\Gamma = GL(2,\mathbb{Z})$ instead of $SL(2,\mathbb{Z})$ to make the picture more uniform. The setup of this paragraph works equally well for $\Gamma = GL(n,\mathbb{Z})$ and $G = GL(n,\mathbb{R})$.

[5]See [113] for a thorough explanation.

Finally we now come to the adeles. The adelic group $GL(n,\mathbb{A})$ is the product of $GL(n,\mathbb{R})$ with $GL(n,\mathbb{A}_f)$, the direct product of all $GL(n,\mathbb{Q}_p)$ with respect to their integral subgroups $GL(n,\mathbb{Z}_p)$. We have already seen – at least in the case $n=2$ – that the first factor, $GL(n,\mathbb{R})$, acts on automorphic functions on $Z\Gamma\backslash G$ on the right and that rational matrices act on the left. Just as with $\mathbb{A}^* = GL(1,\mathbb{A})$ in Tate's thesis, there is a version of the strong approximation theorem for $GL(n,\mathbb{A})$. It states that $GL(n,\mathbb{A}) = GL(n,\mathbb{Q})GL(n,\mathbb{R})K_f$, where $K_f = \prod_{p<\infty} GL(n,\mathbb{Z}_p)$. We now define an action of $GL(n,\mathbb{A}_f)$ on the left that extends the action of $GL(n,\mathbb{Q})$:

$$[\lambda(g_f)F](h) \quad = \quad [\lambda(\gamma)F](h) \quad = \quad F(\gamma^{-1}h)\,, \tag{7.4}$$

where $\gamma \in GL(n,\mathbb{Q})$ is the factor guaranteed by the strong approximation theorem in writing $g_f \in GL(n,\mathbb{A}_f) \subset GL(n,\mathbb{A})$ as a product. This definition is well defined, because any two close "approximants" γ must be related by a multiple of an integral matrix, and F is presumed to be invariant under $GL(n,\mathbb{Z})$. Roughly speaking, the topology on the adeles is given in terms of a basis of products of $GL(n,\mathbb{Z}_p)$ and finite index subgroups, which are related to congruence groups. Thus the adelic topology aligns with the invariance of F under matrices in a congruence subgroup.

Unifying these actions leads to the notion of adelic representations and adelized automorphic forms, where F, instead of being a function on $GL(n,\mathbb{R})$ alone, is padded with extra variables. Namely, we have an adelic function

$$F_{\mathbb{A}}(g_\infty\,;\, g_2\,, g_3\,, g_5\,\ldots) \tag{7.5}$$

such that almost all g_p lie in $GL(n,\mathbb{Z}_p)$, and

$$F_{\mathbb{A}}(g_\infty\,;\, g_2\,, g_3\,, g_5\,\ldots) \quad = \quad [\lambda(g_2)\,\lambda(g_3)\,\lambda(g_5)\,\cdots\lambda(g_p)\,\cdots F](g_\infty)\,, \tag{7.6}$$

where the number of $\lambda(g_p)$'s that act in an nontrivial way is finite, and their actions for various p's commute with each other. The adelized function F has the properties

$$F(g_\infty) \quad = \quad F_{\mathbb{A}}(g_\infty\,;\,1,1,1,\cdots) \qquad \text{and} \qquad F_{\mathbb{A}}(\gamma\,g) \quad = \quad F_{\mathbb{A}}(g)\,, \tag{7.7}$$

for any diagonally embedded rational matrix γ. The center Z and central character ω have analogous adelic versions, which are related to automorphic forms on $GL(1,\mathbb{A})$, in fact. The right regular representation now acts on $GL(n,\mathbb{A})$ by the formula (7.2), but note that this right action of the factor $GL(n,\mathbb{A}_f)$ is really a left action on $GL(n,\mathbb{R})$.

This leads us to our final version of automorphic representation: an irreducible subrepresentation of the action of the right regular representation ρ on $GL(n,\mathbb{A})$ on $L^2_\omega(Z_{\mathbb{A}}\,GL(n,\mathbb{Q})\backslash GL(n,\mathbb{A}))$. The constituents of these subspaces are generalizations of the automorphic forms we encountered previously. All have classical counterparts as functions with transformation properties for various congruence subgroups Γ of $GL(n,\mathbb{Z})$, but the adelic version provides a uniform framework. The role of Γ itself is replaced by right-invariance under finite index subgroups K'_f of $K_f = \prod_{p<\infty} GL(n,\mathbb{Z}_p)$; here $\Gamma = \{\gamma_{\mathbb{R}} \mid \gamma \in GL(n,\mathbb{Q}), \gamma_f \in K'_f\}$, where $\gamma_{\mathbb{R}}$ and γ_f denote the projections of $\gamma \in GL(n,\mathbb{A})$ to the factors $GL(n,\mathbb{R})$ and $GL(n,\mathbb{A}_f)$, respectively. We note that forms for various conjugate subgroups all fall into the same irreducible adelic automorphic representation, as do "newforms". More importantly, the theory of Hecke operators (Section 3.4) for powers of a prime p (e.g. the T_{p^k} from Section 3.4) can be recast as the study of the action of $GL(n,\mathbb{Q}_p)$, which allows the powerful representation theory of this p-adic group to be used. In general, the adelic framework unifies many constructions in automorphic forms and explains their effectiveness. Better yet, it provides insight for new constructions which would seem very difficult to uncover using only the classical perspective.

7.1. **Jacquet-Langlands (1970).** In 1970, a remarkable book was published: *Automorphic Forms on* $GL(2)$, by H. Jacquet and R. Langlands [72]. The irreducible unitary representations π of $GL(n,\mathbb{A})$ discussed above factor into restricted tensor products $\pi \cong \otimes_{p\le\infty}\pi_p$, where π_p is a "local representation" of $GL(n,\mathbb{Q}_p)$. One can treat the case of a number field, or even an arbitrary global field, in a similar way. For $n=2$, Jacquet and Langlands rephrase Hecke's theory from Section 3.1 using adelic machinery, much in the way Tate reworked Riemann and Hecke's classical arguments. In particular, they attach a global L-function $\Lambda(s,\pi)$ (a Dirichlet series times a product of gamma factors, such as the $\pi^{-s/2}\Gamma(s/2)$ that differentiates $\xi(s)$ from $\zeta(s)$) to each automorphic representation of $GL(2)$. They prove that $\Lambda(s,\pi)$ is "nice", meaning that it satisfies the standard properties of **E**ntirety, **B**oundedness in **V**ertical strips, and **F**unctional **E**quation that Hecke's method yields. Secondly they give a criteria for any "nice" L-function of this type to come from an automorphic representation, that is, a converse theorem.

Although the methods of group representations are new, the underlying technique of Jacquet-Langlands is fundamentally Hecke's method, as we shall briefly describe. However, neither the statement nor proof of their converse theorem is really Weil's Theorem 3.2. For example, let us return to the discussion concluding Section 3. Weil's proof of his converse theorem demonstrates that only a finite number of Dirichlet characters are required in his twisting hypothesis (ii). In fact, Piatetski-Shapiro [137], carefully examining this point, discovered an important simplifying feature in the early 1970s which has become one of the most important technical devices in today's applications. He found that Jacquet-Langlands' proof also requires only a finite number of twists by characters – but a completely disjoint set of characters from the ones Weil needed! For a classical treatment, see [144].

Recall how in Section 3.1 we considered the L-functions of modular forms for $SL(2,\mathbb{Z})$. The first example of a modular form whose L-function is entire is *Ramanujan's* Δ *form*

$$\Delta(z) \;=\; e^{2\pi i z}\prod_{n\ge1}(1-e^{2\pi i n z})^{24}\,, \tag{7.8}$$

which has weight $k=12$. (See [160] for a beautiful exposition of Δ in the context of Hecke theory.) Expand the product as $\Delta(z)=\sum_{n\ge1}\tau(n)e^{2\pi i n z}$ and normalize the coefficients by setting $a_n=\frac{\tau(n)}{n^{11/2}}$; in this normalization, the Ramanujan conjecture (established by Deligne [31]) can be stated uniformly as $|a_p|\le 2$ for all primes p. Ramanujan also conjectured[6] that the "standard" (i.e. Hecke) L-series associated to Δ has an *Euler product* over primes, much like ζ:

$$L(s,\Delta) \;=\; \sum_{n\ge1}a_n\,n^{-s} \;=\; \prod_p(1\,-\,a_p\,p^{-s}\,+\,p^{-2s})^{-1}. \tag{7.9}$$

This was proven by Mordell [119], and nowadays we understand the factorization as being equivalent to the assertion that Δ is an eigenfunction of the *Hecke operators* (3.14) from Section 3.4 – in particular, this is (3.15).

Let us now explain the connection between the arguments of Jacquet-Langlands and of Hecke. Our starting point is the Fourier expansion of a modular or Maass

[6]though not in the language of L-functions.

form $\phi(x+iy)$ in the variable x, in which it is periodic (with period 1 in the case of $SL(2,\mathbb{Z})$, as we shall now consider). Recall that if ϕ is a holomorphic cusp form of weight k,

$$\phi(z) \quad = \quad \sum_{n=1}^{\infty} c_n \, e^{2\,\pi\,i\,n\,z}. \tag{7.10}$$

Similarly for a Maass form we have the Fourier expansion (3.12). Up to constants (and a factor of $y^{k/2}$ in the holomorphic case, like in (7.1)), we may write these expansions as

$$\sum_{n\neq 0} \frac{a_n}{|n|^{1/2}} \, W(2\,\pi\,n\,y)\, e^{2\,\pi\,i\,n\,x}\,, \tag{7.11}$$

where as before $a_n = \frac{c_n}{n^{(k-1)/2}}$ in the holomorphic case, and

$$W(y) \quad = \quad \begin{cases} y^{k/2}\, e^{-y}\,, & \phi \text{ holomorphic,} \\ \sqrt{|y|}\, K_\nu(|y|)\,, & \phi \text{ a Maass form.} \end{cases} \tag{7.12}$$

In Section 7 we saw how both holomorphic and Maass forms can be viewed as functions on $GL(2,\mathbb{R})$. With this point of view we can write the corresponding function, up to constants, as

$$F(g) \quad = \quad \sum_{n\neq 0} \frac{a_n}{|n|^{1/2}} \, W\left(\begin{pmatrix} n & \\ & 1 \end{pmatrix} g \right), \tag{7.13}$$

where

$$W\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} k \right) \quad = \quad e^{2\,\pi\,i\,n\,x}\, W(2\,\pi\,y_1/y_2)\,. \tag{7.14}$$

(Here the matrix k on the lefthand side is orthogonal; all matrices in $GL(2)$ can be written in that form according to the *Iwasawa decomposition*.) This $W(g)$ is called a "Whittaker" function in connection with the special functions it is related to. It satisfies a transformation law on the left:

$$W\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g \right) \quad = \quad e^{2\,\pi\,i\,u}\, W(g)\,, \tag{7.15}$$

and thus can be obtained from the integral

$$W(g) \quad = \quad \int_0^1 F\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g \right) e^{-2\,\pi\,i\,u}\, du\,. \tag{7.16}$$

The adelic method of Jacquet and Langlands involves incorporating the coefficient a_n into a cognate Whittaker function which generalizes the properties (7.15) and (7.16). Consider now the adelized version $F_{\mathbb{A}}$ of F defined in (7.5)-(7.7), and define its adelic Whittaker function

$$W_{\mathbb{A}}(g_{\mathbb{A}}) \quad = \quad \int_{\mathbb{Q}\backslash\mathbb{A}} F_{\mathbb{A}}\left(\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} g_{\mathbb{A}} \right) \psi(-u)\, du\,, \quad g_{\mathbb{A}} \in GL(2,\mathbb{A})\,, \tag{7.17}$$

where ψ is a non-trivial character of $\mathbb{A}$ that is trivial on the subgroup $\mathbb{Q}$ (which we recall is diagonally embedded into $\mathbb{A}$). The measure du is normalized to give $\mathbb{Q}\backslash\mathbb{A}$ measure 1. The definition depends on the precise choice of character, but all non-trivial characters can be written as $\psi(qu)$ for some $q \in \mathbb{Q}^*$, and this q can be absorbed into $g_{\mathbb{A}}$ via the matrix $\begin{pmatrix} q & \\ & 1 \end{pmatrix}$; changing variables does not affect the measure since the adelic absolute value $|q|_{\mathbb{A}} = 1$. The result is that $F_{\mathbb{A}}$ can be reconstructed from $W_{\mathbb{A}}$ via the succinct formula

$$F_{\mathbb{A}}(g_{\mathbb{A}}) \;=\; \sum_{q\,\in\,\mathbb{Q}^*} W_{\mathbb{A}}\left(\begin{pmatrix} q & \\ & 1 \end{pmatrix} g_{\mathbb{A}}\right), \tag{7.18}$$

much like (7.13).

Now we shall make a tacit assumption that our original modular form is a Hecke eigenform (see Section 3.4). Our Whittaker function here, like many adelic functions, can be expressed as a product of *local* Whittaker functions W_p on $GL(2, \mathbb{Q}_p)$:

$$W_{\mathbb{A}}(g_{\mathbb{A}}) \;=\; \prod_{p\leq\infty} W_p(g_p)\,, \quad g_{\mathbb{A}} \;=\; (g_\infty\,;\, g_2\,, g_3\,, g_5\,, g_7\,\ldots)\,, \tag{7.19}$$

where each W_p obeys a transformation law similar to (7.15). In fact, just as with the Iwasawa decomposition in (7.14), the local Whittaker functions depend only on diagonal matrices, and actually their value there is related to the original Fourier coefficient by $W\begin{pmatrix} p^k & \\ & 1 \end{pmatrix} = a_{p^k}$. This last fact underlies the connection between (7.11) and (7.18): the extra adelic variables encode the value of the Fourier coefficients; these are very often zero, notably when $k < 0$ and the subscript is no longer an integer. This is why the sum over $\mathbb{Q}$, which appears to be much larger, actually corresponds to the sum over $\mathbb{Z}$.

Jacquet and Langlands use this theory beautifully to write the global L-function as

$$\Lambda(s) \;=\; \int_{\mathbb{Q}^*\backslash\mathbb{A}^*} F_{\mathbb{A}}\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|_{\mathbb{A}}^{s-1/2}\, d^*a\,. \tag{7.20}$$

This has a functional equation $s \mapsto 1 - s$, owing to the invariance of W under $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, just as in Hecke's argument. Now by substituting (7.18) and collapsing the common $\mathbb{Q}^*$ from the quotient and sum together ("unfolding"), the integral

$$\Lambda(s) \;=\; \int_{\mathbb{A}^*} W_{\mathbb{A}}\begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|_{\mathbb{A}}^{s-1/2}\, d^*a \;=\; \prod_{p\leq\infty} \int_{\mathbb{Q}_p^*} W_p\begin{pmatrix} a_p & \\ & 1 \end{pmatrix} |a_p|_p^{s-1/2}\, d^*a_p \tag{7.21}$$

splits as a product of local integrals. The ones for $p < \infty$ separately give the local factors of an Euler product which represents the Dirichlet series for Hecke's L-function $L(s)$, and the integral for $p = \infty$ gives the corresponding Γ-functions $L(s)$ must be multiplied by in order to have a clean functional equation. This is entirely analogous to the situation in Tate's thesis after (6.7), and the computations are deep down identically those needed for the classical treatment in Section 3. Details can be found in [12], [48], [72].

We should emphasize that the method is far more general and has strong advantages in its local precision, in that it gives a very satisfactory treatment of the contribution to the functional equation by each prime. Also the technique works for congruence subgroups, as well as over general global fields. Just as Tate's thesis understood Riemann's ζ-function in terms of $\mathbb{A}^* = GL(1, \mathbb{A})$, Jacquet-Langlands subsumed the theory of modular forms and their L-functions through $GL(2, \mathbb{A})$. Subsequently, efforts were under way to provide a similar theory for general groups, most notably $GL(n, \mathbb{A})$.

7.2. **Godement-Jacquet (1972).** Tate (see Section 6) redid Hecke (Section 2.2) by using adeles, developing a Poisson summation formula, and working with

$$\zeta(f,c) \quad = \quad \int_{\mathbb{A}^*} f(a)\, c(a)\, d^*a\,.$$

R. Godement and H. Jacquet [49] generalized Tate by working with $GL(n)$ for arbitrary n instead of $GL(1)$. In particular, they proved that the global, completed L-functions of automorphic forms on $GL(n)$ satisfy properties **E, BV** and **FE** of Section 1 (actually their integral representation for $GL(2)$ is completely different from Jacquet-Langlands' and Hecke's).

We shall not pursue this avenue here, but will briefly describe the L-functions of cusp forms on $GL(n)$. Recall how after (7.8) we renormalized the coefficients of Ramanujan's Δ-form by a factor of $n^{(k-1)/2}$. This can be carried out for any holomorphic cusp form f of weight k for $SL(2,\mathbb{Z})$, resulting in an Euler product of the same form as (7.9). Writing $a_p = \alpha_p + \alpha_p^{-1}$, the Euler product factors further as

$$L(s,f) \quad = \quad \prod_p (1-\alpha_p\, p^{-s})^{-1}\,(1-\alpha_p^{-1}p^{-s})^{-1}. \tag{7.22}$$

The preceding expression is called a *degree two* Euler product because of its two factors, in comparison with the degree one Euler product $\zeta(s) \;=\; \prod_p (1-p^{-s})^{-1}$.

The L-functions of cusp forms ϕ for $GL(n,\mathbb{A}_{\mathbb{Q}})$ are Euler products of degree n,

$$L(s,\phi) \quad = \quad \prod_p \prod_{j=1}^{n} (1 \,-\, \alpha_{p,j}\, p^{-s})^{-1}. \tag{7.23}$$

To form their global, completed L-functions, they must be multiplied by a product of n Γ-factors,

$$L_\infty(s,\phi) \quad = \quad \prod_{j=1}^{n} \Gamma_{\mathbb{R}}(s \,+\, \mu_j)\,, \tag{7.24}$$

where the μ_j are special complex parameters related to π (for example, the ν from Maass forms in (3.12)), and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ is again the factor which distinguishes $\xi(s)$ from $\zeta(s)$. The completed L-function,

$$\Lambda(s,\phi) \quad = \quad L_\infty(s,\phi)\; L(s,\phi) \tag{7.25}$$

is **E**ntire (unless $n=1$ and $\Lambda(s,\phi)=\zeta(s)$), **B**ounded in **V**ertical strips, and satisfies the **F**unctional **E**quation

$$\Lambda(s,\phi) \quad = \quad w\; Q^{1/2\,-\,s}\; \Lambda(1-s,\tilde{\phi})\,. \tag{7.26}$$

Here w is a complex number of modulus one, the "conductor" Q is a positive integer (related to the congruence subgroup ϕ comes from), and $\tilde{\phi}$ is the "contragredient" automorphic form to ϕ (coming from the automorphic representation dual to ϕ's). The notion of contragredient does not really rear its head in the previous topics we have covered, but is a feature of the more general functional equations.

7.3. **Jacquet-Piatetski-Shapiro-Shalika (1979).** Another proof of the analytic properties of the standard L-functions of cusp forms on $GL(n)$ is a generalization of Hecke's method (Section 3.1). In the 1970s Piatetski-Shapiro and Shalika [131], [169] independently developed their "Whittaker" expansions on $GL(n)$ in order to generalize the expansion (7.18) of Jacquet-Langlands. The Whittaker function on $GL(n, \mathbb{A}_{\mathbb{Q}})$ is given by the integral

(7.27)

$$W_{\mathbb{A}}\left(g_{\mathbb{A}}\right) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} F_{\mathbb{A}}\left(\left(\begin{smallmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ & 1 & u_{23} & \cdots & u_{2n} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & u_{n-1\,n} \\ & & & & 1 \end{smallmatrix}\right) g_{\mathbb{A}}\right) \overline{\psi(u_{12} + u_{23} + u_{n-1\,n})}\, du\,,$$

with the integration over the subgroup N of unit upper triangular matrices. The expansion of $F_{\mathbb{A}}$ in terms of $W_{\mathbb{A}}$ is given by

$$F_{\mathbb{A}}(g_{\mathbb{A}}) \quad = \sum_{\gamma \in N(\mathbb{Q})\backslash P(\mathbb{Q})} W_{\mathbb{A}}(\gamma\, g), \tag{7.28}$$

where P is the subgroup of G consisting of matrices whose bottom row is $(0\,0 \cdots 0\,1)$. This also has an explicit, classical version ("neoclassical", in the terminology of Jacquet), which can be found in [11], [71], both of which are excellent references for this section.

In a series of papers, Jacquet, Piatetski-Shapiro, and Shalika used these expansions to generalize Hecke's construction to L-functions of automorphic forms on $GL(n)$ (including properties **E, BV**, and **FE**) and prove a converse theorem for $GL(3)$ (see [10], [11], [71], [73], [74], and [115], [116] for a different treatment). This is a big advantage over the method of Godement-Jacquet, whose integral is over the large group $GL(n)$. The integrals of Jacquet, Piatetski-Shapiro, and Shalika instead involve integration over one-dimensional subgroups, matching the one complex variable of the L-functions $L(s)$. In later papers of Cogdell and Piatetski-Shapiro, a powerful converse theorem has been established for $GL(n)$ (see [25] and Section 9). These techniques lie close to the heart of the "Rankin-Selberg" method, which uses integral representations to generate a wide variety of the Langlands L-functions we will come to in Sections 8 and 9 (see [11] for a thorough, though slightly out of date, survey). While the statement of the converse theorem is quite technical, it is similar in form to Weil's Theorem 3.2, in that it involves the assumptions of **E**ntirety, **B**oundedness in **V**ertical strips, and **F**unctional **E**quation; as in the above proof of Hecke's theorem Theorem 3.1, these are used to shift a contour integral which reconstructs an automorphic form using Mellin inversion. However, an important difference is that their converse theorem typically involves twisting by automorphic forms on $GL(m)$, not merely Dirichlet characters (which, we have seen, correspond to automorphic forms on $GL(1)$). This is an important topic in Section 9, where we state a typical version in Theorem 9.1; a general account can be found in [20].

8. Langlands-Shahidi (1967-)

This section is meant for readers having some familiarity with Lie groups, but it can be skipped without loss of continuity. References include [3], [7], [44], [56], [118], [166], [168]. Our purpose here is to describe the general method of obtaining analytic properties of L-functions from Eisenstein series, generalizing Selberg's method for $\zeta(s)$ in Section 4.1. Many of the applications in Section 9 are based upon properties yielded by the Langlands-Shahidi method.

The theory of Eisenstein series was widened by Langlands to more general Lie groups in [95], [97]; in particular Langlands proved the analytic continuation and functional equations that were useful in Selberg's proof of the analytic properties of $\zeta(s)$. In his Yale monograph [96], Langlands considered the constant terms of the completely general Eisenstein series. This time, a wide variety of (generalized) L-functions appeared; his analysis gives their meromorphic continuation. The calculations involved are quite complicated and are performed adelically; they led Langlands to define the L-group and ultimately to the formulation of his functoriality conjectures.

Recall the example of $GL(2)$ from Section 4.1 (which we will reconsider through group representations and adeles in Section 8.2). There, analysis of the constant term and first Fourier coefficient already sufficed for the analytic continuation and functional equation of $\zeta(s)$ via Selberg's method. Langlands proposed studying the non-trivial Fourier coefficients in general, and Shahidi has now worked that theory out ([161], [162], [163], [164], [165], [166], [167], [168]) along with Kim and others. In general it has been a difficult challenge to prove the L-functions arising in the constant terms and Fourier coefficients are entire. The analytic continuation of Eisenstein series typically gives the meromorphic continuation to $\mathbb{C}$, except for a finite number of poles on the real axis between 0 and 1; these come from points where the Eisenstein series themselves are not known to be holomorphic. A recent breakthrough came with a clever observation of H. Kim: the residues of the Eisenstein series at these potential singularities are L^2, non-cuspidal automorphic forms, and – as in Section 7 – give rise to unitary representations that can be explicitly described by the Eisenstein series they came from. Kim remarked that results about the classification of irreducible unitary representations show that many of these potential representations do not exist, thus allowing one to conclude the holomorphy of the Eisenstein series at these points in question! When combined with [162], [164], this has recently led to new examples of entire L-functions (more on this in Section 9).

8.1. **An Outline of the Method.** The following is a brief sketch of the main points of the method; a fuller introduction with more definitions and detailed examples can be found in [166]. Detailed examples of constant term calculations can be found in many places, e.g. [44], [96], [100], [114]. Though it is possible to describe the method without adeles (as was done in Section 4.1), their use is key in higher rank for factoring infinite sums and product expansions into L-functions. Because the Langlands-Shahidi method utilizes various algebraic groups, we will have to assume some familiarity with the basic concepts. For this reason we include an example of the $GL(2)$ case in Section 8.2.

Let F be a global field, $\mathbb{A} = \mathbb{A}_F$ its ring of adeles, and G a split algebraic group over F. Much carries over to the quasi-split case as well, and we will highlight the technical changes needed for this at the end. Fix a Borel (= a maximal connected solvable) subgroup $B \subset G$, and a standard maximal parabolic $P \supset B$ defined over F.[7] Decompose $B = TU$, where T is a maximal torus. The parabolic can also be

[7] The theory has an extension to non-maximal parabolic subgroups, but this does not yield any extra information about L-functions. This matches the fact that the Eisenstein series for maximal parabolic subgroups depend on one complex variable, as do L-functions.

decomposed as $P = MN$, where the unipotent radical $N \subset U$, and M is the unique Levi component containing T. Denote by ${}^LG, {}^LM, {}^LN$, etc., the Langlands dual L-groups (see [166] for definitions).

One of the key aspects of this method is that it uses many possibilities of parabolics of different groups G, especially exceptional groups. This is simultaneously a strength (in that there is a wide range of exotic possibilities) and a limitation (in that there are only finitely many exceptional groups).

8.1.1. *Cuspidal Eisenstein Series.* Recall that an automorphic form in $L^2(\Gamma\backslash G)$ is associated to a (unitary) automorphic representation of G. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $M(\mathbb{A})$; we may assume that almost all components π_v are spherical unitary representations (meaning that they have a vector fixed by $G(O_v)$, where O_v is the ring of integers of the local field F_v). For these places v the equivalence class of the unitary representation π_v is determined by a semisimple conjugacy class $t_v \in {}^LG$, the L-group. This conjugacy class is used to define the L-functions below in (8.3). The finite number of exceptional places are where π *ramifies*.

A maximal parabolic subgroup P has a modulus character δ_P, which is the ratio of the Haar measures on $M \cdot N$ and $N \cdot M$. It is related to the simple root of G which does not identically vanish on P. For any automorphic form ϕ in the representation space of π, we can define the Eisenstein series

$$E(s, g, \phi) \quad = \sum_{\gamma \in P(F)\backslash G(F)} \phi(\gamma g)\, \delta_P(\gamma g)^s \tag{8.1}$$

and their constant terms

$$c(s, g, \phi) \quad = \quad \int_{N'(F)\backslash N'(\mathbb{A})} E(s, ng, \phi)\, dn\,, \tag{8.2}$$

where N' is the unipotent radical of the *opposite* parabolic P' to P (it is related by the longest element in the Weyl group). One can view the constant term as an automorphic form on M, and we will shortly relate it to ϕ and π. The measure dn is normalized to give the quotient $N'(F)\backslash N'(\mathbb{A})$ volume 1. The notion of constant term applies to any parabolic, but P' is the most useful one for our purposes.

8.1.2. *Langlands L-functions.* If ρ is a finite-dimensional complex representation of ${}^LM$, and S is a finite set including the archimedean and ramified places of F and π, then the partial Langlands L-function is

$$L_S(s, \pi, \rho) \quad = \quad \prod_{v \notin S} \det(I \, - \, \rho(t_v)\, q_v^{-s})^{-1}\,. \tag{8.3}$$

Here q_v is the cardinality of the residue field of F_v, a prime power. The full, *completed* L-function involves extra factors for the places in S, whose definition is technical and in general difficult. This is connected to the local Langlands correspondence, proven recently by Harris and Taylor for $GL(n)$ and by Jiang and Soudry for $SO(2n+1)$ (see [14], [57], [59], [62], [63], [65], [78], [101]). When ρ is the standard representation of ${}^LGL(n) = GL(n)$ and $F = \mathbb{Q}$, the Euler factors in (8.3) agree with those in (7.23); in general the degree of $L_S(s, \pi, \rho)$ equals the dimension of ρ.

8.1.3. *The Constant Term Formula.* The constant term formula involves the sum of two terms. The first, which only occurs when the parabolic P is its own opposite P', is $\phi(g)\delta_P(g)^s$ – simply the term in (8.1) for $\gamma =$ the identity matrix. Langlands showed that the map from ϕ to the second term is described by an operator

$$M(s,\pi) \quad = \quad \left(\prod_{j=1}^{m} \frac{L(a_j\, s\,,\, \tilde{\pi}\,,\, r_j)}{L(1\,+\,a_j\, s\,,\, \tilde{\pi}\,,\, r_j)} \right) \otimes_{v \in S} A(s, \pi_v)\,, \tag{8.4}$$

where the $A(s,\pi_v)$ are a finite collection of operators, r the adjoint action of ${}^LM$ on the lie algebra of ${}^LN$, $r_1, \ldots, r_m$ the irreducible representations it decomposes into, and a_j integers which are multiples of each other (coming from roots related to the r_j). The variety of decompositions of r is what gives this method much of its power for treating complicated L-functions. See [166] for a fuller discussion, along with an example for the Lie group G_2 and the symmetric cube L-function. Tables listing Lie groups and the representations r_j occurring for them can be found in [96] and [163], for example.

8.1.4. *The Non-Constant Term: Local Coefficients.* We must now make a further restriction on the choice of π involved, namely that it be *generic*, i.e. have a Whittaker model. This means that if ψ is a generic unitary character of $U(F)\backslash U(\mathbb{A})$, we need to require

$$W(g,\psi) \quad = \quad \int_{U_M(F)\backslash U_M(\mathbb{A})} \phi(ng)\, \overline{\psi(n)}\, dn \quad \neq \quad 0\,, \qquad U_M \ = \ U \cap M$$

for some ϕ and g (we have already seen this notion in (7.17) and (7.27)).

Shahidi's formula uses the Casselman-Shalika formula for Whittaker functions (see [18], [171]) to express the following non-constant term at the identity $g = e$ as

$$\int_{N'(F)\backslash N'(\mathbb{A})} E(s, ne, \phi)\, \overline{\psi(n)}\, dn \quad = \quad \prod_{j=1}^{m} \frac{1}{L(1\,+\,a_j\, s\,,\, \tilde{\pi}\,,\, r_j)} \cdot \prod_{v\in S} W_v(e)\,, \tag{8.5}$$

for a certain choice of ϕ. Applying the functional equation of the Eisenstein series (which has the constant-term ratio involved), one gets the "crude" functional equation for the product of m L-functions

$$\prod_{j=1}^{m} L_S(a_j\, s\,,\, \tilde{\pi}\,,\, r_j) \quad = \quad \prod_{j=1}^{m} L_S(1\,-\,a_j\, s\,,\, \pi\,,\, r_j) \cdot \prod_{v\in S} (\text{local factors})\;. \tag{8.6}$$

Shahidi's papers [162] and [164] match all the local factors above to the desired L-functions (cf. the remark after (8.3)). This gives the full functional equation for these m L-functions, but only when multiplied together. His 1990 paper [164] uses an induction argument to isolate the functional equation of each of the above m factors separately.

8.1.5. *Analytic Properties and the Quasi-Split Case.* It still remains to prove that the L-functions are entire, except perhaps at $s = 0$ and 1 (where the order of the poles is understood, like for $\xi(s)$). The theory of Eisenstein series provides this full analyticity for the L-functions arising in the constant term unless π satisfies a self-duality condition; even in this case, it can be shown that the L-functions have only a finite number of poles, all lying on the real axis between 0 and 1. Kim's observation of using the unitary dual has worked in many cases to eliminate this possibility. It is also always possible to remove the potential poles by twisting by a highly-ramified $GL(1)$ character of $\mathbb{A}_F$; this has been crucial for applications to functoriality through the Converse Theorem [23], [24], [25], which we come to in Section 9.

The main difference in the quasi-split case is that the action of the Galois group G_F is no longer trivial. The L-groups are potentially disconnected, inasmuch as they are semi-direct products of a connected component with G_F. Also, the representation ρ used to define the Langlands L-functions in (8.3) may also depend on the place v.

8.2. $GL(2)$ **Example.** Here we reconsider the ζ-function example from Section 4.1, but in the framework of the Langlands-Shahidi method. In this setting, the Eisenstein series on $G = GL(2)$ is defined by

$$E(s,g,f) \;=\; \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g)\,, \tag{8.7}$$

where $P = P' = B = \left\{ \begin{pmatrix} a & x \\ & b \end{pmatrix} \right\} \subset G$ is the Borel subgroup/minimal parabolic ($GL(2)$ is too small a group to afford other interesting choices). The Eisenstein series formed from π are related to the representations *induced* from π, from $M(\mathbb{A})$ to $G(\mathbb{A})$. In (8.7) we may absorb the factor δ_P^s into f by taking a vector in the *induced* representation $I(s) = Ind_{B(\mathbb{A})}^{G(\mathbb{A})}\,|a|^s$, which roughly speaking is the space of functions

$$\left\{ f : G(\mathbb{A}) \to \mathbb{C} \;\middle|\; f\left(\begin{pmatrix} a & x \\ & a^{-1} \end{pmatrix} g \right) = |a|^{s+1} f(g) \right\}.$$

The Eisenstein series (8.7) converges for $\mathrm{Re}(s)$ sufficiently large. In fact, we may choose our $f \in I(s)$ so that $E(s,g,f)$ reduces to just the classical Eisenstein series $E(z,\frac{1+s}{2})$ considered in Section 4.1. To do this, we take f to be identically 1 on $\widehat{K} = O(2,\mathbb{R}) \times \prod_{p<\infty} GL(2,\mathbb{Z}_p)$ and use the fact $B(\mathbb{Q})\backslash G(\mathbb{Q}) \simeq B(\mathbb{Z})\backslash G(\mathbb{Z})$;[8] this, in view of the Iwasawa decomposition $G = B\widehat{K}$, is the simplest choice. It corresponds to (8.7), taking π to be the trivial representation of $M(\mathbb{A})$. In general, Eisenstein series are always induced from automorphic forms on smaller groups, which in the example here is just the constant function on the factors $M(\mathbb{A}) = GL(1,\mathbb{A}) \times GL(1,\mathbb{A}) \subset B$.

To compute the constant term, we appeal to the Bruhat decomposition

$$\begin{gathered} G \;=\; B \sqcup\; BwB = B \sqcup\; BwN\,, \\ w \;=\; \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \quad , \qquad N = \left\{ \begin{pmatrix} 1 & \star \\ & 1 \end{pmatrix} \right\} \subset G \end{gathered} \tag{8.8}$$

[8]Classically speaking, this isomorphism comes from the decomposition of any rational matrix $g \in GL(2,\mathbb{Q})$ as $g = bu$, $b \in B(\mathbb{Q})$, $u \in GL(2,\mathbb{Z})$. The resulting indexing of $B(\mathbb{Z})\backslash G(\mathbb{Z})$ via rational matrices gives powerful insight into how to arrange the summands of the Eisenstein series into a computationally useful form.

which is valid over any field. When applied to $\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})$, it allows us to compute the constant term integral over N :

(8.9)
$$c(s,g,f) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(s,ng,f)\,dn = f(g) + \sum_{\gamma\in N(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f(w\,\gamma\,n\,g)\,dn = f(g) + [M(s)f](g)\,,$$

where $M(s)$ is the *intertwining operator*

$$M(s)f(g) \;=\; \int_{N(\mathbb{A})} f(w\,n\,g)\;dn \tag{8.10}$$

from $I(s)$ to $I(-s)$. If f is chosen to be a product $f(g) = \prod_p f_p(g_p)$, then the integral (8.10) factors further to give an Euler product, in analogy to (6.4) and (6.7). For the details of this example, see Langlands' article [100]; in general, his constant term method [96] gives similar integrals for constant terms over general groups. In any event, for our example here where f is trivial on $\widehat{K}$ (i.e. so that $E(s,f,g)$ recovers the classical Eisenstein series), $[M(s)f](e) = \frac{\xi(s)}{\xi(s+1)}$.

To complete our discussion let us again compute the ψ-th Fourier coefficient of $E(s,g,f)$, where ψ is a non-trivial additive character of $N(\mathbb{A})$ trivial on $N(\mathbb{Q})$ (or, equivalently, a non-trivial additive character of $\mathbb{A}$ trivial on $\mathbb{Q}$). Afterwards we will use the functional equation of $E(s,g,f)$ to get the functional equation of $\zeta(s)$.

We find

$$E_\psi(s,e,f) \;:=\; \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(s,n,f)\,\overline{\psi(n)}\;dn \;=\; \frac{c(s)}{\xi(s+1)}\,, \tag{8.11}$$

and

$$E_\psi(\,-s\,,\,e\,,\,M(s)f\,) \;=\; \frac{\xi(s)}{\xi(s+1)}\,\frac{c(-s)}{\xi(1-s)}\,,$$

where $c(s) = c(-s)$ is non-zero (it is related to the K-Bessel function appearing in (4.6)). So using the functional equation

$$E(s,\,e,\,f) \;=\; E(-s,\,e\,,M(s)f)\,,$$

it follows that

$$\xi(\,s\,) \;=\; \xi(\,1-s\,)\,.$$

8.3. **Boundedness in Vertical Strips and Non-vanishing.** We have just seen how the functional equation and several analytic properties of L-functions can be obtained via Langlands' analytic continuation of Eisenstein series, which itself relies on spectral theory. Through (8.11) and the known holomorphy of $E(s,f,g)$ on the line Re $s=0$ (which follows from the general spectral analysis), one also obtains a new proof of the famous result that $\zeta(s)$ never vanishes along the line Re $s=1$ (see [75], [120], [153], [161]). Among other things, this result is the key to the standard proof of the Prime Number Theorem − as was originally outlined by Riemann himself in [145]! In fact, this proof of the non-vanishing of L-functions on the line Re $s=1$ using Eisenstein series turns out to be the most general method available at present, in some cases working far inside the known range of absolute convergence of certain L-functions.

Intriguingly, it is possible to prove **B**oundedness in **V**ertical strips using the Langlands-Shahidi method. This is striking, because our other examples (Riemann, Hecke, and Tate, e.g. Theorem 2.1) all acquire **BV** through an integral representation of an L-function; here the L-function's analytic properties are obtained very indirectly. Not surprisingly, the argument is more round-about and subtle, but pays off in that it turns out – again – to extend to more general L-functions than treated by other methods alone (see [45], [153]). This has been very important for applications to the Langlands functoriality conjectures through the converse theorem (see Section 9 for further discussion).

Recall that our **BV** assertion is that $s(s-1)\xi(s)$ is bounded for s in any vertical strip. In our situation, the *fact* that

$$r(s) \;=\; \frac{\xi(s)}{\xi(s+1)}$$

satisfies the finite order estimate $O(e^{|s|^\rho})$ in Re $s \geq \frac{1}{2}$ is possible to prove using spectral theory. We recall that $\xi(s)$ satisfies that finite order inequality in the region Re $s \geq 3/2$; this is because $|\zeta(s)| \leq \sum_{n=1}^{\infty} n^{-3/2} = \zeta(3/2)$ is bounded there, and Stirling's formula (2.21) shows that $\Gamma(s/2) = O(e^{|s|^{1+\varepsilon}})$ there. Thus $\xi(s) = r(s)\xi(s+1)$ obeys the finite order inequality in Re $s \geq 1/2$, and the functional equation shows this is true for Re $s \leq 1/2$ as well – giving a new proof that $\xi(s)$ is of finite order. For ζ and the general L-functions considered in [45], additional Eisenstein series and group representations, along with some results of [56] and [123], are required. In particular, more is needed than mere meromorphicity of the Eisenstein series alone. One also needs an estimate on the growth rate of Eisenstein series in vertical strips going beyond the original work of Langlands, which here is not being used as simply a "black box". For a different, though related, method of obtaining **BV** and the non-vanishing of $\xi(s)$ on the line $\mathrm{Re}(s) = 1$ through Eisenstein series, see [151].

9. The Langlands Program (1970-)

Many articles have been addressed to the "Langlands program" (e.g. [1], [41] and the references in the introduction), and it is not our desire to add to these. However, one part of the program is closely connected to our discussion: namely, it "explains" why the analytic continuation and functional equation of the L-functions of automorphic forms on $GL(n)$ *probably* suffice to ensure that *any* L-series in arithmetic has an analytic continuation and functional equation!

9.1. The Converse Theorem of Cogdell-Piatetski-Shapiro (1999).

Let's see why: many arithmetic objects, such as elliptic curves, have an L-series attached to them which are conjectured to be entire and have functional equations similar to those possessed by our L-functions. Some very interesting examples, which we will not touch on here directly, involve the Artin conjecture (which involves L-series of Galois representations); see [80], [147].

For an elliptic curve E defined over the rational numbers, this L-series, called the *Hasse-Weil* L-function $L(s,E)$, is defined by counting points on E over varying finite fields. Here $1 + p - a_p$ is the number of points on the reduced curve modulo p, and $L(s,E)$ is defined by the Euler product in (3.15) with $k = 2$, except for a finite number of exceptional prime factors (see [172]). The resemblance to the L-functions of holomorphic modular forms of weight 2 is the springboard for the celebrated "Modularity Conjecture" of Taniyama, Shimura, and Weil, which was recently proven in [181], [175], and [8]. It asserts that $L(s,E) = L(s,f)$, the L-function of some holomorphic cusp form f of weight 2 on $\Gamma_0(N)$, where N is a subtle invariant (the "conductor") calculable from the arithmetic of the curve.[9] Since the L-functions of modular forms are known to be entire through Hecke's theory (Section 3.1), we therefore now know that the Hasse-Weil L-functions of rational elliptic curves are entire. Among other things, this gives a definition of $L(s,E)$ at the center of its critical strip, where the Birch-Swinnerton-Dyer conjecture asserts deep relations with arithmetic ([5], [54], [90]).

One might ask if the modularity of an elliptic curve might be proved using Weil's Converse Theorem 3.2. Unfortunately, this route requires one to know that the Hasse-Weil L-functions are entire beforehand, which at present seems far beyond reach. Thus the prospect of proving entirety and applying the Converse Theorem to these arithmetic L-functions seems to be begging the question. However, it is an interesting fact the Converse Theorem on $GL(3)$ (proven in 1979 by Jacquet, Piatetski-Shapiro, and Shalika) [73] *does* enter into the proof of the Modularity Conjecture, as it had been earlier used to establish a key step: the Langlands-Tunnell Theorem [99], [177].

Furthermore the Converse Theorem (in the form developed by Piatetski-Shapiro and Cogdell, e.g. [25], Theorem 9.1) has had remarkable success towards the Langlands Program in a different aspect. Roughly speaking, the Langlands conjectures assert correspondences between automorphic forms on different groups. When starting with an automorphic form, it is often possible to prove the **E**ntirety, **B**oundedness in **V**ertical strips, and **F**unctional **E**quations of the (twisted) L-functions the Converse Theorem requires. This has led to very significant progress, especially on liftings of automorphic forms on $GL(n)$ to $GL(m)$, $n < m$. In particular, the recent breakthroughs of Kim and Shahidi (cf. [21], [83], [85], [86], [87], and Section 8), and also of Lafforgue ([37], [92], [105]), have proven many important new examples of Langlands "Functoriality", by showing certain L-functions are entire and then appealing to the Converse Theorem of Cogdell and Piatetski-Shapiro ([25]). Unfortunately the precise statements connected to the use of the Converse Theorem are quite technical and complicated, and so we will just make do with a less technical (but still very useful) version of the Converse Theorem in Theorem 9.1. We will then summarize the main applications in Section 9.3. In short the basic idea, which can also be seen through the standard functoriality conjectures of Langlands, is the following: any Langlands L-function (8.3) – of any automorphic form, on any group, over any field – should itself be the L-function of an automorphic form on $GL(n, \mathbb{A}_\mathbb{Q})$. So $GL(n, \mathbb{A}_\mathbb{Q})$ is speculated to be the mother of all automorphic forms, and its offspring L-functions are already known to have an analytic continuation and functional equation.

9.2. **Examples of Langlands L-functions: Symmetric Powers.** To get the full statements of the Langlands conjectures, one requires even more than the definitions of the Langlands L-functions from (8.3). For simplicity and the benefit of readers who have skipped Section 8, we will explain the relevant L-functions here

[9] Weil's contribution [178] to the conjecture is closely related to Theorem 3.2: in fact his prediction of modularity on $\Gamma_0(N)$, N being the conductor of E, comes from a comparison of the expected functional equations of Hasse-Weil L-functions with the exact form of (3.11) in Theorem 3.2.

in the *everywhere-unramified* case, which corresponds to cusp forms on $GL(n,\mathbb{R})$ invariant under $GL(n,\mathbb{Z})$.

Recall the Euler product for the L-function of a modular form f from (7.22); this formula is also valid for the Maass forms from Section 3.3, of course provided they are eigenforms of all the Hecke operators T_p. To better highlight the symmetries involved (as well as to allow for more generality), let us introduce the parameter $\beta_p = \alpha_p^{-1}$ and rewrite (7.22) as

$$L(s,f) \;=\; \prod_p (1-\alpha_p\, p^{-s})^{-1}\,(1-\beta_p\, p^{-s})^{-1}. \tag{9.1}$$

Langlands has defined higher degree Euler products from $L(s,f)$ called the *symmetric k-th power* L-functions:

$$L(s, Sym^k f) \;=\; \prod_p \prod_{j=0}^{k} (1-\;\alpha_p^{\,j}\;\beta_p^{\,k-j}\;p^{-s})^{-1}. \tag{9.2}$$

The definition of the symmetric power L-functions is a general example of Langlands' method of creating new Dirichlet series out of Euler products. The major challenge, as we shall see, is to derive important analytic properties of these Dirichlet series and thereby put them on the same footing as the other L-functions we have come across. His formulation is in terms of finite dimensional representations, which in this case of $GL(2)$ is the $k+1$-dimensional representation on homogeneous polynomials of degree k. Other examples give rise to Euler products which are very symmetric, like this one on the righthand side of (9.2). In general, one starts by factoring the L-function of an automorphic form on $GL(n)$ as

$$L(s,\pi) \;=\; \prod_p \prod_{j=1}^{n} (1\;-\;\alpha_{p,j}\,p^{-s})^{-1}. \tag{9.3}$$

New Euler products may be taken using symmetric combinations of the $\alpha_{p,j}$ above (the individual $\alpha_{p,j}$ chiefly have meaning only in the context of their aggregate $\{\alpha_{p,j}\}_{1\le j\le n}$). In addition to the symmetric powers for $GL(2)$, there are symmetric and exterior powers for $GL(n)$:

$$L(s,\pi,Sym^k) \;=\; \prod_p \prod_{i_1\le i_2\le\cdots\le i_k} (1-\alpha_{p,i_1}\,\alpha_{p,i_2}\cdots\alpha_{p,i_k}\,p^{-s})^{-1} \tag{9.4}$$

$$L(s,\pi,Ext^k) \;=\; \prod_p \prod_{i_1< i_2<\cdots< i_k} (1-\alpha_{p,i_1}\,\alpha_{p,i_2}\cdots\alpha_{p,i_k}\,p^{-s})^{-1}\,. \tag{9.5}$$

Given another L-function on $GL(m)$

$$L(s,\pi') \;=\; \prod_p \prod_{k=1}^{m} (1\;-\;\beta_{p,k}\,p^{-s})^{-1}\,, \tag{9.6}$$

Langlands forms the "Rankin-Selberg" tensor product

$$L(s,\pi\otimes\pi') \;=\; \prod_p \prod_{j=1}^{n}\prod_{k=1}^{m} (1\;-\;\alpha_{p,j}\,\beta_{p,k}\,p^{-s})^{-1}\,, \tag{9.7}$$

in analogy with the classical constructions [143], [155] for $GL(2)$ (see [11]). There is a complementary theory for Γ-factors and completed, global Langlands L-functions as well. The general Langlands construction is in terms of finite dimensional representations of L-groups (Section 8.1.2); in particular, they can be repeated in various configurations. Now, thanks to the recent proof of the local Langlands correspondence by Harris and Taylor for $GL(n)$ [14], [57], [59], [62], [63], [65], [101], the definitions at the ramified places can be made also. Langlands' deep conjectures, in these cases, assert that each of the L-functions defined above is in fact the L-function of some automorphic form on some $GL(d)$, where d is the degree of the Euler product in each case (i.e. the number of factors occurring for each prime). Or, in other words, if his symmetric-looking Euler products look like the Euler product of an automorphic form as in (9.3), they probably are!

9.3. **Recent Examples of Langlands Functoriality (2000-).** To continue, we now wish to focus on the examples of Langlands' lifting mentioned above. We will describe various lifts which start with automorphic forms on $GL(n)$ and create automorphic forms on some $GL(m)$, $m > n$. Though many examples of Langlands functoriality are known in various types of cases, this class is very analytic and has largely been unapproachable without using the types of analytic properties of L-functions that we have come across in this paper. When considered as correspondences between eigenfunctions on one space and another, the lifts below are quite stunning theorems in harmonic analysis, made possible by a deep use of the arithmetic of $GL(n,\mathbb{Z})$.

Having explained the tensor product L-function (9.7), we can now state a version of the converse theorem (in practice, slightly weaker assumptions are often used, as well as analogs over different number fields):

Theorem 9.1. *($GL(n) \times GL(n-2)$ Converse Theorem – [25])*

Consider the Euler product $L(s,\pi)$ (9.3), and assume that it is convergent for $\mathrm{Re}\, s$ sufficiently large. Suppose that $L(s,\pi)$ along with all possible tensor product L-functions $L(s,\pi\otimes\tau)$, for τ an arbitrary cuspidal automorphic form on $GL(m,\mathbb{A}_{\mathbb{Q}})$, $1 \le m \le n-2$, satisfy **E***ntirety,* **B***oundedness in* **V***ertical strips, and the* **F***unctional* **E***quation. Then $L(s,\pi)$ is in fact the L-function of a cuspidal automorphic form on $GL(n,\mathbb{A}_{\mathbb{Q}})$.*

Of course, in this statement we have not described the global L-function (e.g. Γ-factors) whose analytic properties we are describing, but it is similar to the ones from Section 3.1. Needless to say, Theorem 9.1 is a generalization of Theorem 3.2. When $n = 3$, it is an earlier theorem of Jacquet-Piatetski-Shapiro-Shalika [73]. To use the Converse Theorem to establish lifting to $GL(n)$, one still needs to show that various tensor product L-functions obey the analytic conditions it requires. Such properties are themselves very difficult assertions in their own right, and progress has been hard won. We shall now describe the established lifts from $GL(n)$ to $GL(m)$ that were mentioned at the end of the last subsection.

The first such example is the symmetric square lift $Sym^2 : GL(2) \to GL(3)$, the so-called Gelbart-Jacquet lift [43]. Because it is the simplest of these to explain, we will spend a moment to go over how it is proved. A central role is played by Shimura's integral representation of the symmetric square L-function [170]; one obtains the necessary analytic properties of $L(s, Sym^2\pi\otimes\chi)$, where χ is a Dirichlet character (recall that these are automorphic forms on $GL(1)$). Then the Converse Theorem of [73] (i.e. Theorem 9.1 with $n = 3$) implies the existence of a cuspidal automorphic representation Π whose L-function $L(s,\Pi) = L(s, Sym^2\pi)$ – i.e., the Langlands functorial symmetric square lift from $GL(2)$ to $GL(3)$.

Examples on $GL(n)$ for $n \geq 4$ require the Cogdell-Piatetski-Shapiro versions of the Converse Theorem and are quite technical, even in description. Fortunately, many have been achieved in the last few years, mainly as a consequence of new analytic properties from the Langlands-Shahidi method (Section 8), mined from various configurations of parabolic subgroups in exceptional groups such as E_8. Here is a summary of the recent lifts:

Theorem 9.2. *The following instances of Langlands functoriality are known. That is, in each case there are automorphic forms on the target $GL(n)$ whose standard L-functions agree with the Langlands L-functions on the source group (cf. Section 9.2):*

- *Gelbart-Jacquet* [43]. $Sym^2 : GL(2) \to GL(3)$.
- *Ramakrishnan* [140]. *Tensor Product:* $GL(2) \times GL(2) \to GL(4)$.
- *Kim-Shahidi* [85], [87]. *Tensor product:* $GL(2) \times GL(3) \to GL(6)$.
- *Kim-Shahidi* [85], [87]. $Sym^3 : GL(2) \to GL(4)$.
- *Kim* [83], [85]. $Ext^2 : GL(4) \to GL(6)$ *weakly automorphic (bad at 2 and 3).*
- *Kim* [83], [85]. $Sym^4 : GL(2) \to GL(5)$.
- *Cogdell-Kim-Piatetski-Shapiro-Shahidi* [21], [22]*: Weak functoriality to $GL(n)$ for generic cusp forms on split classical groups.*

The notion of "weak" automorphy means that an automorphic form on the target $GL(n)$ exists whose L-function matches the desired Euler product – but *except* perhaps for a finite number of factors. Much more about these results can be found in these references and also the ICM lectures [26], [167]. Ramakrishnan's result used an integral representation for a triple-product L-function ([39], [58], [66], [133]), but can also now be proven using the Langlands-Shahidi method ([83]). The last example mentioned here is a lift from generic cuspidal automorphic forms on $SO(n)$ or $Sp(2n)$ to some $GL(m)$ (see [21], [22], [26], [167]). A differing "descent method" (i.e. studying the opposite direction of the lift) of Ginzburg, Rallis, and Soudry [47] (see also Jiang and Soudry [78]) can be used to establish the lifts of [21], [22] in strong form; in other words, the adjective "weak" can be removed from the last assertion of Theorem 9.2.

Of course Langlands' conjectures go far beyond these examples involving only $GL(n)$ over a number field. Other routes, through theta liftings (see [2], [106]), and the Arthur-Selberg trace formula (see [1], [42], [89]), have also provided many instances of Langlands Functoriality. In particular, the trace formula is in some sense the most successful when successful, in that it gives a very complete description and characterization of the lifts it treats. Nevertheless, the full force of Langlands' Conjectures seems absolutely beyond current technology (see [104] for intriguing comments by Langlands on the limitations of the trace formula). We shall not describe these nor the exact formulations of the Converse Theorem here, though we hope we have transmitted the flavor of the arguments and technical analytic properties such as **EBV** which have put these recent results within grasp.

9.4. **Applications to Number Theory (2001-).** The coefficients of modular forms on the complex upper half plane $\mathbb{H}$ play a fundamental role in many problems in number theory. For example, the coefficients of holomorphic modular forms

$\phi(z) = \sum_{n \geq 0} c_n \, e^{2\,\pi\,i\,n\,z}$ can be related to various counting problems, such as the number of ways to represent a number as a sum of squares, or the number of points on an elliptic curve (Section 9.1). The coefficients a_n of the non-holomorphic Maass forms in (3.12) are also related to number theory as well, ranging for example from Galois theory to the properties of Kloosterman sums $\sum_{x\bar{x} \equiv 1 \pmod{p}} e^{2\,\pi\,i\,\frac{ax+b\bar{x}}{p}}$ [50], [69], [70], [149], [158]. The sizes of the a_n and eigenvalue parameter ν, along with their distributions, are very important in many instances; in the remainder of this section, we will describe the role of the analytic properties of L-functions in gleaning some of this information.

9.4.1. *Progress towards the Ramanujan and Selberg conjectures.* Recall Ramanujan's Δ form, defined in (7.8). We mentioned that Ramanujan conjectured a bound on the normalized coefficients a_n of his Δ form, a bound which has a natural generalization to the coefficients of modular forms of any weight and to Maass forms as well.

Conjecture 9.3. (Ramanujan Conjecture) Let $\phi(x + iy)$ be either a holomorphic cusp form of weight k with Fourier coefficients $c_n = a_n n^{(k-1)/2}$ as in (7.10), or a Maass form with Fourier coefficients a_n as in (3.12). Then $a_n = O(|n|^{\varepsilon})$ for any $\varepsilon > 0$ (of course the implied constant in the O-notation here may depend on ε). When ϕ is a Hecke eigenform and a_1 is normalized to be 1, then equivalently $|a_p| \leq 2$.

This conjecture was proven in the holomorphic case by Deligne [31] in 1974, but remains open for Maass forms.

Years later after Ramanujan, Selberg made a separate conjecture about the size of the parameter ν that enters into the Fourier expansion of Maass forms. It is related to the Laplace eigenvalue by $\lambda = 1/4 - \nu^2$. Selberg conjectured

Conjecture 9.4. (Selberg, 1965 [158]) Let $\lambda > 0$ be the Laplace eigenvalue of a Maass form for $\Gamma \backslash \mathbb{H}$, where Γ is a congruence subgroup of $SL(2, \mathbb{Z})$. Then $\lambda \geq \frac{1}{4}$ (equivalently, ν is purely imaginary).

Selberg was originally motivated by questions involving cancellation in sums of Kloosterman sums, but his question is a deep one about the nature of the Riemann surfaces $\Gamma \backslash \mathbb{H}$. Their volumes grow to infinity as their index increases, and one would naively expect an accumulation of small Laplace eigenvalues. However, Selberg predicts a barrier at $\lambda = \frac{1}{4}$. This has implications for the geometry of $\Gamma \backslash \mathbb{H}$: intuitively, small eigenvalues are a measure of how close a surface is to being disconnected, since, after all, the multiplicity of the eigenvalue $\lambda = 0$ is the number of disconnected components. These ideas have played a crucial role in the development of *expander graphs*: discrete combinatorial networks which have relatively few edges connecting their vertices, but which are extremely difficult to disconnect by removing only a moderate number of edges. See [107], [108], [112], [124], [149], [150]. By the way, Maass forms with eigenvalue exactly equal to $\frac{1}{4}$ are known to exist, and in fact they will play a role later at the end of this section. So Selberg's conjecture, if true, is sharp!

Not long after Selberg's conjecture, Satake observed a unifying reformulation of both the Ramanujan and Selberg conjectures, in terms of representation theory (more specifically, tempered representations). Suppose ϕ is a Hecke eigenform. A key idea was the parametrization of the Hecke eigenvalues (which are also Fourier coefficients) a_p, for p prime, as $a_p = \alpha_p + \alpha_p^{-1}$, $\alpha_p \in \mathbb{C}$ (cf. (7.22)). This has meaning from the representation theory of the group $GL(2,\mathbb{Q}_p)$ and is analogous to the convention of writing the Laplace eigenvalue $\lambda = \frac{1}{4} - \nu^2$. The statement that $|a_p| \leq 2$ is equivalent to proscribing that the complex modulus satisfy $|\alpha_p| = 1$. Writing α_p as p^{μ_p}, the connection between Ramanujan's and Selberg's conjectures becomes even more clear: both ν and all $\mu_p = \log_p(\alpha_p)$ should be purely imaginary.

The generalized Ramanujan-Selberg conjecture asserts this phenomenon holds for $GL(n)$:

Conjecture 9.5. If π is a cusp form on $GL(n)$ which is unramified at the prime p, the quantities $\alpha_{p,j}$ appearing in the Euler product (9.3) obey $|\alpha_{p,j}| = 1$.

All but a finite number of primes are ramified for π, and none of them are when $\Gamma = GL(n,\mathbb{Z})$. A similar statement for the archimedean case $p = \infty$ is conjectured to be true for the parameters $\mu_{\infty,j}$, generalizing ν, that appear in the Γ-factors that multiply the L-function in its global, completed form (see (7.24)). For the cognoscenti we will note that the Ramanujan conjecture 9.5 has a statement in terms of representation theory which covers the ramified places as well: the associated local representations π_p of $GL(n,\mathbb{Q}_p)$ should all be *tempered*.

Though the generalized conjecture for $GL(n)$ is of course no easier than it was for $GL(2)$, this added perspective has been crucial for two reasons. The first is that we know a "trivial" or "local bound" coming from representation theory [76] that $|\text{Re } \mu_{p,j}| < 1/2$ for all places $p \leq \infty$. For $GL(2)$ this is quite trivial indeed: it states, for example, that the Laplace eigenvalue is merely positive and that the corresponding bound on the Hecke eigenvalue a_p comes directly from the boundedness of a cusp form. However, for $GL(n)$, $n > 2$, this bound actually becomes quite deep, due to a separation feature between the trivial and non-trivial unitary irreducible representations of $GL(n)$.

The second reason is that the Langlands program connects automorphic forms on different $GL(n)$'s, for example through symmetric powers. Notably, the factors $(1 - \alpha_p^{n-1} p^{-s})^{-1}$ and $(1 - \alpha_p^{1-n} p^{-s})^{-1}$ occur in the Euler product for the $n-1$-st symmetric power from $GL(2)$ to $GL(n)$ (formula (9.2)). If this symmetric power L-function was indeed the L-function of a cusp form, we would conclude that $p^{-1/2} < |\alpha_p^{n-1}| < p^{1/2}$ from the "trivial bounds" above. This gives an improved bound towards the Ramanujan-Selberg conjectures for any n for which the symmetric power lifting can be established – a bound which approaches the conjecture $|\alpha_p| = 1$ itself as $n \to \infty$. A similar magnification can be performed with the archimedean parameters $\mu_{\infty,j}$.

Thus the Langlands program (in particular, the symmetric power functorial liftings) implies both the Ramanujan and Selberg conjectures and their generalizations to $GL(n)$! It should be noted that this strategy is different from Deligne's and other arguments coming from algebraic geometry – which themselves have been successful for certain *cohomological* forms, but do not apply to Maass forms, for example. (Actually Deligne's argument uses the "magnification" mechanism of the previous paragraph, but in a different context.) Anytime a new lift is proven or a new bound on the $|\alpha_{p,j}|$ of cusp forms on $GL(n)$ is established, it results in a bound towards the Ramanujan and Selberg conjectures. The following bounds have been proven using results of [109], an analytic technique of [35], and the recent progress of Kim-Shahidi described in the previous subsection.

Theorem 9.6. *(Kim-Sarnak* [83, Appendix 2]*) If π is a cusp form on $GL(2,\mathbb{A}_{\mathbb{Q}})$, then*

$$p^{-7/64} \;\leq\; |\alpha_p| \;\leq\; p^{7/64}, \qquad \text{if } \pi \text{ is unramified at } p, \tag{9.8}$$

and

$$\lambda \;\geq\; \frac{975}{4096} \;=\; \frac{1}{4} - \left(\frac{7}{64}\right)^2 \;\approx\; .238037\,, \qquad \text{if } \pi \text{ comes from a Maass form.} \tag{9.9}$$

This theorem is for $\mathbb{Q}$, but results are also possible over general number fields. A weaker estimate (with $\frac{7}{64}$ replaced by $\frac{1}{9}$) is established by Kim-Shahidi in [86], using their recent progress and techniques from [163].

9.4.2. *The distribution of the Hecke eigenvalues, Sato-Tate.* Having seen that the Hecke eigenvalue parameters α_p for a modular $GL(2)$ Hecke eigenform should lie on the unit circle in the complex plane, we now turn to their distribution over this circle as p varies. The question has its origin in conjectures and investigations made independently by Sato and Tate [173] for the a_p of rational elliptic curves (Section 9.1). Namely, if we consider the phase of α_p, i.e. the angle θ_p such that $a_p = 2\cos\theta_p$, the $\theta_p \in [0,\pi]$ should be equidistributed with respect to the measure $\frac{2}{\pi}\sin^2\theta\,d\theta$. This means that

$$\lim_{X\to\infty} \frac{\#\{\,\alpha < \theta_p < \beta \;|\; p \leq X\}}{\#\{\,p \leq X\}} \;=\; \int_\alpha^\beta \left[\frac{2}{\pi}\sin^2\theta\right] d\theta\,; \tag{9.10}$$

when viewed in terms of the a_p themselves, the conjecture states that a histogram of the a_p is governed by the distribution $\frac{1}{2\pi}\sqrt{4-x^2}$, which looks like a semi-circle (really, semi-ellipse) between -2 and 2 of area 1. The Sato-Tate semi-circle measure occurs in many contexts; here it is related to the Weyl integration formula, which weighs the relative sizes of conjugacy classes in $SL(2,\mathbb{R})$. This conjecture is not meant to be valid for *all* elliptic curves (nor, by extension, to all modular forms via Wiles et al.), but instead only for the "typical" (i.e. non-CM) elliptic curve. In the other cases, the distribution is much simpler and the desired results are known (see [159]). Regardless, the Sato-Tate conjecture is expected to also hold for most modular and Maass forms, as we shall see shortly. See [98, p. 210] for a generalization to $GL(n)$.

As in nearly all distributional questions in number theory, an equivalent formulation of the Sato-Tate conjecture can be made in terms of the *moments*

$$S_m(X) \;:=\; \sum_{p\leq X} a_p^m\,. \tag{9.11}$$

Conjecturally $S_m(X)/\pi(X)$ should tend to the constant

$$\lim_{X\to\infty} \frac{S_m(X)}{\pi(X)} \;=\; \frac{1}{2\,\pi}\int_{-2}^{2} x^m\,\sqrt{4-x^2}\,dx \tag{9.12}$$

($\pi(X)$, as in the introduction, refers to the number of primes $p \leq X$). In fact, the truth of (9.12) for all $m \geq 0$ implies the Sato-Tate conjecture (9.10).

In view of the connection with the powers of $a_p = \alpha_p + \alpha_p^{-1}$, and symmetric power L-functions in the previous subsection, it is not surprising that symmetric power L-functions play a role in the Sato-Tate conjecture as well. In fact, the non-vanishing and holomorphy of the m-th symmetric power L-function $L(s, Sym^m\,\pi)$ in the region Re $s \geq 1$ implies the m-th moment (9.12) (see [159], [165], and also Ogg's paper [126], which shows that the holomorphy actually implies non-vanishing). Kim and Shahidi have now established this for $m \leq 9$ ([86]). Actually the non-vanishing of $\zeta(s)$ in the region Re $s \geq 1$ is essentially what Riemann observed implies the prime number theorem $\pi(X) = \sum_{p \leq X} 1 \sim X/\log X$, so it is natural to see this analytic condition appear in a counting problem. This is a typical way exotic L-functions enter into analytic number theory.

Though the formulation of (9.10) here implicitly assumed the Ramanujan conjectures, (9.12) is more general. It can be viewed as saying that the Ramanujan conjecture is true on average – and much more. We note in passing that various on-average results can be proven using the theory of L-functions. The Rankin-Selberg method [143], [155] has its origin in this issue for $GL(2)$; the generalization of the Rankin-Selberg method to $GL(n)$ ([74], [162]) also gives a weaker on-average version of Ramanujan. One can also give a relatively large lower bound on the percentage of primes p such that the Ramanujan conjecture holds for p ([86], [139], [141]).

Finally, we conclude by describing a result of Sarnak [152]. We had mentioned before that the Maass forms for $SL(2,\mathbb{Z})\backslash\mathbb{H}$ – the non-holomorphic L^2-eigenfunctions of the Laplacian – are quite mysterious in nature, and none has been explicitly described. However Maass, in his original paper [111], constructed some examples for $\Gamma\backslash\mathbb{H}$, where Γ is a congruence subgroup of $SL(2,\mathbb{Z})$. These, and generalizations coming from Galois theory through the Artin conjecture, are very special types of Maass forms and come from algebraic constructions. In particular, several give fascinating examples of Maass forms (3.12) whose coefficients a_n are relatively small integers – bounded in absolute value by the number of divisors of n. This is remarkable because of the discreteness and limitation of the possible coefficients. In [152] Sarnak considers hypothetical Maass forms with integral coefficients that are *not* examples of the known constructions from Galois theory. In these cases, the results of Kim and Shahidi [86] on the non-vanishing and holomorphy of $L(s, Sym^m\pi)$ on the line Re $s = 1$ give the asymptotics of the m-th moment, i.e. (9.12), for $m \leq 9$. If the coefficients are indeed integral, the Ramanujan conjecture asserts that the a_p should assume only one of the five values $\{-2,-1,0,1,2\}$. This constraint makes it difficult to match the predicted moments, and in fact with $m = 6$ it is possible to show the impossibility of all the a_p being integral. Indeed, even without the Ramanujan conjecture, the assumption that all $a_p \in \mathbb{Z}$ can be ruled out simply by taking linear combinations of (9.12) and concluding that

$$(9.13) \qquad \lim_{X\to\infty} \frac{1}{\pi(X)} \sum_{p\leq X} P(a_p) \;=\; \frac{1}{2\pi}\int_{-2}^{2} P(x)\sqrt{4-x^2}\,dx \;=\; 1\,.$$

Here $P(x) = x^2(4-x^2)(x^2-1)$, a sixth degree polynomial which vanishes at the integers $\{-2,-1,0,1,2\}$, and is negative at all others; a contradiction arises because the righthand side is positive. As a result, one obtains the first algebraicity result in the subject of Maass forms:

Theorem 9.7. *(Sarnak* [152]*) Let ϕ be a Maass form for $\Gamma\backslash\mathbb{H}$ as in* (3.12) *with integral coefficients. Assume ϕ is a Hecke eigenform. Then ϕ arises from a Galois representation, and in particular the Laplace eigenvalue of ϕ is $\frac{1}{4}$ (i.e. $\nu = 0$).*

A generalization has been established by Brumley [9]. We leave the reader with some open conjectures – both widely believed to be true and supported by numerical evidence – on which the ideas of functoriality and L-functions have brought an interesting perspective.

Conjecture 9.8. (See [17]) Let ϕ be a Maass form which has Laplace eigenvalue $\frac{1}{4}$. Does ϕ necessarily arise from a Galois representation?

Conjecture 9.9. (Cartier [15]) Is the Laplace spectrum of Maass forms for $SL(2,\mathbb{Z})\backslash\mathbb{H}$ simple? In other words, can there be two linearly independent Maass forms on $SL(2,\mathbb{Z})\backslash\mathbb{H}$ sharing the same eigenvalue?

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