BOOK REVIEWS

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Diffusions, superdiffusions and partial differential equations, by E. B. Dynkin, Colloquium Publications, vol. 50, American Mathematical Society, Providence, RI, 2002, xi + 236 pp., \$49.00, ISBN 0-8218-3174-7

1. Introduction

This book explores the interface of probability and partial differential equations centered around the characterization of solutions of a class of semilinear elliptic equations including the equations

$$\Delta u = u^{\alpha}$$

with $\alpha > 1$ and a class of measure-valued processes called superdiffusions. This research program has flourished since around 1990 and has created a nonlinear analogue of the now classical relation between Brownian motion and potential theory. This is a new chapter in the theory of semilinear parabolic and elliptic equations that complements the purely analytic approach that has been developed through the fundamental contributions of Keller, Osserman, Brezis and Strauss, Loewner and Nirenberg, Gmira and Véron, Brezis and Véron, Baras and Pierre, Marcus and Véron and others (see e.g. [27], and [28] for a recent survey).

2. Diffusions and linear partial differential equations

The main mission of this book is to report on a remarkable research program on semilinear equations that has emerged over the past fifteen years. However, to obtain a perspective on its significance, it is useful to recall the earlier development of Brownian motion and other diffusions and their relation with linear parabolic equations and potential theory. The genesis of this development goes back to the seminal work of Bachelier (1900) and Einstein (1905), who derived the probability density at time $t > s$, $p(s, x; t, \cdot)$ of Brownian motion starting at x at time s and showed that it satisfies the heat equation

$$\frac{\partial u}{\partial t} = D\Delta u$$

where D is the coefficient of diffusivity and Δ is the Laplacian operator. A probabilistic representation for the Dirichlet problem for Laplace's equation in a bounded

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domain $D \subset \mathbb{R}^d$,

$$\Delta u = 0 \qquad (1)$$
$$u = f \text{ on } \partial D$$

where the boundary value f is a continuous function, goes back to 1920's work of Phillips and Wiener [25] and Courant, Friedrichs and Lewy [2], who obtained a solution in the limit as $\varepsilon \to 0$ of the approximating sequence

$$u(x) = \int f(w_{\tau_D}) \Pi_x^\varepsilon(dw) \qquad (2)$$

where τ_D is the first exit time from D and Π_x^ε is the probability measure on the set of paths of a symmetric random walk $\{w_n\}$ on the lattice $\varepsilon\mathbb{Z}^d$ starting at the point x. In 1923 Wiener [30] constructed a probability measure, Π_x, on the space of continuous paths $C([0,\infty),\mathbb{R}^d)$ corresponding to the continuous limit of these random walks and whose finite dimensional distributions are given in terms of the probability transition density $p(s,x;t,\cdot)$. This made possible the next step due to Kakutani [14],[15], who showed that (2) with Π_x^ε replaced by the probability measure Π_x provides the solution of the Dirichlet problem. Classical potential theory, the study of harmonic functions, that is solutions of (1), involves a number of basic objects and concepts. Given a domain D, the Green's function $g_D(\cdot,y)$ with pole y is a positive superharmonic function, harmonic in $D\backslash\{y\}$, $g_D(y,y) = +\infty$, and $g_D(\cdot,y)$ has limit 0 at the boundary. Doob introduced the notion of Greenian domain, that is, a domain in $\mathbb{R}^d$ such that there is a Green's function satisfying

$$g_D(x,y) \le const \cdot \Gamma(x-y)$$

where

$$\begin{aligned}\Gamma(x) &= |x|^{2-d} \text{ for } d \ge 3\\ &= \log_+|x| \text{ for } d = 2\\ &= 1 \text{ for } d = 1.\end{aligned}$$

Given a Greenian domain D, a representation of all positive harmonic functions was obtained by Martin [19] in terms of a compactification $\hat{D}$. Points in the Martin boundary $\hat{\partial} D = \hat{D}\backslash D$ can be identified with harmonic functions, and an arbitrary positive harmonic function can be represented by the Poisson integral

$$h(x) = \int_{D'} k(x,y)\mu(dy)$$

where D' is the subset of $\hat{\partial} D$ corresponding to extreme harmonic functions, $k(\cdot,y)$ is the Poisson kernel and $tr(h) = \mu$ is a finite measure on the Borel subsets of D' called the *boundary trace* of h.

A basic question involves the identification of exceptional sets. A subset A of a Greenian domain D is *polar* if there is a neighborhood of each point in the set that carries a superharmonic function equal to $+\infty$ at each point in the intersection of A with the neighborhood. Polar sets can also be characterized as sets of zero capacity where the capacity $C(A)$ of an analytic subset A of a Greenian domain is defined by

$$C(A) = \sup\{\mu(A) : \mu \text{ supported by } A \text{ and } \int g_D(x,y)\mu(dy) \le 1\}.$$

Kakutani also obtained the probabilistic characterization of a polar set as one that is not hit by Brownian path at a strictly positive time.

During the 1950's consideration of the structure of potential theory and its probabilistic counterparts led to the development of *axiomatic potential theory* (see e.g. [1]) and *probabilistic potential theory* (see e.g. [5],[6],[12]) in which the Laplacian operator is replaced by the infinitesimal generator of a Markov process on a locally compact space or Polish space. This development also included *parabolic potential theory* based on the heat operator $(\frac{1}{2}\Delta - \frac{\partial}{\partial t})$ which has many parallels to classical potential theory.

3. Semilinear equations and superdiffusions

We now turn to the class of semilinear elliptic equations of the form

$$Lu = \psi(u) \tag{3}$$

in a Greenian domain D where ψ satisfies

- for every x, $\psi(x,\cdot)$ is convex and $\psi(x,0)=0$, $\psi(x,u)>0$ if $u>0$;
- $\psi(x,u)$ is continuously differentiable;
- ψ is locally Lipschitz continuous in u uniformly in x.

Equations of this type arise in astrophysics, meteorology, theory of atomic spectra and the Yamabe problem in geometry. An important subclass of equations of this type is given by

$$\Delta u = u^\alpha \tag{4}$$

in a Greenian domain $D \subset \mathbb{R}^d$ where $\alpha > 1$. A natural question is the characterization of the set $\mathcal{U}(D)$ of all positive solutions to equations (3), (4). This has led to a rich theory involving both analytical and probabilistic tools and ideas.

One of the important developments in tackling the problem of characterizing the solutions of semilinear elliptic equations of the form (3) is a relationship between a subclass of these equations and a class of measure-valued Markov processes known as superdiffusions that were first constructed by S. Watanabe [29] in 1968. The book includes an exposition of a construction of superdiffusions, but readers not already familiar with this area would be advised to begin with more complete expositions such as the Saint Flour lecture notes of Perkins [24].

A superdiffusion is defined as follows. Let $\{\xi_t\}_{t\geq 0}$ be a Markov diffusion process generated by the elliptic operator L on $\mathbb{R}^d$ and ψ be a function of the form

$$\psi(x;u) = b(x)u^2 + \int_0^\infty (e^{-\lambda u} - 1 + \lambda) n(x;d\lambda) \tag{5}$$

where b is a bounded nonnegative measurable function and n is a measurable kernel from $\mathbb{R}^d$ to $\mathbb{R}_+$ such that

$$\int_0^\infty \lambda \wedge \lambda^2 n(x;d\lambda)$$

is bounded. Then the associated superdiffusion $\{X_t\}_{t\geq 0}$ is a Markov process with state space $\mathcal{M}(\mathbb{R}^d)$, the space of finite Borel measures on $\mathbb{R}^d$, and transition law given by the Laplace functional

$$P_\mu e^{\langle -f, X_t\rangle} = e^{-\langle V_t f, \mu\rangle}, \quad \mu \in \mathcal{M}(\mathbb{R}^d)$$

for every nonnegative measurable function f and $v_t = V_t f$ satisfies the semilinear parabolic equation

$$\frac{\partial v_t}{\partial t} + Lv_t = \psi(v_t).$$

The special case $L = \Delta,\ \alpha = 2$ corresponds to *super-Brownian motion* that also arises in the scaling limit in high dimensions of a "universality class" including random trees, percolation clusters and a number of interacting particles systems (see Slade [26]). Superprocesses also can be viewed as generalized solutions of a class of stochastic partial differential equations and have been extensively studied over the last 25 years (see [3],[8],[17],[24]).

The superdiffusion X_t can be obtained as the limit of a cloud of particles in which at a constant rate particles die and are replaced by a random number of particles and during their lifetimes the motions of the particles are given by independent L-diffusions. This picture suggests a richer structure than the measure-valued process that has been developed in different ways (see for example [17]). For the purposes of representing the solution of semilinear elliptic equations, Dynkin introduced the notion of a branching exit system. This involves the measures X_Q produced by the particles at the times they exit from a collection of open subsets $Q \subset \mathbb{R}^d$. The resulting exit measure from Q is given by

$$X_Q = \delta_{(t_1,y_1)} + \cdots + \delta_{(t_n,y_n)}$$

where $(t_1, y_1), \ldots, (t_n, y_n)$ denote the exit times and locations of the particles and $\delta_{(t,x)}$ denotes a unit measure at (t, x). Given a class of open sets $\mathbb{O}$, we then obtain a family of random measures $\{X_Q : Q \in \mathbb{O}\}$ with law P_μ. As a consequence of the independence of the particles if $Z = \exp(-\sum_i \langle f_i, X_{Q_i}\rangle)$, it can be shown that

$$P_\mu Z = e^{-\langle u,\, \mu\rangle}$$

where the function defined by

$$u(y) = V_Q(f)(y) = -\log P_{\delta_y} e^{-\langle f, X_Q\rangle}$$

satisfies

$$u + G_Q\psi(u) = K_Q f.$$

Here Green's operator G_Q and the Poisson operator K_Q associated with the L-diffusion ξ_s are given by

$$K_Q f(x) = \Pi_x 1_{\tau_Q<\infty} f(\xi_{\tau_Q})$$
$$G_Q\rho(x) = \Pi_x \int_0^{\tau_Q} \rho(\xi_s)ds.$$

If Q is a smooth domain and f is a continuous function, then u satisfies

$$\text{(6)} \qquad \begin{aligned} &Lu = \psi(u) \text{ in } Q \\ &V_Q(f)(\tilde{x}) \to f(\tilde{x}) \text{ as } x \to \tilde{x} \in \partial Q. \end{aligned}$$

In this special case (6) establishes a link between the semilinear elliptic equation and the superdiffusion. An objective of the research program is to extend this to general Greenian domains and to identify the set of all positive solutions $\mathcal{U}(D)$ of the semilinear elliptic equation (3) when ψ is in the class (5). It can be shown that $\mathcal{U}(D)$ is a complete lattice with partial order $\leq$ as follows. Let $\pi(u) = v$ if $u \in C_+(D)$ and $V_{D_n}(u) \to v$ pointwise for every sequence exhausting D. Then we define $u \wedge v = \pi(\max(u, v))$ and $u \vee v = \pi(\min(u, v))$. We also define $u \oplus v = \pi(u+v)$.

In the special case (4) the nature of the solutions, as well as the techniques used, depends in an important way on the values of d and α and in particular on whether $d < \frac{\alpha+1}{\alpha-1}$ (subcritical case) or $d \geq \frac{\alpha+1}{\alpha-1}$ (supercritical case).

A first consequence of the nonlinearity is the possibility of the existence of a "large solution", that is, a solution $u(x)$ such that

$$\lim_{x \to \partial D} u(x) = \infty.$$

When the large solution is unique, it is a maximal solution and dominates any solution. The existence of the large solution in a ball was used by Iscoe in 1988 to establish the compact support property of super Brownian motion demonstrating the importance of the relationship between properties of the superdiffusion and equation (4).

On the other hand it is natural to consider the class of solutions u dominated by a positive harmonic function in D. Such solutions are called *moderate solutions.* There is a 1-1 correspondence between moderate solutions $\mathcal{U}_1(D)$ and a subclass of positive harmonic functions through the relation

$$u(x) + \int_D g_D(x,y)\psi(u(y))dy = h(x).$$

We write $u = u_\mu$ where $\mu = tr(h)$. Let $\mathcal{N}_1$ denote the set of all measures on D' which are traces of moderate solutions and $\mathcal{N}_0$ be the limits of increasing sequences of measures in $\mathcal{N}_1$.

An essential step in the program to describe all positive solutions are the notions of σ-moderate solution and fine trace. $u \in \mathcal{U}$ is *σ-moderate* if there are moderate solutions u_n such that $u_n \uparrow u$. The fine trace was introduced by Dynkin after a counterexample of Le Gall showed that an earlier definition of trace (the rough trace) was not adequate to uniquely characterize all positive solutions in the general case. The *fine trace* $Tr(u)$ is a pair (Γ, ν). The measure $\nu \in \mathcal{N}_0$ is given by

$$\nu(B) = \sup\{\mu(B) : \mu \in \mathcal{N}_1, \mu(\Gamma) = 0, u_\mu \leq u\}.$$

Γ is a Borel subset of D' that is closed in a fine topology introduced by Dynkin and arises as the set of all singular points, $SG(u)$, of u. A point y belongs to the set $SG(u)$ if $\int_0^\zeta \psi'[u(\xi_s)]ds = \infty$, $\Pi_x^y - a.s. \forall\, x \in D$ where Π_x^y is the law of the diffusion conditioned to exit at y at time ζ.

In 1998 Dynkin and Kuznetsov obtained a classification of all σ-moderate solutions in terms of the fine trace, namely, $u = u_\Gamma \oplus u_\nu$ and showed that given any solution u, $u_\Gamma \oplus u_\nu$ is the maximal σ-moderate solution dominated by u. Here in terms of the lattice order

$$u_\nu = \mathrm{Sup}\{u_\mu : \mu \in \mathcal{N}_1, u_\mu \leq u \text{ and } \mu(\Gamma) = 0\}$$

and

$$u_\Gamma = \mathrm{Sup}\{u_\mu : \mu \in \mathcal{N}_1, \mu(D' \backslash \Gamma) = 0\}.$$

Since the characterization of all σ-moderate solutions is known in these cases, a key question is whether every positive solution is σ-moderate, and this is stated as an open problem in the Epilogue of the book. In the case $d = 2, \alpha = 2$, this had been proved by Le Gall [16], but further progress was made only after the publication of this book and largely stimulated by it. We briefly indicate these recent developments.

In the 2002 doctoral thesis of Benoit Mselati [20],[21], Le Gall's result was extended to the case $d \geq 2$, $\alpha = 2$ using Le Gall's Brownian snake process (refer to Le Gall [17]). In the subcritical case ($d < \frac{\alpha+1}{\alpha-1}$) Le Gall and Mytnik [18] obtained a probabilistic representation using the exit measure density. In (2003) Dynkin [9],[10] announced progress on an extension of Mselati's result to $1 < \alpha \leq 2$ using a probabilistic representation of solutions and a new inequality for superdiffusions. Dynkin's superdiffusion representation is as follows. The *range* $\mathcal{R}_D$ of a superdiffusion in a domain D is a minimal closed set that supports an exit measure X_O for an arbitrary open set O. The random variable $Z_\nu \geq 0$ is the *stochastic boundary value* of u_ν, $Z_\nu = SBV(u_\nu)$, if for every sequence D_n exhausting D

$$\lim_{n\to\infty} \langle u_\nu, X_{D_n} \rangle = Z_\nu, \ a.s.$$

Then if $Tr(u) = (\Gamma, \nu)$, Dynkin's approach is to show that

$$\begin{aligned} u(x) &= -\log P_{\delta_x}\{\mathcal{R}_D \cap \Gamma = \emptyset, e^{-Z_\nu}\} \\ &= (u_\Gamma \oplus u_\nu)(x) \end{aligned}$$

which implies that u is σ-moderate. The Brownian snake and superdiffusion representations provide powerful tools for the analysis of these equations but are limited in that they are only applicable for the case $1 < \alpha \leq 2$. There has been a parallel development due to Marcus and Véron [22] on the analytical side, including an extension of the boundary trace to the case $\alpha > 2$ and the characterization of solutions in the subcritical case. See [28] for a recent exposition of the analytic results on nonlinear elliptic equations. Marcus and Véron (2003) [23] have also announced progress on the extension of Mselati's result to the supercritical case. This is one of a number of results obtained first by probabilistic methods that have stimulated analytical developments in the more general case and led to cross-fertilization between the two approaches. Moreover the probabilistic approach has been successful in studying equations with singular coefficients (arising from branching in random catalytic media) for which there are at present no comparable analytical tools (see [4]).

4. The book

The primary focus of the book is a systematic exposition of the results obtained in a series of papers of Dynkin, and Dynkin and Kuznetsov going back to the early 1990's (e.g. [7],[11]) but also includes basic background on the necessary analytic and probabilistic methods.

The book is divided into two parts. The first part includes a review of finite dimensional diffusions and the associated linear parabolic equations, and superdiffusions and the associated semilinear parabolic equations. In addition there is an elegant development of branching exit Markov systems that starts with branching particle systems and provides the basis for a development of superdiffusions tailored for the study of boundary value problems for elliptic equations.

The second part provides a systematic development of both the analytic and probabilistic aspects of linear and semilinear elliptic equations based largely on the work of Dynkin, and Dynkin and Kuznetsov. To a remarkable extent this results in a theory parallel to the beautiful classical potential theory. However, the technical machinery required is substantial, and a whole range of concepts are required. This includes the analytic notions of moderate and σ-moderate solutions described above, as well as trace, following both the earlier definition of rough trace and the fine trace, and the probabilistic notions of range of a superdiffusion and stochastic boundary values. In addition, classes of exceptional boundary sets are introduced. This includes polar sets, that is, sets not hit by the range of a superdiffusion, and null capacity sets with respect to Martin capacity, which is a Choquet capacity defined for $B \subset \hat{\partial} D$ by

$$CM_\alpha(B) = \sup\{\nu(B), \int g_D(c,x)dx[\int_B k(x,y)\nu(dy)]^\alpha \leq 1\}.$$

Not surprisingly, the theory is not yet as complete as the classical linear case, and a number of important open problems remain.

The notes at the end of chapters provide historical comments and a guide to the literature. The last chapter includes additional historical comments and a brief survey of the extensive literature on branching measure-valued processes, superprocesses and snakes.

The final chapter in the book is an epilogue that presents three open problems and directions for research. The first is the question "Is every solution σ-moderate?" which as indicated above has already produced fruit. The second deals with relations between the different classes and exceptional boundary sets, and the third is a far-reaching direction of research to identify all functions on the space of measures with compact support that are harmonic for the superdiffusion.

To conclude, this book provides the reader with an in-depth introduction to a rich and rapidly developing research area that has already produced remarkable results.

References

[1] M. Brélot, Éléments de la théorie classique du potentiel, *Les cours de Sorbonne*, Centre de Documentation Universitaire, Paris V, 1959. MR **21:**5099

[2] R. K. Courant, K. Friedrichs and H. Lewy, Über die partiellen Differenzengleichungen der mathematischen Physik. Math. Ann. 100 (1928), 32-74.

[3] D. A. Dawson, Measure-valued Markov processes, in *École d'Été de Probabilités de Saint Flour XXI*, Lecture Notes in Mathematics 1541, Springer-Verlag, 1993, 1-261. MR **94m:**60101

[4] D. A. Dawson and K. Fleischmann, Catalytic and mutually catalytic branching, in *Infinite Dimensional Stochastic Analysis*, eds. Ph. Clément, F. den Hollander, J. van Neerven and B. de Pagter, Royal Netherlands Academy, Amsterdam, 145-170, 2000. MR **2002f:**60164

[5] J. L. Doob, Classical potential theory and its probabilistic counterpart, Springer, 1984. MR **85k:**31001

[6] E. B. Dynkin, Markov Processes, I, II, Springer-Verlag, 1965. MR **33:**1887

[7] E. B. Dynkin, A probabilistic approach to one class of nonlinear differential equations, Probab. Th. Rel. Fields 89 (1991), 89-115. MR **92d:**35090

[8] E. B. Dynkin, An introduction to branching measure-valued processes, CRM Monograph Series, vol. 6, AMS, 1994. MR **96f:**60145

[9] E. B. Dynkin, A new inequality for superdiffusions and its applications to nonlinear differential equations, manuscript, 2003.

[10] E. B. Dynkin, Superdiffusions and positive solutions of nonlinear partial differential equations, to appear in Uspekhi Mat. Nauk., 2003.

[11] E. B. Dynkin and S. E. Kuznetsov, Superdiffusions and removable singularities for quasilinear partial differential equations, Comm. Pure Appl. Math. 49 (1996), 125-176. MR **97m:**60114

[12] G. Hunt, Markov processes and potentials I, II, III, Illinois J. Math. 1 (1957), 44-93; 1 (1957), 316-369; 2 (1958), 151-213. MR **19:**951g, MR **21:**5824

[13] I. Iscoe, On the supports of measure-valued critical branching Brownian motions, Ann. Probab. 16 (1988), 200-221. MR **88j:**60097

[14] S. Kakutani, Two dimensional Brownian moion and harmonic functions, Proc. Imp. Acad. Tokyo 20 (1944), 227-233. MR **7:**315b

[15] S. Kakutani, Markov processes and the Dirichlet problem, Proc. Imp. Acad. Tokyo 21, 227-233, 1945. MR **11:**357h

[16] J.-F. Le Gall, A probabilistic Poisson representation for positive solutions $\Delta u = u^2$ in a planar domain, Comm. Pure Appl. Math. 50 (1997), 69-103. MR **98c:**60144

[17] J.-F. Le Gall, Spatial branching processes, random snakes and partial differential equations, Birkhäuser, 1999. MR **2001g:**60211

[18] J.-F. Le Gall and L. Mytnik, Regularity and irregularity of the exit measure density for $(1+\beta)$ stable super-Brownian motion, 2003 preprint.

[19] R. S. Martin, Minimal positive harmonic functions, Trans. Amer. Math. Soc. 49 (1941), 137-172. MR **2:**292h

[20] M. B. Mselati, Classification et répresentation probabiliste des solutions positives de $\Delta u = u^2$ dans un domaine, Thèse Doctorat de l'Université Paris 6, 2002.

[21] M. B. Mselati, Classification et répresentation probabiliste des solutions positives d'une équation elliptique semi-linéaire, C. R. Acad. Sci. Paris Ser. I 335, 733-738, 2002.

[22] M. Marcus and L. Véron, The boundary trace of positive solutions of semilinear elliptic equations, I. The subcritical case, Arch. Rat. Mech. Anal. 144 (1998), 201-231. MR **2000a:**35077

[23] M. Marcus and L. Véron, Capacitary estimates of solutions of a class of nonlinear elliptic equations, C. R. Acad. Sci. Paris Ser. I 336, 2003.

[24] E. A. Perkins, Dawson-Watanabe Superprocesses and Measure-valued Diffusions, Ecole d'été de probabilités (Saint Flour, 1999), Lecture Notes in Math. 1781, Springer, 2002, 125–324. MR **2003k:**60104

[25] H. B. Phillips and N. Wiener, Nets and Dirichlet problem, J. Math. Phys. 2, 105-124, 1923.

[26] G. Slade, Scaling limits and super-Brownian motion, Notices AMS, 49, 1056-1067, 2002. MR **2003g:**60170

[27] L. Véron, Singularities of Solutions of Second Order Quasilinear Equations, Pitman Research Notes in Math. 353, Addison Wesley Longman Inc., 1966. MR **98b:**35053

[28] L. Véron, Generalized boundary value problems for nonlinear elliptic equations, Electron. J. Diff. Equ. Conf. 6, 313-342, 2001. MR **2001j:**35099

[29] S. Watanabe, A limit theorem on branching processes and continuous state branching processes, J. Math. Kyoto Univ. 8, 141-167, 2001. MR **38:**5301

[30] N. Wiener, Differential space, J. Math Phys. 2, 131-174, 1923.

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