BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 2, Pages 253–256
S 0273-0979(03)01003-6
Article electronically published on December 16, 2003

Subgroup growth, by Alexander Lubotzky and Dan Segal, Birkhäuser, Basel, 2003, 476 pp., $148.00, ISBN 3-7643-6989-2

The book under review is one of the first books on Asymptotic Group Theory—a new, quickly developing direction in modern mathematics which has links to many topics in Algebra, Analysis, Probability and Discrete Mathematics.

Typically the subject of Asymptotic Group Theory is the study of the type of growth of various functions involving a natural parameter related to the group. Among the most important functions of this nature are the word growth and the subgroup growth functions, along with various modifications. While the word growth function $\gamma_G(n)$ counts the number of elements of length no greater than n in the group, the subgroup growth function $s_G(n)$ counts the number of subgroups of index no greater than n in G.

From a group-theoretical point of view, the natural questions are:

1. What are the general features of growth functions?

2. Which algebraic features of the group are reflected in the growth function?

More broadly one might ask:

3. What are the applications of growth functions and what is their connection to other topics in mathematics and science?

A number theorist might add to the above list:

4. What are the arithmetic properties of the sequences $\{s_G(n)\}_{n=1}^{\infty}$, $\{\gamma_G(n)\}_{n=1}^{\infty}$, or any other growth function associated to a group G?

In addition, Logicians, Computer Scientists and Geometers might ask questions that relate the growth to their own areas.

The book of A. Lubotzky and D. Segal, leading specialists in group theory, answers these questions in a beautiful way in the context of subgroup growth.

The subject was started about 20 years ago by the pioneering efforts and articles of F. Grunewald, A. Lubotzky, A. Mann, A. Shalev, D. Segal, and others, although some sporadic results, such as the recursive formula of M. Hall [Hal49] from 1949 for $s_F(n)$ in a free group F, appeared much earlier. During the last 20 years the subject of subgroup growth went through a period of tremendous development, as many mathematicians were involved in the process. As a result we now have a powerful theory, with new methods and tools, which has strong connections to many topics in Group Theory and Algebra in general, such as finite groups (including the classification of simple groups), nilpotent groups, solvable groups, groups acting on p-adic trees (including rooted trees), profinite groups, lattices in Lie groups and in algebraic groups, associative and Lie graded algebras, probability (especially random generation), number theory and p-adic analysis (zeta functions), and many others.

After such a successful period, it was natural to expect a text on the subject that would summarize the achievements in the field, and we are very lucky to witness the appearance of this wonderful book. Before describing the contents of this book,

2000 *Mathematics Subject Classification.* Primary 20E07.
Key words and phrases. Subgroup growth.

let me stop for a moment to describe some of the most fascinating results in the area.

The subgroup growth of a finitely generated infinite group cannot be slower than polynomial. For example, for every positive integer m, $\mathbb{Z}$ has exactly one subgroup of index m, and therefore $s_{\mathbb{Z}}(n) = n$. On the other end of the spectrum, the subgroup growth of a finitely generated infinite group cannot be faster than the growth of type $n!$ (as in the case of a free group of rank greater than 1). So the first natural question is to characterize the groups that exhibit such extreme asymptotic behavior. A fundamental result of Lubotzky, Mann and Segal [LMS93], known as the Polynomial Subgroup Growth Theorem, characterizes the finitely generated groups of polynomial subgroup growth as the groups for which $G/R(G)$ is virtually solvable of finite rank, where $R(G)$ denotes the intersection of all finite index subgroups in G. In fact, since the numbers $s_G(n)$ only relate to groups above $R(G)$, it is a standard practice to assume $R(G) = 1$ when one studies subgroup growth; i.e., G is considered to be residually finite. The class of residually finite groups is one of the most important classes in Group Theory, and many basic problems in Algebra (such as the Restricted Burnside Problem solved by E. Zelmanov [Zel90], [Zel91]) naturally arise in or reduce to this class.

The proof of the Polynomial Subgroup Growth Theorem shares some features with another remarkable result in Asymptotic Group Theory, namely the celebrated result of Gromov [Gro81] that characterizes the finitely generated groups of polynomial word growth as the virtually nilpotent groups. Gromov's proof uses the solution of the 5-th Hilbert Problem (the characterization of topological groups having the structure of a real Lie group), and the proof of the Polynomial Subgroup Growth Theorem uses Lazard's solution of the p-adic version of the 5-th Hilbert Problem. While Gromov's proof uses Tit's Alternative [Tit72] (a finitely generated linear group either contains the free group of rank two or is virtually solvable), the proof of the Polynomial Subgroup Theorem uses the so-called Lubotzky Alternative (see below for a statement). Both proofs reduce the question to the case of solvable groups and get the final answer in a well known and understood subclass of solvable groups. However, despite the similarities in structure, the two proofs are logically completely independent. In addition, the proof of the Polynomial Subgroup Growth Theorem uses the classification of finite simple groups, the theory of p-adic Lie groups and the Prime Number Theorem. Important steps in the proof already appear in the earlier works by the same authors [MS90], [LM91].

Another fundamental result is a theorem of Pyber and Shalev [PS96] from 1996 stating that if G is a finitely generated group that does not include all finite groups in its upper sections, then G has at most exponential subgroup growth type.

The question of possible types of growth, called the growth spectrum, is of fundamental importance. There are important results on this topic for word growth, growth of Dehn functions, Kesten growth (asymptotics of return probabilities for random walks in groups), etc. An almost complete and satisfactory answer has been obtained for the subgroup growth. The main result claims that if a non-decreasing function $g(n)$ satisfies one of the conditions

$$\log n = O(g(n)) \qquad \text{and} \qquad g(n) = o(n) \tag{1}$$

or

$$g(n^{\log n}) = O(g(n)), \tag{2}$$

then there exists a 4-generator group whose subgroup growth has type $n^{g(n)}$. This result shows that the subgroup growth spectrum is essentially complete. That is, for any reasonably nice non-decreasing unbounded function $g : \mathbb{N} \to (0, \infty)$ such that $g(n) = o(n)$ there exists a finitely generated residually finite group whose subgroup growth is of type $n^{g(n)}$. This essentially means that there are no gaps in the spectrum of possible subgroup growth types between n and n^n. The result is due to Pyber [P] and Segal [Seg01], who proved it in separate papers for the fast case (1) and the slow case (2).

One of the big themes in the study of subgroup growth is the study of analytic and arithmetic properties of the zeta function given by the Dirichlet series

$$\zeta_G(s) = \sum_{n=1}^{\infty} \frac{a_G(n)}{n^s},$$

where $a_G(n)$ is the number of subgroups of index n in G (thus $\sum_{i=1}^{n} a_G(i) = s_G(n)$). This function is in direct analogy to the zeta function of a number field. In case G is a finitely generated nilpotent group the zeta function has excellent properties, including (a) an Euler product expansion, (b) the property that, for each prime p, the local factor

$$\zeta_{G,p}(s) = \sum_{n=0}^{\infty} \frac{a_G(p^n)}{p^{ns}}$$

is a rational function of p^{-s}, (c) partial analytic continuation, and (d) the asymptotics $s_G(n) \sim Cn^{\alpha}(\ln n)^{\beta}$, for some $C > 0$, $0 < \alpha = \alpha(G) \in \mathbb{Q}$ and $0 \leq \beta \in \mathbb{Z}$. The properties (a)-(d) represent a combination of results of Grunewald, Segal, Smith and du Sautoy (see [GSS88], [dSG00]; see [dSS00] for a survey). In particular, finitely generated nilpotent groups have polynomial growth of degree which is a rational number and is not necessarily an integer (in contrast with the word growth in nilpotent groups, which is always polynomial of integer degree as reflected in the formula of H. Bass [Bas72]).

I listed the above four remarkable results just to illustrate the beauty and the depth of the theory of subgroup growth. However, this is only the tip of the iceberg of results on subgroup growth, and the book of Lubotzky and Segal provides a perfect guide for beginners and a research/reference tool for more advanced readers. The book is excellently written. Both authors are not only leading experts with great contributions in the area but also excellent writers of mathematical (and other types of) texts. The Ferran Sunyer i Balaguer Prize further confirms and acknowledges this. This international prize is awarded once per year by the Fundació Ferran Sunyer i Balaguer and the Institut d'Estudis Catalans for a mathematical monograph of expository nature.

Without going into a list of all sections, let me mention that readers will be impressed with the encyclopedic scope of the text. It includes all, or almost all, topics related to subgroup growth, from growth in nilpotent and solvable groups, free groups, linear groups, profinite groups, pro-p groups, analytic pro-p groups, and ideas of growth in general coming from Probability, Number Theory and Combinatorics.

The book also includes plenty of general information on topics that are well known to algebraic audiences and should be part of the background for every modern researcher in mathematics. Such information is organized in twelve "windows", starting with a window on Finite Group Theory and ending in a window on p-adic

Integrals and Logic. These windows are a wonderful methodological tool introduced by the authors. They can be opened during the reading of the main body of the text whenever the reader needs a reference to more basic material, but can also be used independently to quickly brush up on several important topics in Group Theory, including Finite Simple Groups, Permutation Groups, Linear Groups, etc. The windows also include an account on the Lubotzky Alternative, which states that a finitely generated linear group either has a subgroup of finite index whose profinite completion maps onto the congruence completion of some semisimple arithmetic group or it is a virtually solvable group.

The book ends with a section on open problems which contains 35 problems related to subgroup growth. The list will be useful and interesting to both established mathematicians and young researchers. There is no doubt that the list includes the most important and illuminating problems in the area, and we eagerly anticipate solutions of at least some of them in the near future.

The book will surely have a big impact on all readers interested in Group Theory, as well as in Algebra and Number Theory in general.

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