BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 1, Pages 121–125
S 0273-0979(03)00997-2
Article electronically published on October 2, 2003

Classical and celestial mechanics, the Recife lectures, Hildeberto Cabral and Florin Diacu (Editors), Princeton Univ. Press, Princeton, NJ, 2002, xviii+385 pp., \$49.50, ISBN 0-691-05022-8

The photographs of Recife which are scattered throughout this book reveal an eclectic mix of old colonial buildings and sleek, modern towers. A clear hot sun shines alike on sixteenth century churches and glistening yachts riding the tides of the harbor. The city of 1.5 million inhabitants is the capital of the Pernambuco state in northeastern Brazil and home of the Federal University of Pernambuco, where the lectures which comprise the body of the book were delivered. Each lecturer presented a focused mini-course on some aspect of contemporary classical mechanics research at a level accessible to graduate students and later provided a written version for the book. The lectures are as varied as their authors. Taken together they constitute an album of snapshots of an old but beautiful subject.

Perhaps the mathematical study of mechanics should also be dated to the sixteenth century, when Galileo discovered the principle of inertia and the laws governing the motion of falling bodies. It took the genius of Newton to provide a mathematical formulation of general principles of mechanics valid for systems as diverse as spinning tops, tidal waves and planets. Subsequently, the attempt to work out the consequences of these principles in specific examples served as a catalyst for the development of the modern theory of differential equations and dynamical systems. Part of the tradition of the subject is that the examples themselves are given center stage. Each mechanical system has special features which give it its unique character. Such features are often exceedingly interesting and beautiful, but can easily be missed if one approaches the system as a mere special case of a general theory. The Recife lectures reflect this spirit.

For example, Alain Albouy provides a fascinating account of the classical two-body problem of celestial mechanics. The problem could be treated as a simple case of reduction of a Hamiltonian system with symmetry [5], [7]. For motion in the plane, the relative position of the bodies is described by a vector $x \in \mathbf{R}^2 \setminus 0$, so it is a system of two degrees of freedom. Taking into account the corresponding velocity variables (or rather, from the Hamiltonian viewpoint, the momenta), one finds that the phase space is the cotangent bundle $T^*(\mathbf{R}^2 \setminus 0)$, a four-dimensional symplectic manifold. The rotation group SO(2) acts as a symmetry group, and one expects by general theory that one can use this symmetry to reduce to a system of one degree of freedom. Using the conservation of energy, it is possible to explicitly solve such a reduced system and then to recover the solutions on the original four-dimensional phase space. In a typical problem of this kind, the result is a foliation into two-dimensional invariant tori which support quasi-periodic motions. But this general analysis makes no use of the special Newtonian force law; it is equally valid for any two-body interaction with SO(2) symmetry. For the Newtonian $1/r^2$ force law, a miracle occurs — all of the solutions are periodic instead of just quasi-periodic. To put it another way, the two-dimensional tori are further decomposed into invariant circles. This highly degenerate situation seems unbelievable from the point of view

2000 *Mathematics Subject Classification.* Primary 70-06; Secondary 37N10, 70F15, 70Hxx.

of general theory, yet it is the most interesting feature of the problem. Without it, the familiar Keplerian ellipses which provide the first approximations for the motions of the planets around the sun would precess. Albouy describes several "non-Hamiltonian" explanations for the miracle. Other remarkable but atypical features of the problem receive a similarly unconventional treatment.

Many of the other lectures also present topics from celestial mechanics. In the book's first article, Dieter Schmidt describes some classical and modern results on central configurations. It turns out that although the n-body problem for $n > 2$ cannot be solved by reduction, the symmetry of the problem still gives rise to certain special solutions for which the individual masses move on non-precessing elliptical orbits as in the two-body case, while the geometrical configuration formed by the n points always remains similar to the initial configuration. An initial configuration for which such motions are possible is called a *central configuration*. For example, in the rather artificial case where all n masses are equal, it is easy to see that the regular n-gon is a central configuration. One can also put a not-necessarily-equal mass at the center to obtain a central configuration of $n + 1$ bodies. For general masses, it is not immediately obvious that any central configurations exist. For a configuration to be central, the positions of the bodies must solve a complicated system of algebraic equations with the masses appearing as parameters. It can be shown that the system does indeed have solutions for any choice of masses, but determining the number of solutions and describing their possible shapes is a difficult problem which is a topic of current research interest. Lagrange worked out the three-body case in the eighteenth century [4]. Rather surprisingly, the equilateral triangle turns out to be central even for non-equal masses, and, in addition, there are three collinear central configurations whose shape depends on the masses. An elegant formulation of the algebraic equations using the mutual distances of the bodies as coordinates was introduced by Otto Dziobek in 1900 [3]. This formulation has proved especially useful for $n = 4, 5$, and Schmidt uses it to derive several surprising geometrical constraints on the possible shapes that central configurations can have. For example, if a convex quadrilateral is a central configuration (for some choice of four positive masses), then the ratio of the diagonals of the quadrilateral is bounded by $\sqrt{3}$ and each diagonal is longer than all four exterior edges. In a later lecture he studies the problem of bifurcation of the n-gon with a central mass, m. As m is varied, families of central configurations with less symmetry bifurcate from the "centered n-gon". Calculating the bifurcation values and studying the nature of the bifurcations require some symbolic computation.

The lectures of Hildeberto Cabral treat the problem of stability of Lagrange's equilateral triangle solutions as an application of KAM theory and the Birkhoff normal form. The planar circular restricted three-body problem (one mass is assumed to be zero and the other two move on circular orbits of the two-body problem) is a Hamiltonian system of two degrees of freedom. In a rotating coordinate system, the two non-zero masses are fixed, and one tries to study the motion of the third mass in the plane. No further reductions are possible and the problem is highly non-trivial. The positions for which the three masses form an equilateral triangle are equilibria in the rotating coordinate system — the net gravitational attraction of the third mass by the two non-zero masses is exactly balanced by the centrifugal force of the rotation. The problem is to determine the behavior of solutions near the equilibrium. Cabral uses this question as the motivation for a general discussion of the stability problem for equilibria in Hamiltonian systems. After giving a brief introduction to Hamiltonian formalism, he discusses linear stability and develops the Birkhoff normal form theory using the method of Lie series. The problem of non-linear stability for Hamiltonian systems is notoriously difficult, but in the two-degree of freedom case, one can attack the problem using KAM theory or, more precisely, the invariant curve theorem [8], [9]. Although the phase spaces of such systems are four-dimensional, the manifolds of constant energy have dimension three, and a Poincaré section to the flow on such a manifold has dimension two. The existence of invariant curves in an appropriately constructed section prevents orbits from drifting away from the equilibrium, thereby proving stability. In addition to presenting the necessary theoretical background, Cabral describes how to carry out this program for the equilateral triangle points. The Birkhoff normal form near the equilibrium is needed to check the hypotheses of the invariant curve theorem for the Poincaré section. Cabral, by the way, is also one of the editors of the book and the initiator of the lecture series in Recife.

Of course not all mechanics is celestial, and several lectures in the book are devoted to other kinds of systems. Mark Levi's lectures cover a number of geometrically oriented themes. First he describes the optical-mechanical analogy [2, Ch.9]. Light rays and particle trajectories both obey variational principles: Fermat's principle of least time and the principle of least action, respectively. In both cases, there is an alternative description in terms of wave fronts, leading to first-order PDE's. Levi shows how to go back and forth between the wave and ray viewpoints and also explains the underlying unity of the optical and mechanical worlds. In a later section he turns to the problem of *geometric phase*. A familiar example occurs in the rotation of the free rigid body. Imagine a football in flight. The motion of the ball relative to its center of mass can be understood as a combination spinning around the symmetry axis and a precession of this axis in space. The total angular momentum vector is constant throughout the motion and so determines a direction of reference in space; this is the axis around which the symmetry axis precesses. To study the relation between these two motions one can ask how much precession around the angular momentum vector takes place during one complete rotation of the ball around the symmetry axis. The formula for the change in precession angle turns out to have a "dynamical" term involving the component of angular velocity along the direction of the angular momentum and another less obvious "geometrical" term which can be interpreted in several intriguing ways. One approach is to view the motion from the point of view of an observer rotating along with the football. Then the angular momentum vector appears to move around the axis of symmetry. It always maintains the same length, however, so as the ball spins once the angular momentum vector sweeps out a closed curve on a sphere. Using some elementary differential geometry, Levi shows that the second term in the formula for the change of precession angle is just the area on the sphere enclosed by this curve. He also gives amusing, intuitive explanations for other geometric phase problems. Among other things, the reader will find out how to use old bicycle wheels to measure the area of spherical regions and to prove the Gauss-Bonnet theorem!

The theme of geometric phase is taken up again in several lectures by Jair Koiller (and collaborators). First he considers the problem of adiabatic phases, that is, changes in the angles for systems with slowly varying parameters. A good example is the Foucault pendulum, which can be viewed as a spherical pendulum under the influence of a slowly changing gravitational force. One can ask how much the pendulum rotates around the axis determined by gravity in one period of its natural oscillation. In this case the dynamical term in the answer is zero — if the earth were not rotating, the pendulum would oscillate in a fixed vertical plane. The small geometrical phase term is what causes the gradual evolution of the plane of oscillation. One can abstract this example to get a general setting for which geometric phase is the dominant effect. Suppose a certain system has a symmetry axis, like the axis of the gravitational field for the Foucault pendulum. There will be an associated angular momentum around the axis which is a constant of motion. After fixing a value of this constant, one can form a reduced system on a quotient space of the original phase space. Given that the reduced system exhibits a periodic motion (one oscillation of the pendulum), one can try to reconstruct the corresponding change in the angle around the symmetry axis. Normally there will be two contributions to this change: a dynamical one involving the value of the angular momentum constant and another contribution reflecting the geometry of the quotient map. If the angular momentum term happens to vanish, as in the Foucault case, the angle will be determined by the geometric contribution. This setup can be generalized to more complex symmetry groups [6]. The quotient map will determine a fiber bundle with the symmetry group as the fiber. Given a periodic motion in the quotient space (the base space of the bundle), one finds that the problem is to reconstruct what is happening in the fiber. One can generalize the idea of angular momentum to this setting, and in the zero momentum case, the geometry of the bundle will determine the behavior. Koiller describes several interesting problems which fit into this scheme. A familiar example is the falling cat problem. The cat is released upside down with zero angular momentum and as it falls, the angular momentum remains zero. Yet the cat is able to rotate itself by changing shape (perhaps in a periodic way). Similarly a bacterium initially at rest in a fluid medium is able to translate itself through the medium by changing shape periodically. Koiller describes an ongoing research program to explain such "microswimming" using the theory of geometric phase.

The rest of the lectures will just be briefly noted. Florin Diacu, the book's other editor, contributes an accessible survey on singularities of the n–body problem which includes proofs of many of the classical results, some simple examples of McGehee's technique for "blowing-up" collisions and references to the recent work on non-collision singularities. Ernesto Pérez-Chavela describes Poincaré's compactification method and uses it to study motion near infinity in the Kepler and Hill problems. There is a second article by Dieter Schmidt, this time on the calculations involved in approximating the motion of the moon. Finally, Jack Hale and Plácido Táboas present a functional analytic approach to some problems in bifurcation theory.

To summarize, this is a nice guidebook to some of the interesting sights in contemporary mechanics research. It would be a rewarding book for browsing, but a typical reader will probably be looking for an introduction to one of the chosen topics. So it is especially recommended as a reference. The Recife lecture series continues; as more Recife lectures are delivered and written up, perhaps further volumes of this kind will appear.

References

1. R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd ed., Benjamin/Cummings, Reading, MA (1978). MR **81e:**58025
2. V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York (1989). MR **90c:**58046
3. O. Dziobek, *Über einen merkwürdigen Fall des Vielkörperproblems*, Astron. Nach. **152** (1900), 33–46.
4. J.L. Lagrange, *Œuvres*, vol. 6, Gauthier-Villars, Paris (1873), 272.
5. K. Meyer, *Symmetries and integrals in mechanics*, in Dynamical Systems, ed. M. Peixoto, Academic Press, New York (1973), 259–272. MR **48:**9760
6. J.E. Marsden, R. Montgomery, T. Ratiu, *Reduction, symmetry and phases in mechanics*, Memoirs AMS, **88** (1990). MR **91b:**58074
7. J.E. Marsden, A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys., **5** (1974), 121–130. MR **53:**6633
8. J.K. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nach. Akad. Wiss. Göttingen Math-Phys. Kl. (1962), 1–10. MR **26:**5255
9. C.L. Siegel and J.K. Moser, *Lectures on Celestial Mechanics*, Springer-Verlag, Berlin (1971), 225–250. MR **58:**19464
10. A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton Math. Series 5, Princeton University Press, Princeton, NJ (1941). MR **3:**215b

Richard Moeckel
University of Minnesota
E-mail address: rick@math.umn.edu