BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 2, Pages 205–244
S 0273-0979(04)01004-3
Article electronically published on February 2, 2004

MATHEMATICAL TOOLS FOR KINETIC EQUATIONS

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Abstract. Since the nineteenth century, when Boltzmann formalized the concepts of kinetic equations, their range of application has been considerably extended. First introduced as a means to unify various perspectives on fluid mechanics, they are now used in plasma physics, semiconductor technology, astrophysics, biology.... They all are characterized by a density function that satisfies a Partial Differential Equation in the phase space.

This paper presents some of the simplest tools that have been devised to study more elaborate (coupled and nonlinear) problems. These tools are basic estimates for the linear first order kinetic-transport equation. Dispersive effects allow us to gain time decay, or space-time L^p integrability, thanks to Strichartz-type inequalities. Moment lemmas gain better velocity integrability, and macroscopic controls transform them into space L^p integrability for velocity integrals.

These tools have been used to study several nonlinear problems. Among them we mention for example the Vlasov equations for mean field limits, the Boltzmann equation for collisional dilute flows, and the scattering equation with applications to cell motion (chemotaxis).

One of the early successes of kinetic theory has been to recover macroscopic equations from microscopic descriptions and thus to be able theoretically to compute transport coefficients. We also present several examples of the hydrodynamic limits, the diffusion limits and especially the recent derivation of the Navier-Stokes system from the Boltzmann equation, and the theory of strong field limits.

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Received by the editors January 15, 2003, and, in revised form, November 6, 2003.
2000 *Mathematics Subject Classification.* Primary 35F10, 35L60, 35Q35, 76P05, 82B40.

1. Introduction

Kinetic physics appeared in the second half of the nineteenth century with Maxwell and Boltzmann, later with Enskog, Vlasov, and Grad. Searching for a form of matter which could explain Saturn's rings, Maxwell imagined that they were made of rocks colliding and gravitating around the planet. The density of matter is then parametrized by the space position x and the velocity ξ of the rocks, the so-called phase space. A few years later, Boltzmann completely formalized the process, giving a general representation of a 'dilute gas' as particles undergoing collisions and with free motion between collisions, and he wrote the famous equation which is now named after him. He also discovered the asymptotic process that allows one to relate his model to classical fluid dynamics. Vlasov wrote another kinetic equation for plasmas of charged particles. There, each particle undergoes a collective Coulombic attraction from others. Nowadays kinetic equations appear in a variety of sciences and applications such as astrophysics, aerospace engineering, nuclear engineering, particle-fluid interactions, semi-conductor technology, social sciences or in biology like chemotaxis and immunology. The common feature of these models is that the underlying Partial Differential Equation is posed in the phase space $(x, \xi) \in \mathbb{R}^{d+d}$ (or a subset of it). The space dimension d is 3 for many applications, but dimensions 2 and 1 are also relevant sometimes; for many mathematical results it is possible to consider every dimension $d \in \mathbb{N}^*$.

Very early, mathematicians tried to justify rigorously the validity of the Boltzmann equation and of the fluid dynamics limit. Hilbert gave his name to the so-called Hilbert expansion which aimed to recover compressible flows from dilute gases. Chapman and Enskog proposed an alternative method to take into account viscous effects. Grad also tried to derive the Boltzmann equation from a system of particles like hard spheres when the cross-section (the surface of particles that one can see in a unit cube) remains finite in a scaling process. Several pieces of these programs have now been fulfilled, and the last twenty years saw many questions answered. But more interesting for mathematical science itself is that general methods and approaches now exist which allow us to address more and more challenging problems.

Although most of the problems of interest are nonlinear, the common factor behind models of kinetic physics comes from the (linear) transport of particles. In other words, they are based on the kinetic-transport equation

$$\begin{aligned} &\frac{\partial}{\partial t} f(t,x,\xi) + \xi \cdot \nabla_x f(t,x,\xi) = 0, \qquad t \in \mathbb{R},\, x \in \mathbb{R}^d,\, \xi \in \mathbb{R}^d, \\ &f(t=0,x,\xi) = f^0(x,\xi), \end{aligned}$$

or variants of it. The apparent simplicity of this equation and of its explicit representation solution is misleading. When considering so-called macroscopic quantities

$$\int_{\mathbb{R}^d} \psi(\xi)\, f(t,x,\xi)\, d\xi,$$

one discovers that the problem undergoes really complex dynamics. The most noticeable general result is the regularization by averaging on velocities which states that for $f^0 \in L^p$ with bounded support in ξ, macroscopic quantities belong to Sobolev or Besov spaces with positive numbers of derivatives; see §3. But each variable of the problem, t, x, ξ, and the different roles they play in the dynamics make various effects arise, especially coming from dispersion for instance.

These regularizing effects on macroscopic quantities by velocity averaging are useful for studying more elaborate models than simple kinetic-transport. Indeed, in most of the nonlinearities arising from applications, the nonlinear term is built on an integral in ξ. It is therefore natural that this tool plays a central role for proving existence of weak solutions, an area where recent progress is substantial with, for instance, the global solution of the Boltzmann equation, one the most noticeable recent achievements of the domain. The justification of asymptotic limits has also benefited from this compactness method, and a remarkable recent result is the complete justification of the incompressible hydrodynamic limit of the Boltzmann equation; see §5.2. But, among the subjects that have emerged recently, one can also mention the recent development of strong field limits in Vlasov equations, a new way of deriving macroscopic equations from kinetic equations, the theory of smooth solutions, or applications to biology.

Indeed, a particular feature that makes kinetic equations so fascinating arises from the numerous fields of applications; the variety of nonlinear problems is ever-growing, and new methods have to be found for each of them. But the number of questions and methods is extremely large, and it is nearly impossible to treat all of them here. For instance we do not speak here of numerical methods and their convergence theory, nor of discrete kinetic equations, nor of semigroup and spectral approaches for linear models, nor of probabilistic approaches, nor of relations with Schrödinger type equations (Wigner transform), nor of the homogeneous equations, nor of boundary value problems; and we address an extremely limited range of applications.

We first present general methods for linear kinetic equations in section 2; this covers time decay and dispersion effects as Strichartz inequalities, moment lemmas—all of these improve the obvious integrability derived from conservations laws. Always in the context of the linear equation it is also possible to gain regularity thanks to averaging lemmas that are treated in a separate section. We give several statements beginning with the simplest regularizing effect in $H^{1/2}$. These tools have been used to treat nonlinear models in the last few years. We present them in section 4, which includes Vlasov equations of plasma physics, scattering models (including microscopic chemotaxis modeling) and the Boltzmann equation, but no physics background is assumed from the reader. The last section deals with asymptotic problems and the derivation of macroscopic models, especially through the diffusion, hyperbolic and high field limits which are also recent discoveries in the topic.

We have given proofs of the most elementary results presented in the first two sections, which can be skipped in order to have a faster overview, but they illustrate for nonexperts the complex structures of the seemingly simple linear kinetic equation.

2. Kinetic-transport equation (gain of integrability)

The simplest kinetic equation is the pure kinetic-transport equation. Although it has little practical interest, we give an exhaustive theory in order to illustrate several simple ideas that have been (or might be) used in nonlinear equations. The kinetic-transport equation describes the evolution of the density $f(t,x,\xi)$ of particles which at time t and position $x \in \mathbb{R}^d$ moves with velocity ξ. The associated Eulerian equation is

$$\begin{aligned}&\tfrac{\partial}{\partial t} f(t,x,\xi) + \xi \cdot \nabla_x f(t,x,\xi) = 0,\\ &f(t=0,x,\xi) = f^0(x,\xi).\end{aligned} \tag{1}$$

Its solution is given by

$$f(t,x,\xi) = f^0(x - \xi t, \xi). \tag{2}$$

This already shows a first class of a priori bounds in (x,ξ) variables. We have for all $t \in \mathbb{R}$,

$$\|f(t)\|_{L^p(\mathbb{R}^{2d})} = \|f^0\|_{L^p(\mathbb{R}^{2d})}, \qquad 1 \le p \le \infty. \tag{3}$$

Although this equation seems very simple because of its representation formula, there are several properties which cannot be seen directly in (2). Mostly these properties arise in the so-called *macroscopic* quantities, i.e. ξ integrals, and especially moments like the macroscopic density ϱ defined by

$$\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,\xi)\, d\xi; \tag{4}$$

or like the momentum $q(t,x) \in \mathbb{R}^d$ and the macroscopic velocity $u(t,x) \in \mathbb{R}^d$, defined by

$$q(t,x) = \varrho\, u(t,x) = \int_{\mathbb{R}^d} \xi\, f(t,x,\xi)\, d\xi; \tag{5}$$

or like the total energy E, internal energy $e(t,x)$ defined by

$$E(t,x) = \frac{1}{2}\varrho\, |u(t,x)|^2 + \varrho\, e(t,x) = \int_{\mathbb{R}^d} \frac{|\xi|^2}{2} f(t,x,\xi)\, d\xi; \tag{6}$$

or in other words, using rather temperature T,

$$\varrho\, e(t,x) = d\, \varrho T = \int_{\mathbb{R}^d} \frac{|\xi - u(t,x)|^2}{2} f(t,x,\xi)\, d\xi. \tag{7}$$

Here and for simplicity, we have forgotten some physical constants.

In this section we give several elementary properties of solutions that express dispersion effects (by analogy with similar estimates for Schrödinger equations) and time decay (macroscopic quantities vanish in long time), gain of integrability and higher velocity moments. We postpone to the next section the presentation of the averaging lemma.

Before we begin, we notice that formula (2) obviously gives the unique solution when $f^0 \in C^1(\mathbb{R}^{2d})$, but formula (2) also holds when $f^0 \in L^p(\mathbb{R}^{2d})$, $1 \le p < \infty$, for the unique weak solution (in distributions) $f \in C(\mathbb{R}_t; L^p(\mathbb{R}^{2d}))$. We have stated our results within this general framework, but they can be considered for smooth solutions to begin with. We also refer to Glassey [61] and to Dautray and Lions [36, Ch. 11], for a general introduction based on a different point of view and for the presentation of many nonlinear problems.

2.1. **A dispersion lemma.** The first tool we present, a time decay estimate related to dispersion in the full space, has been used widely after its introduction by Bardos and Degond [8]. Especially, it was used in nonlinear problems for proving existence of small, global in time solutions that decay by dispersion in the full space. For studying small classical solutions it is a basic tool; see [61].

Lemma 2.1. *The macroscopic density ϱ defined by* (1), (4) *satisfies the inequality*

$$|\varrho(t,x)| \le \frac{1}{t^d} \|f^0\|_{L^1(\mathbb{R}^d_x; L^\infty(\mathbb{R}^d_\xi))}.$$

Proof of Lemma 2.1. We have, using the above representation formula

$$\begin{aligned} |\varrho(t,x)| &\le \int_{\mathbb{R}^d} |f^0(x-\xi t,\xi)|\, d\xi \\ &\le \int_{\mathbb{R}^d} \sup_{w\in\mathbb{R}^d} |f^0(x-\xi t,w)|\, d\xi \\ &= \frac{1}{t^d}\int_{\mathbb{R}^d} \sup_{w\in\mathbb{R}^d} |f^0(x-\xi t,w)|\, d(\xi t) \\ &= \frac{1}{t^d}\int_{\mathbb{R}^d} \sup_{w\in\mathbb{R}^d} |f^0(y,w)|\, dy. \end{aligned}$$

And the inequality is proved. □

2.2. **Strichartz inequality.** Strichartz inequality, derived in Castella and Perthame [31] (see also the improvement in Keele and Tao [85]) for kinetic equations, is a more elaborate way to express dispersion in the full space. Its advantage is to use initial data in classical Lebesgue spaces. Although it has not yet been used for the nonlinear model, in the context of the nonlinear Schrödinger equation, the same method has proven to be very fruitful.

Theorem 2.2. *The macroscopic density ϱ defined by* (4) *satisfies the inequality*

$$\|\varrho\|_{L^q(\mathbb{R}_t; L^p(\mathbb{R}^d_x))} \le C(d)\|f^0\|_{L^a(\mathbb{R}^{2d})},$$

for any real numbers a, p and q such that

$$1 \le p < \frac{d}{d-1}, \qquad \frac{2}{q} = d(1-\frac{1}{p}), \qquad 1 \le a = \frac{2p}{p+1} < \frac{2d}{2d-1}.$$

Remark 2.3. Of course the values $a=1$, $p=1$ and $q=\infty$ are obvious (see (3)), and the other limiting values are the interesting ones. They are given by $p = \frac{d}{d-1}$, i.e. $p = d'$ and thus $q = 2$, and also $a = \frac{2d}{2d-1}$.

Remark 2.4. As we will see in §3, Sobolev regularity can also be proved on macroscopic quantities and better integrability follows by Sobolev injections [54]. The main difference is that this method only applies for bounded velocities and does not provide information for large times.

Proof of Theorem 2.2. The proof essentially expresses that this inequality is the dual of the previous dispersion inequality in Lemma 2.1. It is divided into three steps.
First step, duality method. We argue by duality,

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \varrho(t,x)\,\Phi(t,x)\,dx\,dt &= \int_{\mathbb{R}}\int_{\mathbb{R}^{2d}} f^0(x-\xi t,\xi)\,\Phi(t,x)\,dx\,d\xi\,dt \\ &= \int_{\mathbb{R}^{2d}} f^0(x,\xi)\Big(\int_{\mathbb{R}}\Phi(t,x+\xi t)\,dt\Big)\,dx\,d\xi \\ &\le \|f^0\|_{L^a(\mathbb{R}^{2d})}\|\int_{\mathbb{R}}\Phi(t,x+\xi t)\,dt\|_{L^{a'}(\mathbb{R}^{2d})},\end{aligned}$$

and one has

$$a' = 2p' = \frac{2p}{p-1}.$$

Second step, apply dispersion lemma. Therefore, we may next compute

$$\begin{aligned}\|\int_{\mathbb{R}}\Phi(t,x+\xi t)\,dt\|^2_{L^{a'}(\mathbb{R}^{2d})} &= \|\int_{\mathbb{R}}\Phi(t,x+\xi t)^2\,dt\|_{L^{p'}(\mathbb{R}^{2d})} \\ &= \|\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(t,x+\xi t)\,\Phi(t,x+\xi s)\,ds\,dt\|_{L^{p'}(\mathbb{R}^{2d})} \\ &= \|\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(t,x+\xi(t-s))\,\Phi(t,x)\,ds\,dt\|_{L^{p'}(\mathbb{R}^{2d})} \\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}}\|\Phi(t,x+\xi(t-s))\,\Phi(t,x)\|_{L^{p'}(\mathbb{R}^{2d})}\,ds\,dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^{2d}}\Phi(t,x+\xi(t-s))^{p'}\,\Phi(t,x)^{p'}\,dx\,d\xi\Big)^{1/p'}\,ds\,dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^{2d}}\Phi(t,y)^{p'}\,\Phi(t,x)^{p'}\,\frac{dx\,dy}{|t-s|^d}\Big)^{1/p'}\,ds\,dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\|\Phi(t)\|_{L^{p'}(\mathbb{R}^d)}\|\Phi(s)\|_{L^{p'}(\mathbb{R}^d)}\,\frac{dt\,ds}{|t-s|^{d/p'}} \\ &\le \|\Phi(t)\|^2_{L^{q'}(\mathbb{R};L^{p'}(\mathbb{R}^d))}\end{aligned}$$

(see explanations in the remark below).
Third step, conclusion. Putting together the first two steps, we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}\varrho(t,x)\,\Phi(t,x)\,dx\,dt \le \|f^0\|_{L^a(\mathbb{R}^{2d})}\|\Phi(t)\|_{L^{q'}\big(\mathbb{R};L^{p'}(\mathbb{R}^d)\big)},$$

for all functions $\Phi(t,x)$. By duality we have proved Theorem 2.2. □

Remark 2.5. The above proof follows closely that given by Ginibre and Velo for Schrödinger equation [60]. It uses the Riesz-Sobolev-Young inequality (in one space dimension) which states (see Evans [54]) that the operator $\phi \to \frac{1}{|x|^\alpha} \star \phi$ maps L^r into L^σ with

$$\frac{1}{\sigma} = \frac{1}{r} + \frac{\alpha}{d} - 1, \qquad 0 < \alpha < d,\ 1 < \sigma < \infty,\ 1 < r < \infty.$$

2.3. **Strichartz inequality (steady case).** The same type of duality method as for the evolution Strichartz inequalities allows us to prove L^p estimates for the steady transport equation

$$f + \xi \cdot \nabla_x f = g(x, \xi). \tag{8}$$

This equation defines a unique distributional solution $f \in L^p(\mathbb{R}^{2d})$ for $g \in L^p(\mathbb{R}^{2d})$, $1 \le p \le \infty$, thanks to the representation formula

$$f(x, \xi) = \int_0^\infty g(x - \xi s, \xi)\, e^{-s}\, ds. \tag{9}$$

Theorem 2.6. *The macroscopic density $\varrho(x)$ defined by (4) satisfies the inequality*

$$\|\varrho\|_{L^q(\mathbb{R}^d)} \le C(d, p)\, \|g\|_{L^p(\mathbb{R}^{2d})}, \qquad 1 \le p < \frac{2d}{2d-1}, \quad q = \frac{p}{2-p} < \frac{d}{d-1}.$$

Theorem 2.7. *The macroscopic density on the sphere*

$$\varrho_S(x) = \int_{|\xi|=1} f(x, \xi)\, d\xi$$

satisfies the inequality

$$\|\varrho_S\|_{L^q(\mathbb{R}^d)} \le C(d, p)\, \|g\|_{L^p(\mathbb{R}^d \times \mathbb{S}^{d-1})}, \qquad 1 \le p \le 2, \quad q = \frac{dp}{d+1-p} \le \frac{d-1}{2d}.$$

Remark 2.8. As we will see in section 3, some L^p regularity on ϱ_S can be derived from Averaging Lemmas thanks to Sobolev injections. It turns out that the result of Theorem 2.7 gives a better exponent q (except for $p = 1$ or 2 where they are equivalent).

Proof of Theorem 2.6. First step (duality). We have, for any test function $\Phi(x)$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \varrho(x)\Phi(x)\, dx &= \int_{\mathbb{R}^{2d}} f(x, \xi)\, \Phi(x)\, dx\, d\xi \\
&= \int_{\mathbb{R}^{2d}} \int_0^\infty g(x - \xi s, \xi)\, \Phi(x)\, dx\, d\xi\, e^{-s}\, ds \\
&= \int_{\mathbb{R}^{2d}} g(x, \xi) \Big(\int_0^\infty \Phi(x + \xi s)\, e^{-s}\, ds \Big)\, dx\, d\xi \\
&\le \|g\|_{L^p(\mathbb{R}^{2d})}\, \| \int_0^\infty \Phi(x + \xi s)\, e^{-s}\, ds \|_{L^{p'}(\mathbb{R}^{2d})}.
\end{aligned}$$

Second step (estimate of the dual operator). Again, we denote $p' = 2r$, for some $r \geq 1$, and we can compute, as long as we have $d < r$, i.e. $d < p'/2$, i.e. $p < (2d)'$,

$$\begin{aligned}
\| \int_0^\infty \Phi(x+\xi s)\, e^{-s}\, ds\|^2_{L^{p'}(\mathbb{R}^{2d})} &= \|\Big(\int_0^\infty \Phi(x+\xi s)\, e^{-s}\, ds\Big)^2\|_{L^r(\mathbb{R}^{2d})} \\
&= \| \int_0^\infty \int_0^\infty \Phi(x+\xi s)\, e^{-s}\Phi(x+\xi t)\, e^{-t}\, ds\, dt\|_{L^r(\mathbb{R}^{2d})} \\
&\leq \int_0^\infty \int_0^\infty \|\Phi(x+\xi s)\, \Phi(x+\xi t)\|_{L^r(\mathbb{R}^{2d})}\, e^{-s}\, e^{-t}\, ds\, dt \\
&= \int_0^\infty \int_0^\infty \|\Phi(x)\, \Phi(y)\|_{L^r(\mathbb{R}^{2d})}\, \frac{e^{-(s+t)}}{|s-t|^{d/r}}\, ds\, dt \\
&\leq C(d,r)\|\Phi(x)\, \Phi(y)\|_{L^r(\mathbb{R}^{2d})}.
\end{aligned}$$

Third step (conclusion). Putting together the first two steps, we obtain

$$\int_{\mathbb{R}^d} \varrho(x)\, \Phi(x)\, dx \leq \|g\|_{L^p(\mathbb{R}^{2d})}\, \|\Phi\|_{L^r(\mathbb{R}^d)}.$$

This means that

$$\|\varrho\|_{L^{r'}(\mathbb{R}^d)} \leq \|g\|_{L^p(\mathbb{R}^{2d})}$$

which concludes the proof with $q = r' = (p'/2)'$. □

Proof of Theorem 2.7. We follow the same steps and use the same notations as in the proof of Theorem 2.6 .
First step (duality). We have, for any test function $\Phi(x)$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \varrho(x)\, \Phi(x)\, dx &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} f(x,\xi)\, \Phi(x)\, dx\, d\xi \\
&= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \int_0^\infty g(x-\xi s,\xi)\, \Phi(x)\, dx\, d\xi\, e^{-s}\, ds \\
&= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} g(x,\xi)\Big(\int_0^\infty \Phi(x+\xi s)\, e^{-s}\, ds\Big)\, dx\, d\xi \\
&\leq \|g\|_{L^p(\mathbb{R}^d\times\mathbb{S}^{d-1})}\, \| \int_0^\infty \Phi(x+\xi s)\, e^{-s}\, ds\|_{L^{p'}(\mathbb{R}^d\times\mathbb{S}^{d-1})}.
\end{aligned}$$

Second step (estimate of the dual operator). Again, we denote that $p' = 2r$, for some $r \geq 1$, and we can compute

$$\begin{aligned}
\| \int_0^\infty \Phi(x+\xi s)\, e^{-s}\, ds\|^2_{L^{p'}(\mathbb{R}^d\times\mathbb{S}^{d-1})} &= \|\Big(\int_0^\infty \Phi(x+\xi s)\, e^{-s}\, ds\Big)^2\|_{L^r(\mathbb{R}^d\times\mathbb{S}^{d-1})} \\
&= \| \int_0^\infty \int_0^\infty \Phi(x+\xi s)\, e^{-s}\Phi(x+\xi t)\, e^{-t}\, ds\, dt\|_{L^r(\mathbb{R}^{2d})} \\
&\leq \int_0^\infty \int_0^\infty \|\Phi(x+\xi s)\, \Phi(x+\xi t)\|_{L^r(\mathbb{R}^d\times\mathbb{S}^{d-1})}\, e^{-(s+t)}\, ds\, dt \\
&\leq C(r)\, \|\Phi(x+\xi s)\, \Phi(x+\xi t)\, e^{-(s+t)/r}\|_{L^r(\mathbb{R}^d\times\mathbb{S}^{d-1}\times(0,\infty)^2)} \\
&= C(r)\, \|\Phi(x)\, \Phi(x+\xi(s-t))\, e^{-(s+t)/r}\|_{L^r(\mathbb{R}^d\times\mathbb{S}^{d-1}\times(0,\infty)^2)}.
\end{aligned}$$

Here we argue differently:

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}\times(0,\infty)^2))} \Phi(x)^r\,\Phi(x+\xi(s-t))^r\,e^{-(s+t)}\,dx\,d\xi\,ds\,dt$$

$$= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\int_{s>t>0} \Phi(x)^r\,\Phi(x+\xi(s-t))^r\,e^{-(s+t)}\,dx\,d\xi\,ds\,dt + \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\int_{t>s>0}[\ldots].$$

These two terms can be treated in the same way; they are equal to (changing variables $u = s-t$)

$$\begin{aligned}
&= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\int_{u>0,\,t>0} \Phi(x)^r\,\Phi(x+\xi u)^r\,e^{-(u+2t)}\,dx\,d\xi\,du\,dt \\
&= \int_{\mathbb{R}^{2d}}\int_{t>0} \Phi(x)^r\,\Phi(x+y)^r\,\frac{e^{-|y|}}{|y|^{d-1}}\,e^{-}2t\,dx\,dy\,dt \\
&= C\int_{\mathbb{R}^{2d}} \Phi(x)^r\,\Phi(x+y)^r\,\frac{e^{-|y|}}{|y|^{d-1}}\,e^{-}2t\,dx\,dy \\
&\le C\|\Phi^r\|^2_{L^\gamma(\mathbb{R}^d)}
\end{aligned}$$

with $\gamma = \frac{2d}{d+1}$, using the Riesz-Sobolev-Young inequality.

Third step (conclusion). Putting together the first two steps, we obtain

$$\int_{\mathbb{R}^d} \varrho(x)\,\Phi(x)\,dx \le \|g\|_{L^p(\mathbb{R}^d\times\mathbb{S}^{d-1})}\,\|\Phi\|_{L^{r\gamma}(\mathbb{R}^d)}.$$

This means that

$$\|\varrho\|_{L^{(r\gamma)'}(\mathbb{R}^d)} \le \|g\|_{L^p(\mathbb{R}^d\times\mathbb{S}^{d-1})}$$

which concludes the proof with $q = (p'\gamma/2)'$, i.e. $\frac{1}{q} = 1-\frac{2(p-1)}{\gamma p} = 1-\frac{(d+1)(p-1)}{dp}$. $\square$

2.4. **Macroscopic controls and time decay.** It is also possible to prove control of macroscopic moments by a direct computation in terms of the L^p norms of the microscopic density f and of the energy.

These controls, very elementary and robust, are widely used in the analysis of kinetic equations. We give here a presentation which is based on the preliminary question: which macroscopic quantities do we control in terms of the L^p_ξ norms of f only? Then we apply these controls to additional t and x dependency and prove a time decay result.

Proposition 2.9. *For any measurable function $f(\xi)$ defined on $\mathbb{R}^d$, we have, using notations* (4), (5), (6) *for macroscopic quantities:*
(i) $\varrho\,T^{-d/2} \le C_\infty(d)\|f\|_{L^\infty(\mathbb{R}^d)}$*;*
(ii) *for $\theta = 1/p$, $1\le p\le\infty$, we have more generally*

$$\varrho^\theta\,\Big(\frac{\varrho}{T^{d/2}}\Big)^{1-\theta} \le C_p(d)\|f\|_{L^p(\mathbb{R}^d)}.$$

We do not prove this result which follows from classical methods. We just mention that the best constants are attained by specific 'equilibrium densities'. For instance, in (i) the equality holds for the family labeled by $u\in\mathbb{R}^d$,

$$f_\infty(\xi) = \begin{cases} \alpha\frac{\varrho}{T^{d/2}} & \text{for } |\xi-u|^2\le\beta T, \\ 0, & \text{otherwise,}\end{cases}$$

where α and β are the only two parameters compatible with the macroscopic definitions of ϱ and T in (4)–(7).

Lemma 2.10. *For any measurable function $f(x,\xi)$ defined on $\mathbb{R}^{2d}$, we have, for all $a(x) \in \mathbb{R}^d$,*

$$\|\varrho\|_{L^{(d+2)/d}(\mathbb{R}^d)} \leq C(d)\,\|f\|^{2/(d+2)}_{L^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^{2d}} |\xi - a(x)|^2\,|f(x,\xi)|\,dx\,d\xi\right)^{d/(d+2)}, \tag{10}$$

$$\|9\|_{L^{(d+2)/(d+1)}(\mathbb{R}^d)} \leq C(d)\,\|f\|^{1/(d+2)}_{L^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^{2d}} |\xi|^2\,|f(x,\xi)|\,dx\,d\xi\right)^{(d+1)/(d+2)}. \tag{11}$$

Proof of Lemma 2.10. We deduce the first inequality by some kind of diagonal interpolation with moments and integrability in the same time. From point (i) of Proposition 2.9 we may write

$$\varrho^{(d+2)/d} = \left(\frac{\varrho}{T^{d/2}}\right)^{2/d}\Big(\varrho T\Big) \leq C(d)\,\|f\|^{2/d}_{L^\infty(\mathbb{R}^d)}\Big(\varrho T\Big).$$

The result follows after integration in x, noticing that, for all $a \in \mathbb{R}^d$, we have

$$\varrho T \leq \int_{\mathbb{R}^d} |\xi - a|^2\,|f(x,\xi)|\,d\xi.$$

Next we prove inequality (11) following the same lines:

$$\begin{aligned} |9|^{2(d+2)} &\leq \left(\varrho \int_{\mathbb{R}^d} |\xi|^2\,|f(x,\xi)|\,d\xi\right)^{(d+2)} \\ &\leq \left(\frac{\varrho}{T^{d/2}}\right)^2 \left(\int_{\mathbb{R}^d} |\xi|^2\,|f(x,\xi)|\,d\xi\right)^{(d+2)} \Big(\varrho T\Big)^d. \end{aligned}$$

Therefore

$$|9|^{(d+2)/(d+1)} \leq C(d)\|f\|^{1/(d+1)}_{L^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |\xi|^2\,|f(x,\xi)|\,d\xi\right).$$

The result follows again after integration in x. □

As a direct consequence of these inequalities, we mention the decay in L^p spaces for the macroscopic density. We have for instance

Proposition 2.11. *Weak solutions to the kinetic-transport equation* (1) *satisfy*

$$\|\varrho(t)\|_{L^{(d+2)/d}(\mathbb{R}^d)} \leq \frac{C(d)}{t^{2d/(d+2)}}\,\|f^0\|^{2/(d+2)}_{L^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^{2d}} |x|^2\,|f^0(x,\xi)|\,dx\,d\xi\right)^{d/(d+2)}. \tag{12}$$

This inequality follows from (10) with the choice $a = x/t$ and noticing that, thanks to the exact formula (2), we have

$$\int_{\mathbb{R}^{2d}} |x - \xi t|^2\,|f(t,x,\xi)|\,dx\,d\xi = \int_{\mathbb{R}^{2d}} |x|^2\,|f^0(x,\xi)|\,dx\,d\xi$$

and

$$\|f(t)\|_{L^\infty(\mathbb{R}^d)} = \|f^0\|_{L^\infty(\mathbb{R}^d)}.$$

Of course, the propagation of L^p bounds (3) indicates that there is no chance for decay of f itself.

This kind of method can be used in nonlinear contexts to estimate time decay of macroscopic moments in the Boltzmann or repulsive Vlasov-Poisson system [78], [101].

2.5. **Velocity moments lemma.** We can illustrate another classical method for PDEs on the kinetic transport equation, namely the *method of multipliers*. We present here a sharp version of a result of Lions and Perthame [91], who improved the following ideas from [104].

Proposition 2.12. *Let $f^0 \in L^1(\mathbb{R}^d)$. Then we have*

$$\frac{1}{R}\int_{-\infty}^{\infty}\int_{B(R)}\int_{\mathbb{R}^d} |\xi|\,|f(t,x,\xi)|\,dt\,dx\,d\xi \le 2\,\|f^0\|_{L^1(\mathbb{R}^{2d})} \qquad \forall R>0,$$

$$\lim_{R\to\infty}\frac{1}{R}\int_{-\infty}^{\infty}\int_{B(R)}\int_{\mathbb{R}^d} |\xi|\,|f(t,x,\xi)|\,dt\,dx\,d\xi = 2\,\|f^0\|_{L^1(\mathbb{R}^{2d})}.$$

Proof of Proposition 2.12. A possible proof of this lemma is an illustration of the method of *multipliers*. We choose the multiplier $\frac{\xi}{|\xi|}\cdot\Psi_R(x)$, where the vector valued function Ψ_R is given by

$$\Psi_R(x) = \begin{cases} \frac{x}{R} & \text{for } |x| \le R, \\ \frac{x}{|x|} & \text{for } |x| \ge R. \end{cases}$$

Consider first the case where $f^0 \in \mathcal{S}(\mathbb{R}^{2d})$. After integrating the transport equation on $[-T,T]\times\mathbb{R}^{2d}$, we set $\hat{x} = x/|x|$ and obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} [|f(T,x,\xi)| - |f(-T,x,\xi)|]\,\frac{\xi}{|\xi|}\cdot\Psi_R(x)\,dx\,d\xi \\
&\qquad = \int_{-T}^{T}\int_{\mathbb{R}^{2d}} |f(t,x,\xi)|\,\xi\cdot\nabla_x\Big(\frac{\xi}{|\xi|}\cdot\Psi_R(x)\Big)\,dx\,d\xi\,dt \\
&\qquad = \int_{-T}^{T}\int_{\mathbb{R}^{2d}} |f(t,x,\xi)|\,\frac{\xi}{|\xi|}\cdot D\Psi_R(x)\cdot\xi\,dx\,d\xi\,dt \\
&\qquad = \frac{1}{R}\int_{-T}^{T}\int_{B(R)}\int_{\mathbb{R}^d} |f(t,x,\xi)|\,|\xi|\,dx\,d\xi\,dt \\
&\qquad\quad + \int_{-T}^{T}\int_{|x|\ge R}\int_{\mathbb{R}^d} |f(t,x,\xi)|\,\frac{|\xi - \hat{x}\,\xi\cdot\hat{x}|^2}{|\xi|}\,dx\,d\xi\,dt.
\end{aligned} \tag{13}$$

On the other hand we can compute

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} |f(T,x,\xi)|\,\frac{\xi}{|\xi|}\cdot\Psi_R(x)\,dx\,d\xi &= \int_{\mathbb{R}^{2d}} |f^0(x-\xi T,\xi)|\,\frac{\xi}{|\xi|}\cdot\Psi_R(x)\,dx\,d\xi \\
&= \int_{\mathbb{R}^{2d}} |f^0(y,\xi)|\,\frac{\xi}{|\xi|}\cdot\Psi_R(y+\xi T)\,dy\,d\xi \\
&\to \pm\int_{\mathbb{R}^{2d}} |f^0(y,\xi)|\,|\xi|\,dy\,d\xi
\end{aligned}$$

as $T\to\pm\infty$. This limit together with the identity (13) concludes the proof of the first inequality for $f^0 \in \mathcal{S}(\mathbb{R}^{2d})$. A classical approximation argument leads to the same inequality for $f^0 \in L^1(\mathbb{R}^{2d})$.

In order to prove the second statement it is again enough to do it for $f^0 \in \mathcal{S}(\mathbb{R}^{2d})$, since the quantity in the left hand side converges if the initial data converges in L^1. Then, proving this equality is equivalent to proving that the quantity

$$\int_{-\infty}^{\infty}\int_{|x|\ge R}\int_{\mathbb{R}^d} |f(t,x,\xi)|\,\frac{|\xi - \hat{x}\,\xi\cdot\hat{x}|^2}{|\xi|}\,dx\,d\xi\,dt$$

vanishes as $R \to 0$, which is readily checked by dominated convergence. □

2.6. **Stationary equation in $\mathbb{R}^d$.** We conclude this section with an application of this velocity moments lemma to steady states of kinetic equations in the full space without absorption. This kind of problem arises naturally in kinetic physics (for instance flows around an obstacle share the same mathematical aspects) and in the high frequency limit of dispersive equations; see [100], [59], [17] and the references therein. We consider equation

$$\xi \cdot \nabla_x f = g(x, \xi), \qquad x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d. \tag{14}$$

The difficulties here are twofold. Firstly, there are no direct a priori bounds by lack of absorption, i.e. zeroth order term that could provide a priori bounds (by opposition to (15) below). Secondly, uniqueness does not hold and the 'right solution' has to be selected. Indeed, particles could come in from infinity, and thus only the so-called *outgoing* solution is intuitively unique. As a counterexample to uniqueness consider for instance any two smooth functions F and G; then a family of solutions to (14) with $g = 0$ is given by

$$f(x, \xi) = F(x - x \cdot \xi \frac{\xi}{|\xi|^2}) G(\xi).$$

We notice for future reference that f satisfies a bound of weighted L^1 type, namely

$$\frac{1}{R} \int_{|x| \le R} \int_{\mathbb{R}^d} (1 + |\xi|^2) f(x, \xi) dx d\xi \le \|F\|_{L^1(\mathbb{R}^d)} \, \|(1 + |\xi|^2) G\|_{L^1(\mathbb{R}^d)}.$$

This norm is closely related to the quantities introduced in Proposition 2.12.

Of course this lack of uniqueness disappears when considering the problem with an absorption coefficient $\alpha > 0$,

$$\alpha f_\alpha + \xi \cdot \nabla_x f_\alpha = g(x, \xi), \qquad x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d. \tag{15}$$

Indeed for $g \in L^1(\mathbb{R}^{2d})$ there is a unique solution f_α in the distribution sense such that $f_\alpha \in L^1(\mathbb{R}^{2d})$, and it is given by $f_\alpha(x, \xi) = \int_0^\infty g(x - \xi t, \xi) e^{-\alpha t} dt$. Therefore, one has

$$\alpha \int_{\mathbb{R}^{2d}} f_\alpha(x, \xi) dx \, d\xi = \int_{\mathbb{R}^{2d}} g(x, \xi) dx \, d\xi,$$
$$\alpha \int_{\mathbb{R}^{2d}} |f_\alpha(x, \xi)| dx \, d\xi \le \int_{\mathbb{R}^{2d}} |g(x, \xi)| dx \, d\xi.$$

See (20) below for this last point.

Theorem 2.13. *Assume that the source g belongs to $L^1(\mathbb{R}^{2d})$. Then, the a priori bounds hold, for all $x_0 \in \mathbb{R}^d$,*

$$\frac{1}{R} \int_{|x - x_0| \le R} \int_{\mathbb{R}^d} |\xi| \, |f_\alpha(x, \xi)| dx d\xi \le 2 \|g\|_{L^1(\mathbb{R}^{2d})}, \tag{16}$$

$$\int_{\mathbb{R}^{2d}} \frac{1}{|\xi|} |\xi - \frac{(x - x_0) \cdot \xi}{|x - x_0|^2} (x - x_0)|^2 \, |f_\alpha(x, \xi)| dx d\xi \le 2 \|g\|_{L^1(\mathbb{R}^{2d})}. \tag{17}$$

Also, we have the Sommerfeld type of conditions at infinity: as $R \to \infty$ and uniformly in α,

$$\frac{1}{R} \int_{\{|x| \le R\}} \int_{\mathbb{R}^d} |\xi| \, |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2 |f_\alpha(x, \xi)| dx d\xi \to 0, \tag{18}$$

and also, as $\alpha \to 0$,

$$\alpha \int_{\mathbb{R}^{2d}} |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2 \, |f_\alpha(x,\xi)| dx d\xi \to 0. \tag{19}$$

This theorem gives two types of information. The two inequalities are a priori bounds which show that the solution is locally integrable. It is easy to find counterexamples showing that these estimates are optimal and one cannot hope for better integrability. On the other hand, the two limits are conditions of Sommerfeld type; they are known (at least for wave equations) to provide the boundary conditions at infinity expressing that energy as outgoing at infinity. We do not prove here that these conditions imply uniqueness; we refer to Perthame and Vega [105], where the case of Liouville equations with an homogeneous potential is treated.

Proof of Theorem 2.13. In order to prove (16) and (17), we first notice that

$$\alpha |f_\alpha| + \xi \cdot \nabla_x |f_\alpha| \le |g(x,\xi)|, \qquad x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d. \tag{20}$$

Then, we just argue as before and use the multiplier $\frac{\xi}{|\xi|} \cdot \nabla \Psi_R(x - x_0)$; for motivation recall the proof of Proposition 2.12. It remains to notice that

$$\begin{aligned} \alpha \int_{\mathbb{R}^{2d}} \frac{\xi}{|\xi|} \cdot \nabla \Psi_R(x-x_0) |f_\alpha| \, dx d\xi &\le \alpha \int_{\mathbb{R}^{2d}} |f_\alpha| \, dx d\xi \\ &\le \int_{\mathbb{R}^{2d}} |g| \, dx d\xi, \end{aligned}$$

and this leads to the first two estimates. We skip the details of the computation because they closely follow those of Section 2.5.

Next, we prove (18). To do that, we define

$$\rho_R(x) = \inf(1, \frac{|x|}{R}), \qquad (\text{or also } \nabla \Psi_R = \frac{x}{|x|} \, \rho_R),$$

and, following the method in the proof of Proposition 2.12, we use the combination of multipliers:

$$-2 \frac{\xi}{|\xi|} \cdot \frac{x}{|x|} \rho_R + 2\rho_R = \rho_R |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2.$$

Using it in (20), we obtain, with $\xi^t = \xi - \xi \cdot \frac{x}{|x|^2} x$,

$$\begin{aligned} &\alpha \int_{\mathbb{R}^{2d}} \rho_R(x) |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2 |f_\alpha(x,\xi)| dx d\xi \\ &\quad + \int_{\mathbb{R}^{2d}} \big[\frac{|\xi|}{R} \, |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2 \, \mathbf{1}_{\{|x| \le R\}} + 2 \frac{|\xi^t|^2}{|\xi| \, |x|} \, \mathbf{1}_{\{|x| \ge R\}} \big] \, |f_\alpha(x,\xi)| dx d\xi \\ &\le \int_{\mathbb{R}^{2d}} \rho_R \, |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2 \, |g(x,\xi)| dx d\xi. \end{aligned} \tag{21}$$

Therefore, we have obtained

$$\frac{1}{R} \int_{\{|x| \le R\}} \int_{\mathbb{R}^d} |\xi| \, |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2 |f_\alpha(x,\xi)| dx d\xi \le 2 \int_{\{|x| \le R\}} \int_{\mathbb{R}^d} |g(x,\xi)| dx d\xi.$$

From this (18) follows directly.

Finally, we prove (19) following [105] again. Here we assume that $g \ge 0$ and thus $f_\alpha \ge 0$ which does not affect the generality of the proof. Since f_α is an increasing sequence it converges to a function f that satisfies the same equation with $\alpha = 0$ and the already proved bounds. We first derive the identity

$$\int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|\xi|\,|x|} f(x,\xi) dx d\xi = \frac{1}{2} \int_{\mathbb{R}^{2d}} |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2 \, |g(x,\xi)| dx d\xi. \tag{22}$$

This requires us to use a truncation function

$$\varphi_R(x) = \varphi(\frac{|x|}{R}), \quad \varphi(r) = \begin{cases} 1 & \text{for } 0 \le r \le 1, \\ 2-r & \text{for } 1 \le r \le 2, \\ 0 & \text{for } r \ge 2. \end{cases} \tag{23}$$

Then we have, using the multiplier $\frac{\xi}{|\xi|} \cdot \frac{x}{|x|}\varphi_R$,

$$-\int_{\mathbb{R}^{2d}} [\frac{|\xi^t|^2}{|\xi|\,|x|}\varphi_R - \frac{|\xi|}{R}\; \mathbf{1}_{\{R \le |x| \le 2R\}} (\frac{\xi}{|\xi|} \cdot \frac{x}{|x|})^2] f(x,\xi)\, dx d\xi$$
$$\to \int_{\mathbb{R}^{2d}} \frac{\xi}{|\xi|} \cdot \frac{x}{|x|} g \, dx d\xi,$$

as $R \to \infty$. On the other hand, using (18),

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \frac{|\xi|}{R}\; \mathbf{1}_{\{R \le |x| \le 2R\}} (\frac{\xi}{|\xi|} \cdot \frac{x}{|x|})^2 f(x,\xi)\, dx d\xi \\ &\quad = \int_{\mathbb{R}^{2d}} \frac{|\xi|}{R}\; \mathbf{1}_{\{R \le |x| \le 2R\}} \frac{\xi}{|\xi|} \cdot \frac{x}{|x|} f(x,\xi)\, dx d\xi + o(R) \\ &\quad = -\int_{\mathbb{R}^{2d}} \xi \cdot \nabla\varphi_R f(x,\xi)\, dx d\xi + o(R) \to -\int_{\mathbb{R}^{2d}} g \, dx d\xi. \end{aligned}$$

These two limits combined give (22).

Next, we use again the equation with $\alpha > 0$ and just compute

$$\begin{aligned} &\alpha \int_{\mathbb{R}^{2d}} |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2 \; f_\alpha(x,\xi) dx d\xi + 2 \int_{\mathbb{R}^{2d}} \frac{|\xi^t|^2}{|\xi|\,|x|} \; f_\alpha(x,\xi) dx d\xi \\ &\quad = \int_{\mathbb{R}^{2d}} |\frac{\xi}{|\xi|} - \frac{x}{|x|}|^2 \; g(x,\xi) dx d\xi. \end{aligned}$$

But we can identify the limit as $\alpha \to 0$ in the second term thanks to the a priori bound (17), and inserting (22) in the above equality we obtain (19). $\square$

3. Kinetic-transport equation (velocity averaging lemma)

Because kinetic-transport equations are first order hyperbolic, regularity (and compactness) of the solutions does not follow from a priori bounds, unlike parabolic equations. It is wrong that the solution operator $f^0 \to f(t)$ is compact in L^p spaces. However, compactness can be proved for macroscopic quantities as

$$\varrho_\psi(t,x) := \int_{\mathbb{R}^d} \psi(\xi) f(t,x,\xi)\; d\xi, \tag{24}$$

with ψ a smooth test function. This is the theory of averaging lemmas for kinetic equations. It began with the papers [66], [65] in order to solve compactness problems (see also [1]). Then, several extensions with a right hand side which can be derivatives of L^2 functions were given in [47], and this was extended to a general L^p framework in [50]. This step allowed one to use averaging lemmas for Vlasov type equations and for kinetic formulations. There have been several variants, extensions and generalizations that can be found in Gérard [57], Gérard and Golse [58], Perthame and Souganidis [103], Bouchut [23], Bouchut and Desvillettes [24], Devore and Petrova [45], Jabin and Perthame [81], Lions [88], Westdickenberg [118]; and other improvements can be found in the references therein. The method of proof of averaging lemmas is always based on Fourier arguments in L^2, and singular integrals in L^p. Notice, however, that a recent approach which only uses duality arguments is given in Jabin and Vega [82].

3.1. **A simple case of averaging.** In order to explain in a simple example the interest of averaging lemmas, we begin with a very simple case in L^2 for kinetic-transport. It uses the Hilbert space $H^{1/2}$ defined by the norm $\|\varrho\|_{\dot{H}^{1/2}} + \|\varrho\|_{L^2}$, and the homogeneous seminorm $\|\cdot\|_{\dot{H}^{1/2}}$ is defined in the proof below.

Theorem 3.1. *Let $\psi \in L^\infty(\mathbb{R}^d)$, supp$\psi \subset B(R)$ (ball of radius R centered at origin) in* (24) *and let $f, g \in L^2(\mathbb{R}^{1+2d})$ satisfy*

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla_x f = g. \tag{25}$$

Then, we have

$$\|\varrho_\psi\|_{\dot{H}^{1/2}(\mathbb{R}^{d+1})} \leq C(d, R, \|\psi\|_{L^\infty})\, \|f\|^{1/2}_{L^2(\mathbb{R}^{2d+1})} \|\, g\|^{1/2}_{L^2(\mathbb{R}^{2d+1})}. \tag{26}$$

Remark 3.2. As it is, this statement does not apply to the Cauchy problem (1), but after time truncation by a function $\psi(t)$ with compact support, we find

$$\frac{\partial}{\partial t}(f\psi) + \xi \cdot \nabla_x (f\psi) = g := f\psi'(t),$$

and thus we recover the situation (26) with L^2 solution and right hand side.

Proof of Theorem 3.1. The proof is based on Fourier Transform in space and time. We define

$$\widehat{f}(\tau, k, \xi) = \int_{\mathbb{R}^d} f(t, x, \xi) e^{i(t\tau + x\cdot\xi)} dx\, dt,$$

and similarly $\widehat{g}$, $\widehat{\varrho}_\psi$. We use below the property that

$$\widehat{\varrho}_\psi(\tau, k) = \int_{\mathbb{R}^d} \psi(\xi) \widehat{f} d\xi.$$

Equation (25) becomes very simple after Fourier Transform and reads

$$i(\tau + k\cdot\xi)\widehat{f} = \widehat{g}, \quad (\tau, k, \xi) \in \mathbb{R}^{2d+1}.$$

This allows us to invert the symbol, and in order to avoid the singular hyperplane $\tau + \xi \cdot k = 0$, we add $\beta \widehat{f}$ on each side of the equality, with β a positive real number to be chosen later. We obtain

$$\widehat{f}(\tau, k) = \frac{\widehat{g} + \beta \widehat{f}}{\beta + i(\tau + \xi\cdot k)}.$$

After using Cauchy-Schwarz inequality, we deduce

$$|\widehat{\varrho}|^2 \leq 2 \int_{\mathbb{R}^d} \Big(|\widehat{g}|^2 + \beta^2 |\widehat{f}|^2\Big) d\xi \int_{\mathbb{R}^d} \frac{\psi^2}{\beta^2 + |\tau + \xi\cdot k|^2} d\xi.$$

After using Lemma 3.3 below we deduce

$$|\widehat{\varrho}|^2 \leq \int_{\mathbb{R}^d} \Big(|\widehat{g}|^2 + \beta^2 |\widehat{f}|^2\Big) d\xi \frac{C}{\beta(|\tau| + |k|)},$$

and after integrating in τ, ξ,

$$\|(|\tau|+|k|)\widehat{\varrho}\|^2_{L^2(\mathbb{R}^{d+1})} \le C\frac{\|\widehat{g}\|^2_{L^2(\mathbb{R}^{d+1})}+\beta^2\|\widehat{f}\|^2_{L^2(\mathbb{R}^{d+1})}}{\beta}.$$

After choosing $\beta = \|\widehat{g}\|_{L^2(\mathbb{R}^{d+1})}/\|\widehat{f}\|_{L^2(\mathbb{R}^{d+1})}$, we obtain

$$\|(|\tau|+|k|)\widehat{\varrho}\|^2_{L^2(\mathbb{R}^{d+1})} \le C\|\widehat{g}\|_{L^2(\mathbb{R}^{d+1})}\ \|\widehat{f}\|_{L^2(\mathbb{R}^{d+1})}.$$

This is exactly the definition of the homogeneous $\dot{H}^{1/2}(\mathbb{R}^{d+1})$ seminorm

$$\|\varrho\|_{\dot{H}^{1/2}(\mathbb{R}^{d+1})} = \|(|\tau|+|k|)\widehat{\varrho}\|^2_{L^2(\mathbb{R}^{d+1})},$$

and thus Theorem 3.1 is proved. □

Lemma 3.3. *With the above notations, we have for all $\beta > 0$,*

$$\int_{\mathbb{R}^d}\frac{\psi^2}{\beta^2+|\tau+\xi\cdot k|^2}d\xi \le \frac{C(\psi)}{\beta(|\tau|+|k|+\beta)},$$

with (we recall that supp$\psi \subset B(R)$),

$$C(\psi) = C\ R\sup_{\omega\in\mathbb{S}^{d-1}}\int_{\omega\cdot\xi=0}\psi(\xi)^2d\xi', \qquad d\xi' \text{ the Lebesgue measure on } \{\omega\cdot\xi=0\}.$$

We leave this lemma without proof.

Among possible extensions, one can consider more general integrals than (24). For instance velocities on the sphere arise in several applications, and a remarkable phenomena occurs then. To explain it, we define

$$\varrho_S(t,x) = \int_{\xi\in\mathbb{S}^{d-1}} f(t,x,\xi)d\xi.$$

In dimension $d \ge 3$, one can prove that Theorem 3.1 still holds and a half derivative is gained on averages. However, in two dimensions, due to scaling properties which are particular, the averages on the sphere ϱ_S are smoother than f only by $1/4$ derivatives. More precisely

$$\left\|(1+|\tau|+|\xi|)^{1/4}\widehat{\varrho_S}(\tau,\xi)\right\|_{L^2(\mathbb{R}^{1+2})} \le C\left(\|f\|_{L^2(\mathbb{R}^{1+2}\times\mathbb{S}^1)}+\ \|g\|_{L^2(\mathbb{R}^{1+2}\times\mathbb{S}^1)}\right). \tag{27}$$

However, a more careful analysis will show that away from the light cone $\{|\tau| = |\xi|\}$ the average ϱ_S does have $1/2$ derivatives in L^2. More precisely, the inequality holds:

$$\left\|(1+|(|\tau|-|\xi|)|)^{1/4}\,(1+|\tau|+|\xi|)^{1/4}\widehat{\varrho_S}(\tau,\xi)\right\|_{L^2(\mathbb{R}^{1+2})}$$
$$\le C\left(\|f\|_{L^2(\mathbb{R}^{1+2}\times\mathbb{S}^1)}+\ \|g\|_{L^2(\mathbb{R}^{1+2}\times\mathbb{S}^1)}\right).$$

Weighted Sobolev spaces of this type are classical for wave equations. We refer to Bournaveas and Perthame [27] for this result and references on these spaces. An interesting question is to know whether a similar gain of extra regularity is possible in other situations.

3.2. **General case of compactness by averaging.** We now consider the most unfavorable case of a transport equation with a singular right hand side where we allow a full space-time derivative and as many ξ derivatives as we wish. Let $f(t,x,\xi)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$ be a given global solution to a transport equation where we allow a singular source

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla_x f = \frac{\partial^{|k|}}{\partial \xi^k} \left[\frac{\partial}{\partial t} g_0 + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} g_i \right]. \tag{28}$$

Here, we have used the notations $k = (k_1, \cdots, k_p)$, $|k| = k_1 + \cdots + k_p$, $k_i \in \mathbb{N}$. This structure appears in numerous nonlinear problems: Vlasov equations in §4.1 (Vlasov-Maxwell system was the motivation in [47] for the first study of this kind of structure), scalar conservation laws in §4.4 for instance, and Fokker-Planck operators.

In such a situation, generic regularity on the averages, in the spirit of Theorem 3.1, is not true. Nevertheless compactness can still be proved. The new difficulty is that given the right hand side with $g_i \in L^p$, the above equation cannot be solved; more precisely one cannot define an operator $g_i \to f$ in Lebesgue spaces in order to use interpolation.

The version we give below is that of [103] with the limiting exponent proved by Bouchut in [26].

Theorem 3.4. *Let f, g_i belong to $L^p(\mathbb{R}^{1+2d})$ for some $1 < p < \infty$ and satisfy equation* (28)*, and let $\psi \in \mathcal{D}(\mathbb{R}^d)$ in* (24)*. Then, we have*

$$\|\varrho_\psi\|_{L^p(\mathbb{R}^{d+1})} \leq C(d,p,\psi)\, \|f\|^{1-\alpha}_{L^p(\mathbb{R}^{2d+1})} \| g\|^{\alpha}_{L^p(\mathbb{R}^{2d+1})}, \tag{29}$$

for all α satisfying

$$0 \leq \alpha \leq \frac{1}{|k|+1} \min(\frac{1}{p}, \frac{1}{p'}), \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Remark 3.5. Notice that since the Hölder inequality also gives

$$\|\varrho_\psi\|_{L^p(\mathbb{R}^{d+1})} \leq \|f\|_{L^p(\mathbb{R}^{2d+1})}\, \|\psi\|_{L^{p'}(\mathbb{R}^d)},$$

the boundedness of ϱ_ψ in $L^p(\mathbb{R}^{d+1})$ is obvious without the help of equation (28). Interest comes from the control of the average in terms of g rather than f, and of course the most interesting case is $\alpha = \frac{1}{|k|+1} \min(\frac{1}{p}, \frac{1}{p'})$.

We now state a direct consequence which explains why compactness is contained in the above theorem.

Corollary 3.6. *Consider two sequences $\{f_n\}$, $\{g_{i,n}\}$ of solutions to the transport equation* (28) *and let ψ belong to $L^{p'}(\mathbb{R}^d)$. Assume that for some $1 < p < \infty$ the sequence $\{f_n\}$ is bounded in $L^p(\mathbb{R}^{1+2d})$ and the sequence $\{g_{i,n}\}$ is relatively compact in $L^p(\mathbb{R}^{1+2d})$. Then the averages $\varrho_{\psi,n}$ are relatively compact in $L^p(\mathbb{R}^{1+d})$.*

In situations where less than a full space derivative occurs in the right hand side of the transport equation (28), not only compactness is gained. Regularity in the scales of Sobolev or Besov spaces can be proved, extending the result of Theorem 3.1; we refer to [50], and the chapter by Bouchut in [26]. The precise gain in regularity, and not only compactness, can be useful for regularity questions. This appears for instance in the topic of nondegenerate hyperbolic scalar balance law, where the kinetic formulation provides a method for proving regularizing effects [92],

[81] and the corresponding Sobolev spaces are deduced from those at the kinetic level.

3.3. **Proof of the general compactness result.** We only prove the case of L^2 spaces here. The main idea to obtain other values of p is to use interpolation, but this is not possible directly on equation (28), which does not define an operator (remember it does not have a L^2 solution for all g). For that reason, interpolation has to be performed on the different operators that arise after inverting the main symbol. It turns out that one can prove that these operators are of Calderón-Zygmund type and thus treat also the Hardy spaces (which serve as L^1 for interpolation purposes). But there is an additional difficulty, which is that the proof requires us to estimate a precise Hardy norm, and this is again not possible directly due to the special form of the Fourier multiplier. A possibility consists in using L^p spaces, $p > 1$ instead, but then the limiting exponent $\alpha = \frac{1}{|k|+1}\min(\frac{1}{p}, \frac{1}{p'})$ is lost (see [103]). The idea to use the so-called Hardy-product spaces was introduced in this context by [21] and formalized for the pure compactness case by Bouchut in [26]. It allows us to complete the proof of Theorem 3.4.

First of all and in order to simplify notations, we do not consider the term $\frac{\partial}{\partial t}g_0$. Then, we divide this proof of the L^2 case into three steps. First, we present the method together with technical lemmas. Then, we prove these lemmas in steps 2 and 3.

First step. Method of proof. Denoting $\widehat{f}(\tau, k, \xi)$ the Fourier transform of f in the (t, x) variables, equation (28) yields

$$(\tau + k \cdot \xi)\widehat{f} = \sum_{j=1}^{d} k_j \frac{\partial^{|k|}}{\partial \xi^k}\widehat{g}_j,$$

which can be rewritten for $\beta > 0$ as

$$\widehat{f}\,[(\tau + \xi \cdot k)^2 + \beta^2|k|^2] = \beta^2|k|^2\widehat{f} + \sum_{j=1}^{d} k_j(\tau + \xi \cdot k)\frac{\partial^{|k|}}{\partial \xi^k}\widehat{g}_j.$$

In other words

$$f = f_0 + \sum_{j=1}^{d} f_j \tag{30}$$

with

$$\widehat{f_0} = \frac{\beta^2|k|^2}{(\tau + \xi \cdot k)^2 + \beta^2|k|^2}\widehat{g}_0, \quad \text{where } g_0 = f, \tag{31}$$

and, for $1 \le j \le d$,

$$\widehat{f_j} = \frac{k_j(\tau + \xi \cdot k)}{(\tau + \xi \cdot k)^2 + \beta^2|k|^2}\frac{\partial^{|k|}}{\partial \xi^k}\widehat{g}_j. \tag{32}$$

We study separately the operators $(T_j)_{0\le j\le d}$ which are defined by

$$\varrho_j(t, x) = \int \psi(\xi) f_j(t, x, \xi) d\xi := T_j g_j(x, t).$$

We need the following two lemmas, which we state below and then prove in steps 2 and 3 below.

Lemma 3.7. *Let $\beta > 0$. Then*

$$\|\varrho_0\|_{L^2(\mathbb{R}^{d+1})} \leq C(d, R, \|\psi\|_\infty)\beta^{1/2}\|f\|_{L^2(Q_R)}.$$

Lemma 3.8. *Let $0 < \beta \leq 1$. Then, for all $1 \leq j \leq d$,*

$$\|\varrho_j\|_{L^2(\mathbb{R}^{d+1})} \leq C(d, R, \|\psi\|_{k,\infty})\beta^{-|k|-\frac{1}{2}}\|g_j\|_{L^2(Q_R)}.$$

We now continue with the proof of Theorem 3.4. It is obtained by combining these two lemmas. The average in (24) is exactly

$$\varrho = \sum_{j=0} \varrho_j,$$

and it is upper bounded by

$$\|\varrho\|_{L^2(\mathbb{R}^{d+1})} \leq C(d, R, \|\psi\|_{k,\infty})\Big[\beta^{1/2}\|g_0\|_{L^2(Q_R)} + \frac{\beta^{1/2}}{\beta^{|k|+1}}\|g\|_{L^2(Q_R)}\Big]. \tag{33}$$

Next, we recall that $g_0 = f$ and we choose β as

$$\beta^{|k|+1} = \|g\|_{L^2}/\|f\|_{L^2}. \tag{34}$$

We obtain the value $\alpha = \frac{1}{2(|k|+1)}$ in Theorem 3.4 for $p = 2$ when $\|g\|_{L^2}/\|f\|_{L^2} \leq 1$, i.e. $\beta \leq 1$, a limitation needed to apply Lemma 3.8. If $\beta \geq 1$, we just use Remark 3.5. This completes the proof.

Second step. The Proof of Lemma 3.7. Using the averaging technique from (31) and defining $\lambda = \frac{\tau}{\beta|k|}$ and $\xi_1 = \frac{\xi\cdot k}{|k|}$, we obtain

$$\begin{aligned} |\widehat{\varrho}_0(k,\tau)|^2 &\leq \int |\widehat{g}_0|^2 d\xi \int \psi^2(\xi)\frac{\beta^4|k|^4}{[(\tau+\xi\cdot k)^2 + \beta^2|k|^2]^2}d\xi \\ &\leq C(d)\|\psi\|_\infty^2 R^{d-1}\int \frac{d\xi_1}{[(\lambda + \frac{\xi_1}{\beta})^2 + 1]^2}\int |\widehat{g}_0|^2 d\xi. \end{aligned}$$

Since the above integral in ξ_1 is proportional to β, we obtain

$$\|T_0 g_0\|_{L^2(\mathbb{R}^{d+1})} \leq C(R)\beta^{1/2}\|g_0\|_{L^2(Q_R)}, \tag{35}$$

and Lemma 3.7 is proved. □

Third step. The Proof of Lemma 3.8. We begin with the proof of the case $k = 0$. It follows that of Lemma 3.7. We change variables and obtain, with the same notations as in step 2,

$$\widehat{F}_j(k,\sigma) = \frac{1}{\beta}\,\frac{k_j\sigma}{|k|^2+\sigma^2}\widehat{G}_j(k,\sigma).$$

We have

$$\widehat{\varrho}_j(\tau,k) = \int \psi\frac{k_j k_1}{(\tau+\xi\cdot k)^2 + \beta^2|k|^2}\widehat{g}_j d\xi,$$

and thus

$$|\widehat{\varrho}_j(\tau,k)|^2 \leq \frac{C(R)}{\beta^2}\int |\widehat{g}_j|^2 d\xi \int \frac{|\lambda+\xi_1/\beta|^2}{[|\lambda+\xi_1/\beta|^2+1]^2}d\xi_1,$$

and Lemma 3.8 is proved in the case $k = 0$.

Next, we pass to the case $|k| = 1$; for instance $k = (1, 0, \ldots)$. We need a preliminary step. Using Green's formula, we have

$$\widehat{\varrho}_j(\tau, k) = -\int \frac{\partial \psi}{\partial \xi_1} \frac{k_j(\tau + \xi \cdot k)}{(\tau + \xi \cdot k)^2 + \beta^2 |k|^2} \,\widehat{g}_j d\xi$$
$$- \int \psi \frac{k_j k_1}{(\tau + \xi \cdot k)^2 + \beta^2 |k|^2} \widehat{g}_j d\xi + 2 \int \psi \frac{k_j(\tau + \xi \cdot k)^2 k_1}{[(\tau + \xi \cdot k)^2 + \beta^2 |k|^2]^2} \,\widehat{g}_j d\xi.$$

This defines three operators, S_1, S_2, S_3, for which we may apply the same proof as before, but with different powers of β. The only scaling factors in β play a role to estimate the different operators S_k. We obtain as above that S_1, like T_j for $k = 0$, has L^2-norm of order $C(R)\beta^{-1/2}$ (this gives the dominant term for $\beta > 1$), while the L^2-norms of S_2 and S_3 are of order $C(R)\beta^{-3/2}$ and give the dominant terms for $\beta < 1$. Again, this generalizes to other values of k as indicated above, thus proving Lemma 3.8. □

4. Nonlinear kinetic equations

In this section we review the theory and applications of three large classes of nonlinear kinetic equations: Vlasov equations, scattering equations and the Boltzmann equation. We also present the kinetic formulation as a method to reduce general nonlinear parabolic equations to a singular kinetic equation.

4.1. **Vlasov equations.** A first extension of free transport consists in the Lagrangian dynamics for particles undergoing a force $F(x, \xi)$

$$\left\{ \begin{array}{lll} \frac{d}{dt} X(t) & = \zeta(t), & X(t = 0) = x, \\ \frac{d}{dt} \zeta(t) & = F(X(t), \zeta(t)), & \zeta(t = 0) = \xi. \end{array} \right. \tag{36}$$

The general Liouville equation describing the evolution of the density is then written

$$\frac{\partial}{\partial t} f(t, x, \xi) + \xi \cdot \nabla_x f + \mathrm{div}_\xi [F(x, \xi)\, f] = 0, \qquad t \geq 0,\ x,\ \xi \in \mathbb{R}^d.$$

This equation is still linear when F is given, but usually nonlinearities arise because the force field is created by the particles themselves. Then, the equations are called *Vlasov equations* or 'mean field' equations and express that forces act with a long range by opposition to Boltzmann type equations; see §4.3. A classical case of Vlasov equations is as follows:

$$\frac{\partial}{\partial t} f(t, x, \xi) + \xi \cdot \nabla_x f + \mathrm{div}_\xi [E(t, x)\, f] = 0, \tag{37}$$

$$\left\{ \begin{array}{l} E(t, x) = -\nabla_x U(t, x), \quad U = V(|x|) \star \varrho(t, x), \\ \varrho(t, x) = \int_{\mathbb{R}^d} f(t, x, \xi)\, d\xi. \end{array} \right. \tag{38}$$

System (37)–(38) has several properties, and the conservation laws play an essential role. Notice that in the phase plane (x, ξ) the field $(\xi, E(t, x))$ is divergence free. Therefore one readily checks that for strong solutions one has the equalities

$$\begin{array}{lll} \int_{\mathbb{R}^{2d}} f(t, x, \xi)\, dx d\xi & = \int_{\mathbb{R}^{2d}} f^0(x, \xi)\, dx d\xi, & (\textit{mass conservation}), \\ \|f(t)\|_{L^p(\mathbb{R}^d)} & = \|f^0\|_{L^p(\mathbb{R}^d)}, & (\textit{Liouville principle}), \\ \mathcal{E}_c(t) + \mathcal{E}_p(t) & = \mathcal{E}_c(t = 0) + \mathcal{E}_p(t = 0), & (\textit{energy conservation}), \end{array}$$

with the kinetic energy $\mathcal{E}_c$ and the potential energy $\mathcal{E}_p$ defined by

$$\begin{cases} \mathcal{E}_c(t) &= \int_{\mathbb{R}^{2d}} \frac{|\xi|^2}{2} f(t,x,\xi)\,dxd\xi \\ \mathcal{E}_p(t) &= \int_{\mathbb{R}^{2d}} \frac{1}{2} V(x-y)\varrho(t,x)\varrho(t,y)dxdy. \end{cases}$$

It is called a *Vlasov-Poisson* equation when used with the particular and singular potential

$$V(r) = \alpha \frac{1}{r^{d-2}}, \quad \alpha = \pm 1 \qquad (-\alpha \ln(r) \; for \; d = 2). \tag{39}$$

The idea here is that particles attract ($\alpha = -1$, Newtonian force) or repulse each other ($\alpha = +1$, Coulombic forces) along with a potential. It arises in astrophysics; then V is Coulomb potential (see Batt and Rein [13]). It also arises in plasma physics and semiconductor modelling (Markowich, Ringhofer and Schmeiser [94], Degond [38], Arnold, Carrillo, Gamba and Shu [6], Ben Abdallah *et al.* [16]).

Existence of weak solutions to Vlasov-Poisson, for both Newtonian (in dimensions less than 3) and Coulombic potentials in any dimension, goes back to Horst and Hunze [76] under the assumption that the initial data satisfying $f^0(t=0) \in L^1 \cap L^\infty(\mathbb{R}^{2d})$ has finite energy (the difficulties are to control the kinetic and potential energy separately in the Newtonian case, and to pass to the weak limit in the nonlinear term in both cases). An extension to more general initial data (in terms of L^p spaces) has been carried out by DiPerna and Lions [46]. As for classical solutions, in one or two dimensions the situation has been settled since the 80's (see Horst [75] and the references therein). Three dimensional classical solutions are obtained by a method of characteristics in a series of papers by Pfaffelmoser [107], by Schaeffer [110] and by Batt and Rein [12] in the periodic case (see also the references therein). A method based on PDE arguments which allows one to propagate in time any given initial velocity moments (and existence of unique classical solutions follows) was obtained in three dimensions by Lions and Perthame [90] (for moments larger than 3) and by Gasser, Jabin and Perthame [56] for any moment (this uses a combination of Lemma 2.1 and Proposition 2.12). A subject of current interest is the case of infinite mass (initial data does not decay at infinity); see Caprino, Marchioro and Pulvirenti [30].

More recently models of Vlasov type have been considered in fluid mechanics to describe sprays, bubbles or solid particles interacting through a fluid [74], [108], [69], [15]. It is then natural to restrict to dimension $d = 3$. For instance the Vlasov-Stokes system reads [80], [56]:

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla_x f + \mathrm{div}_\xi[(U-\xi)f] = 0, \qquad t \geq 0,\, x,\, \xi \in \mathbb{R}^d, \tag{40}$$

$$U(x,t) = A(x) \star j(t,x), \qquad j(t,x) = \int_{\mathbb{R}^d} \xi f(t,x,\xi) d\xi. \tag{41}$$

Here, the matrix $A \in \Big(C^\infty(\mathbb{R}^d \backslash 0)\Big)^{d\times d}$ is typically the fundamental solution of the Stokes equation,

$$\begin{cases} \displaystyle\sum_{j=1}^{d} \frac{\partial}{\partial x_j} A_{ij}(x) &= 0, \\ \Delta A_{ij}(x) + \frac{\partial}{\partial x_j} p_i(x) &= \delta(x=0)e_i, \end{cases}$$

for all $1 \le i \le d$ and where e_i denotes the $i-th$ basis vector and p_i the pressure. The matrix A is assumed to satisfy two properties. The first property gives a limitation on the possible singularity at the origin; the second expresses the dissipation of the kinetic energy of the system (a natural condition since it is realized for the particle system)

$$|A(x)| \le \frac{C}{|x|^\beta}, \quad 0 < \beta < 2, \tag{42}$$

$$\int_{\mathbb{R}^d} j(x) \cdot A(x) \star j(x) dx \le 0, \quad \forall j \in \mathcal{D}(\mathbb{R}). \tag{43}$$

As for other related equations, let us quote the case of the Vlasov-Poisson-Fokker-Planck model

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla_x f + \mathrm{div}_\xi (E\, f) = \Delta_\xi f + \sigma \mathrm{div}_\xi (\xi\, f).$$

Additionally to the potential forces as in the Vlasov-Poisson case, this equation expresses the diffusion in ξ (collisions with external gas molecules for instance) and friction with coefficient σ. It is fully treated in Bouchut [23], and the regularizing effect makes solutions that are smooth; the long time asymptotics is treated in Dolbeault [51], Bouchut and Dolbeault [25]. As for other coupling let us quote Desvillettes and Dolbeault [42] and Mischler [95] for the coupled Vlasov-Boltzmann system.

The linear case is not so simple; see Desvillettes and Villani [43], Glassey and Strauss [64]. The subject of long time asymptotics has attracted much interest, also for the pure repulsive Vlasov case, where it is expected that the combination of transport dispersion and repulsion create a faster time decay than pure transport (see §2.4 and equation (12)), but this is not the case; see Illner and Rein [78], Perthame [101]. Existence of weak solutions to the Vlasov Maxwell system are obtained in DiPerna and Lions [47], and this was the first example, after Radiative Transfer equations (§4.2), where the existence proof requires compactness that can only be obtained through averaging lemmas. Also the notion of renormalized solution is essential here to get the most general initial data [48]. Strong solutions with small initial data have also undergone considerable progress by Glassey and Schaeffer [62], Glassey and Strauss [64], [63], and Guo and Strauss [73], [72].

4.2. **Scattering and chemotaxis.** One of the most classical kinetic models arises in the description of transport of particles (neutrons for instance, energy waves more generally) that only deviate from kinetic-transport by collisions with a fixed media (say random scatterers). This raises the following *scattering equation*, which is posed for $t \ge 0$, $x \in \mathbb{R}^d$, $\xi \in V \subset \mathbb{R}^d$:

$$\frac{\partial}{\partial t} f(t,x,\xi) + \xi \cdot \nabla_x f + \int_{\mathbb{R}^d} [K(\xi,\xi') f(t,x,\xi) - K(\xi',\xi) f(t,x,\xi')] d\xi' = 0. \tag{44}$$

The set V denotes the possible allowed velocities: it can be $\mathbb{R}^d$ itself, the unit ball B, or naturally the unit sphere $\mathbb{S}^{d-1}$ if one thinks of photons for instance. The function $K(\xi,\xi') \ge 0$ is called the scattering kernel and may also depend on (t,x) through quantities related to the density f itself (we give two examples below). It has been widely studied, and classical references are Bardos, Santos, Sentis [11], and Dautray and Lions [36]. Homogenization of such equations is important for nuclear engineering: Bensoussan, Lions and Papanicolaou [20], Allaire *et al.* [3], Goudon and Poupaud [70].

In terms of mathematical theory, this linear model also exhibits the maximum principle (if $f(t=0) \geq 0$, then $f(t) \geq 0$), and mass conservation

$$\int_{\mathbb{R}^d \times V} f(t,x,\xi) dx\, d\xi = \int_{\mathbb{R}^d \times V} f(0,x,\xi) dx\, d\xi.$$

As a consequence, it generates an L^1 semigroup of solutions, and therefore when K is a given bounded function we have a unique distributional solution $f \in C(\mathbb{R}^+; L^1(\mathbb{R}^d \times V)$. When K is symmetric, i.e. $K(\xi', \xi) = K(\xi, \xi')$, we have additionally

$$\|f(t)\|_{L^p(\mathbb{R}^d \times V} \leq \|f(t=0)\|_{L^p(\mathbb{R}^d \times V)} \quad \forall p, \quad 1 \leq p \leq \infty,$$

but these last a priori bounds are lost in the nonsymmetric case (except for $p = 1$, mass conservation).

A nonlinear model of the same type arises in *radiative transfer*. Here f denotes the density of photons which are scattered by a media itself heated by the photons. In the simplest modeling case we arrive at

$$\begin{cases} K(t,x;\xi,\xi') = k(\langle f(t,x) \rangle), & V = \mathbb{S}^{d-1}, \\ \langle f(t,x) \rangle = \int_{\mathbb{S}^{d-1}} f(t,x,\xi) d\xi, \end{cases}$$

with $k(\cdot) > 0$ a smooth function. See Bardos *et al.* [10], and Dubroca and Feugeas [52]. When the nonlinearity $k(\cdot)$ is nonincreasing, the solution operator is a strong contraction in L^1 and existence of a solution follows from BV bounds. In more general cases the existence theory relies on compactness derived from averaging lemmas; see [10] for details.

An interesting nonlinearity also arises in models from biology and describes the chemotactic motion of bacteria or more general cells. It was introduced in Othmer, Dunbar and Alt [99] as a microscopic version of the Keller-Segel model. The interpretation of K is now the probability that bacteria turn from a direction of motion ξ' to a direction ξ (the set of velocities is choosen as $V = B$ the unit ball of $\mathbb{R}^d$). This usually occurs with a uniform law $K = Cst$ say. But in the case of chemotactic motion, an external (say chemical) signal S can produce a small deviation from this law. An example of the scattering kernel produced is then

$$K(t,x;\xi,\xi') = \alpha_+ k(S(x+\varepsilon\xi, t)) + \alpha_- k(S(x - \varepsilon\xi', t)), \tag{45}$$

for some given positive and increasing function $k : \mathbb{R} \to \mathbb{R}$ and nonnegative parameters $\alpha_\pm$ and ε. This creates a nonsymmetric kernel which allows some drift in the mean motion. When the signal is emitted by the bacteria themselves a coupling arises; for instance, recalling the definition of ϱ in (4),

$$-\Delta S = \varrho(t,x). \tag{46}$$

This is again a nonlinear mean field equation since the interaction is long range. Existence of strong solutions has been proved, and the difficulty lies in the lack of strong a priori bounds since K is not symmetric. Also these global strong solutions show a fundamental difference with the macroscopic model drift-diffusion model (the Keller-Segel equation). The latter exhibits 'chemotactic collapse' (concentration as a pointwise Dirac mass) in 2 dimensions at least and more generally blow-up (some L^p norm is unbounded) in finite time. This global existence result and the asymptotic limit towards the Keller-Segel model can be found in Chalub *et al.* [34]; see also §5.2.

4.3. **The Boltzmann equation.** On the other hand, for short range potentials, collisional models are used and the upmost classical equation is the Boltzmann equation. It reads

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla f = Q(f), \quad t \geq 0,\ x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^d, \tag{47}$$

where Q denotes Boltzmann's quadratic collision operator

$$Q(f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f' f'_* - f f_*) B(|n.(\xi - \xi_*)|, |\xi - \xi_*|) d\xi_* dn = Q^+(f) - Q^-(f), \tag{48}$$

where we have used the notation $f' = f(t, x, \xi')$, $f_* = f(t, x, \xi_*)\ldots.$ For elastic collisions, the postcollisional velocities are defined for $n \in \mathbb{S}^{d-1}$ by

$$\begin{cases} \xi' = \xi \ - n[(\xi - \xi_*) \cdot n], \\ \xi'_* = \xi_* + n[(\xi - \xi_*) \cdot n]. \end{cases} \tag{49}$$

These rules are dictated by the conservation laws for momentum and energy

$$\begin{cases} \xi + \xi_* = \xi' + \xi'_*, \\ |\xi|^2 + |\xi_*|^2 = |\xi'|^2 + |\xi'_*|^2. \end{cases} \tag{50}$$

The linear micro-collision operator defined for $n \in \mathbb{S}^{d-1}$ by

$$T_n : (\xi, \xi_*) \mapsto (\xi', \xi'_*)$$

is an involution of $\mathbb{R}^{2d}$,

$$T_n \circ T_n = I_{2d \times 2d};$$

in other words $(\xi')' = \xi$, $(\xi'_*)' = \xi_*$. This property and direct computations show the following identities, which are fundamental for the study of the properties of the collision operator:

$$\begin{aligned} &d\xi\ d\xi_* = d\xi'\ d\xi'_*, \\ &n \cdot (\xi - \xi_*) = -n \cdot (\xi' - \xi'_*), \\ &|\xi - \xi_*| = |\xi' - \xi'_*|. \end{aligned}$$

The function B which arises in (48) contains the physics of the collisions. Hard sphere collisions (billard balls) lead to

$$B_{HS}(|n \cdot V|, |V|) = |n \cdot V|,$$

and this type of kernel (with internal energy though) is used for neutral molecules in aerospace engineering or for condensation-evaporation problems. Charged particles lead to interactions with longer range

$$B_{CP}(|n \cdot V|, |V|) = \varphi(|V|) \beta(\frac{|n \cdot V|}{|V|}),$$

and β is strongly unbounded close to $\frac{|n \cdot V|}{|V|} \approx 1$, but it is usual to make the so-called Grad cut-off assumption to regularize β.

Based on these assumptions, one derives the following property for all test functions φ:

(51)
$$\int_{\mathbb{R}^d} Q(f)\varphi(\xi)d\xi = -\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} ff_*[\varphi+\varphi_*-\varphi'-\varphi'_*]B(\dots)d\xi d\xi_* dn$$

$$= \frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} [f'f'_*-ff_*][\varphi+\varphi_*-\varphi'-\varphi'_*]B(\dots)d\xi d\xi_* dn.$$

From this we derive the conservation laws (and one can prove there are no other choices)

$$\begin{aligned} \textstyle\int_{\mathbb{R}^d} Q(f)d\xi = 0 \quad &, \qquad (\varphi = 1,\ \textit{mass conservation}),\\ \textstyle\int_{\mathbb{R}^d} \xi Q(f)d\xi = 0 \quad &, \qquad (\varphi = \xi,\ \textit{momentum conservation}),\\ \textstyle\int_{\mathbb{R}^d} |\xi|^2 Q(f)d\xi = 0 \quad &, \qquad (\varphi = |\xi|^2,\ \textit{energy conservation}). \end{aligned} \tag{52}$$

As a consequence, strong solutions also satisfy several conservation laws:

$$\begin{cases} \int_{\mathbb{R}^{2d}} f(t)dxd\xi = \int_{\mathbb{R}^{2d}} f^0 dxd\xi, \quad \int_{\mathbb{R}^{2d}} \xi f(t)dxd\xi = \int_{\mathbb{R}^{2d}} \xi f^0 dxd\xi,\\ \int_{\mathbb{R}^{2d}} |\xi|^2 f(t)dxd\xi = \int_{\mathbb{R}^{2d}} |\xi|^2 f^0 dxd\xi. \end{cases} \tag{53}$$

The last important property which follows from (51) with $\varphi = \ln(f)$ is called Boltzmann's H-theorem, which states that

(54)
$$-\int_{\mathbb{R}^d} Q(f)\ln(f)d\xi$$
$$= \frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} [ff_*-f'f'_*][\ln(ff_*)-\ln(f'f'_*)]B(\dots)d\xi d\xi_* dn := D(f) \geq 0.$$

It has the consequence that, again for strong solutions,

$$\int_{\mathbb{R}^d} f\ln(f)d\xi + \int_0^\infty\int_{\mathbb{R}^d} D(f(t,x,\cdot))dx\,dt \leq \int_{\mathbb{R}^d} f^0\ln(f^0)d\xi.$$

This inequality is important because, combined with (53), one deduces weak L^1 compactness for families of solutions with initial data that satisfy uniformly the condition

$$\int_{\mathbb{R}^d} f^0[1+|\xi|^2+|x|^\alpha+|\ln(f^0)|]d\xi < \infty, \qquad \alpha > 0. \tag{55}$$

The entropy dissipation also contains much hidden information in terms of trend to equilibrium and related controls; see Villani [117].

Another important consequence of Boltzmann's H-theorem is to identify the kernel of Q. Consider a function $f(\xi)$ such that $(1+|\xi|^2)f(\xi) \in L^1(\mathbb{R}^d)$ and $D(f) < \infty$. If $Q(f) = 0$, we deduce from (54) (which can be established rigorously then) that

$$ff_* = f'f'_* \quad \forall \xi_* \in \mathbb{R}^d,\ \forall n \in \mathbb{S}^{d-1}.$$

From this relation we can deduce that f takes the form of a so-called *Maxwellian* *distribution*: for some real parameters ϱ, u, T (the notations are compatible with (4)–(7))

$$Q(f) = 0 \quad \Leftrightarrow \quad f(\xi) = \frac{\varrho}{(2\pi T)^{d/2}} e^{-\frac{|\xi-u|^2}{2T}}. \tag{56}$$

Proof of (56). See [37]. It uses the C_0^2 Fourier transform $g(k)$ of f. The relation $ff_* = f'f'_*$ is transformed into

$$\begin{aligned} g(k)g(k_*) &= \int_{\mathbb{R}^d} f(\xi')f(\xi'_*)e^{i(k\cdot\xi+k_*\cdot\xi_*)}d\xi d\xi_* \\ &= \int_{\mathbb{R}^d} f(\xi)f(\xi_*)e^{i(k\cdot\xi'+k_*\cdot\xi'_*)}d\xi d\xi_* \\ &= \int_{\mathbb{R}^d} f(\xi)f(\xi_*)e^{i(k\cdot\xi+k_*\cdot\xi_*)\ -\ i(k-k_*)\cdot n\ (\xi-\xi_*)\cdot n}d\xi d\xi_* \ . \end{aligned}$$

We now fix k and k_* and choose a unit vector n_o orthogonal to $k-k_*$. With $n = n_o$ the above formula just yields the trivial identity $gg_* = gg_*$. We thus consider a unit vector $n = n_o + \eta$ with η small and thus not orthogonal to $k - k_*$. Then, the first term in the Taylor expansion of the above formula gives

$$n_o \cdot (\nabla_k g g_* \ - \ \nabla_{k_*} g g_*) = 0. \tag{57}$$

At this stage, we may assume that $g(0) = 1$ and $\nabla_k g(0) = 0$. This is equivalent to normalizing f by $\int_{\mathbb{R}^d} f(\xi)\ d\xi = 1$ and $\int_{\mathbb{R}^d} \xi f(\xi)\ d\xi = 0$. Then, taking first $k_* = 0$ in (57), we find

$$n_o \cdot \nabla_k g \ = \ 0$$

for all unit vectors n_o orthogonal to k. This means that $\nabla_k g$ is proportional to k, which also means that the function g is radially symmetric, i.e., $g = \bar{g}(|k|^2)$. Finally, we set $r = |k|^2$ and insert this information in (57). We obtain that, as long as $\bar{g}\bar{g}_*$ does not vanish,

$$k\frac{\bar{g}'(r)}{\bar{g}(r)} \ - \ k_*\frac{\bar{g}'(r_*)}{\bar{g}(r_*)}$$

is proportional to $k - k_*$. But this means that $\ln(\bar{g}(r))'$ is independent of r. In other words $\bar{g}(r) = e^{-\beta r}$ as long as it does not vanish, and by continuity this holds everywhere thus proving that $g(k) = e^{-\beta|k|^2}$. This is exactly like saying that f is a Maxwellian. □

The Boltzmann equation has been widely studied. For physical motivations and mathematical properties one can consult Truesdell and Muncaster [113], Cercignani [32], and Sone [111]. An overview book is Cercignani, Illner and Pulvirenti [33], which contains in particular a derivation of the Boltzmann equation from particle systems, the so-called Boltzmann-Grad limit; see also Ukai [114]. Several variants of the Boltzmann equation arise in physics. For instance recent interest has been to understand inelastic collisions, because such models arise in some regimes of granular flows; see Bobylev, Carillo and Gamba [22], Toscani [112], and Benedetto *et al.* [18], [19]. Related is the Enskog equation for dense gases; see Arkeryd and Cercignani [4], and Chapter 2 in [37], where several references from physics and astrophysics are given. Quantum models, with Pauli exclusion principles, and relativistic models lead to different kernels, and one can consult Escobedo and Mischler [53]. Tumor growth and the modeling of immune response lead also to nonlinear transport equations of integral type; see Bellomo and Presiozi [14]. Traffic flows also lead to Boltzmann type equations; see for instance Illner *et al.* [77] and the references therein.

A survey of recent progress can be found in Villani [116] and we therefore restrict ourselves to very fundamental aspects. Historical progress in the mathematical theory of the Boltzmann equation has been the theory of DiPerna and Lions [49] (see also [87]) which proves global existence of weak solutions (so-called renormalized solutions) in the physical space, i.e. using only a priori estimates like (55). This theory provided tools which allowed one to handle numerous new questions; as an example let us note the regularizing effect of the operator Q^+ [87]. Especially the case without cut-off has been treated, with the so-called grazing collisions leading to the Landau operator; see Degond and Lucquin [41], Alexandre *et al.* [2] and the references in [116]. Stationary Boltzmann equations are also important; see Arkeryd and Nouri [5] for weak solutions in a slab, Ukai and Asano [115] for strong solutions in an exterior domain with given Maxwellian at infinity and nearly compatible boundary conditions, or Liu and Yu [93] and the references therein for shock profiles. Bounded domains for the evolution problem are treated in Mischler [95]. For large time behavior, we refer to [87] (part I), Desvillettes and Villani [44] for bounded domains, and to [101] for the whole space.

In order to describe the existence result of renormalized solutions we need some assumptions:

$$\int_{|\xi|<R}\int_{\mathbb{S}^{d-1}} B(|n.\xi|,|\xi|)d\xi\; dn < \infty \qquad \forall R>0, \quad B\geq 0, \tag{58}$$

$$(1+|z|)^{-2}\int_{|\xi-z|<R}\int_{\mathbb{S}^{d-1}} B(|n.\xi|,|\xi|)d\xi\; dn \to 0 \qquad as\ |z|\to\infty, \quad \forall R>0. \tag{59}$$

Also renormalized solutions are defined as functions $f\in C\Big([0,\infty);L^1(\mathbb{R}^{2d})\Big)$ such that

$$\int_{\mathbb{R}^d} f(t)[1+|\xi|^2+|x|^\alpha+|\ln(f(t))|]d\xi<\infty, \qquad \text{for some } \alpha>0; \tag{60}$$

$$\int_{\mathbb{R}^d}\frac{Q^+(f)}{1+f}dx\in L^\infty\Big(0,\infty;L^1_{\text{loc}}(\mathbb{R}^d)\Big), \quad \int_{\mathbb{R}^d}\frac{Q^-(f)}{1+f}dx\in L^1_{\text{loc}}\Big([0,\infty)\times\mathbb{R}^d\Big); \tag{61}$$

$$\frac{\partial}{\partial t}\beta(f)+\xi\cdot\nabla\beta(f)=\beta'(f)Q(f), \quad \text{in } \mathcal{D}'\Big((0,\infty)\times\mathbb{R}^{2d}\Big), \tag{62}$$

for all $\beta\in C^1\Big((0,\infty)\ \mathbb{R}\Big)$; and finally (see (54)),

$$\int_0^\infty\int_{\mathbb{R}^d} D(f(t,x,\cdot))dx\,dt<\infty. \tag{63}$$

Theorem 4.1. *Let f^0 satisfy* (55)*, and let B satisfy* (58) *and* (59)*. Then there exists a renormalized solution to the Boltzmann equation with initial data f^0. If there is a strong solution, then the renormalized solution is unique.*

We refer to [87] (part II) for the proof of this result. There are still open problems on the properties of renormalized solutions besides the propagation of regularity. For instance, the solution is built passing to the (strong) limit in a family of smooth solutions for regularized kernels, but second ξ moments are only weakly convergent. Therefore the conservation of energy or the local form of the momentum equation do not follow. We also refer to §5.2 for other aspects of the Boltzmann equation.

4.4. **Kinetic formulation of hyperbolic-parabolic conservation laws.** Kinetic equations also arise as a mathematical representation of nonlinear conservation laws, the so-called *kinetic formulation*, which was introduced in Lions, Perthame and Tadmor [92]. A recent overview of the subject is given in [102]. Kinetic formulations can be seen as a method to replace nonlinear parabolic equations (and some systems) by a linear equation, acting on a nonlinear quantity, and this is useful because one can use linear tools such as convolution and Fourier transform. It is also a method which allows one to derive numerical schemes for nonlinear equations. We present here, and following [35], the example of degenerate nonlinear parabolic conservation laws in the scalar case.

To explain this concept, we consider first the nondegenerate case, a case where existence of a unique family of smooth solutions with decay at infinity is standard to establish. Then the problem is to find a smooth, vanishing at infinity, real-valued function $u(t,x)$ defined for $t \geq 0$, $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, which solves the quasilinear parabolic equation

$$\begin{cases} \frac{\partial}{\partial t} u + \sum\limits_{i=1}^{d} \frac{\partial}{\partial x_i} A_i(u) - \sum\limits_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} A_{ij}(u) = 0, \\ u(t=0,x) = u^0(x). \end{cases} \tag{64}$$

We define and assume the following:

$$a_i(\cdot) = A_i'(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}), \quad a_{ij}(\cdot) = A_{ij}'(\cdot) \in L^\infty_{\mathrm{loc}}(\mathbb{R}),$$

$$a(\xi) = (a_1(\xi), a_2(\xi), \cdots, a_d(\xi)) \quad (\text{a mapping} : \mathbb{R} \to \mathbb{R}^{\mathrm{d}}),$$

$$a_{ij} \quad \text{is a symmetric matrix}, \qquad a_{ij}(\cdot) \geq \nu \, I_{d\times d} \quad (\nu > 0). \tag{65}$$

This equation satisfies a simple so-called *entropy property.* That is for all smooth function $S(\cdot)$, after multiplying the equation (64) by $S'(u)$, we obtain

$$\frac{\partial}{\partial t} S(u) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \eta_i^S(u) - \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} B_{ij}^S(u) = -S''(u) a_{ij}(u) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}, \tag{66}$$

where the entropy fluxes η_i^S, B_{ij}^S are defined (up to an additive constant) by

$$\left(\eta_i^S\right)'(\cdot) = a_i(\cdot) S_i'(\cdot), \qquad \left(B_{ij}^S\right)'(\cdot) = a_{ij}(\cdot) S_i'(\cdot). \tag{67}$$

Notice also that a priori bounds follow from the entropy inequalities. We can choose $S(u) = |u|^p$, $1 \leq p < \infty$, to obtain

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq \|u^0\|_{L^p(\mathbb{R}^d)}, \qquad 1 \leq p < \infty,$$

and $S(u) = (u-K)_+^2$ with an appropriate choice of K yields

$$\min u^0 \leq u(t,x) \leq \max u^0.$$

We can also use the entropy dissipation, for $S \geq 0$ and $S'' \geq 0$; we then have

$$\int_0^\infty \int_{\mathbb{R}^d} S''(u) \, a_{ij}(u) \, \frac{\partial u}{\partial x_i} \, \frac{\partial u}{\partial x_j} \, dx \, dt \leq \frac{1}{2} \int_{\mathbb{R}^d} S(u^0) \, dx. \tag{68}$$

In particular the choice $S(u) = u^2/2$ also leads to the energy estimate

$$\int_0^\infty \int_{\mathbb{R}^d} a_{ij}(u) \, \frac{\partial u}{\partial x_i} \, \frac{\partial u}{\partial x_j} \, dx \, dt \leq \frac{1}{2} \|u^0\|^2_{L^2(\mathbb{R}^d)}. \tag{69}$$

The choice $S_\xi(u) = \max(0, u-\xi)$ for $\xi > 0$, and $S_\xi(u) = \max(0, \xi - u)$ for $\xi < 0$, leads to $S''_\xi(u) = \delta(u=\xi)$ and gives the (more interesting although not so clearly defined) a priori estimate

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^d} \delta(\xi = u(t,x))\, a_{ij}(\xi)\, \frac{\partial u}{\partial x_i}\, \frac{\partial u}{\partial x_j}\, dx\, dt \\ \leq \mu(\xi) := \begin{cases} \|(u^0-\xi)_+\|_{L^1(\mathbb{R}^d)} & \text{for } \xi \geq 0, \\ \|(\xi - u^0)_+\|_{L^1(\mathbb{R}^d)} & \text{for } \xi \leq 0. \end{cases}\end{aligned} \tag{70}$$

At least this inequality makes sense as a measure in ξ, i.e. testing it against nonnegative continuous functions of ξ.

To proceed further in analyzing the consequences of these entropy inequalities, we introduce the function (from $\mathbb{R}^2$ into $\mathbb{R}$),

$$\chi(\xi;u) = \begin{cases} +1, & \text{for } \ 0 < \xi < u, \\ -1, & \text{for } \ u < \xi < 0, \\ \ 0, & \text{otherwise.} \end{cases}$$

This function is related to several deep problems, especially to weak limits of oscillating bounded sequences $u_n(x)$ and to Young measures (see [102]). Indeed, it gives a representation of any function $S(u)$ as follows:

$$\int_{\mathbb{R}} S'(\xi)\chi(\xi;u)d\xi = S(u) - S(0). \tag{71}$$

Therefore, when a sequence of functions u_n converges in $L^\infty - w*$, it allows us to study the limit of all the limits $S(u_n)$ for all smooth functions S.

We claim that the family of equalities (66) is equivalent to writing in $\mathcal{D}'((0,\infty)\times\mathbb{R}^{d+1})$ the so-called *kinetic formulation* (recall that ξ is a real-valued variable here).

Proposition 4.2. *Equation* (66) *is equivalent to the problem of finding a function $u(t,x)$ such that, in the sense of distributions,*

$$\begin{aligned}\frac{\partial}{\partial t}\chi\Big(\xi, u(t,x)\Big) + a(\xi)\cdot\nabla_x\chi\Big(\xi, u(t,x)\Big) - a_{ij}(\xi)\sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j}\chi\Big(\xi, u(t,x)\Big) \\ = \frac{\partial}{\partial \xi} n(t,x,\xi),\end{aligned} \tag{72}$$

$$n(t,x,\xi) = \delta(\xi = u(t,x))\, a_{ij}(\xi)\, \frac{\partial u}{\partial x_i}\, \frac{\partial u}{\partial x_j}. \tag{73}$$

In fact n is a bounded nonnegative measure thanks to (69), also thanks to inequality (70) and the usual continuity arguments,

$$n \in C_0\Big(\mathbb{R}_\xi; w - M^1((0,+\infty)\times\mathbb{R}^d)\Big),$$

where M^1 denotes the Banach space of bounded Radon measures.

We now indicate the reason why the *kinetic formulation* holds true.

Derivation of the kinetic formulation (72). Using the chain rule in (72) we have

$$\frac{\partial}{\partial t}\chi(\xi;u) = \delta(\xi = u)\, \frac{\partial u}{\partial t};$$

therefore equation (72) can also be written

$$\delta\Big(\xi = u(t,x)\Big)\Big[\frac{\partial u}{\partial t} + a(u)\cdot\nabla_x u\Big] - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}[a_{ij}(u(t,x))\delta(\xi = u)\frac{\partial u}{\partial x_j}]$$
$$= \frac{\partial}{\partial \xi}[\delta\Big(\xi = u(t,x)\Big) \sum_{i,j=1}^{d} a_{ij}(u(t,x))\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}].$$

On the other hand, we have

$$\frac{\partial}{\partial x_i}[a_{ij}(u(t,x))\delta(\xi = u)\frac{\partial u}{\partial x_j}] = \delta'(\xi = u)a_{ij}(u)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \delta(\xi = u)\frac{\partial}{\partial x_i}[a_{ij}(u)\frac{\partial u}{\partial x_j}],$$

and therefore (72) is also equivalent to

$$\delta\Big(\xi = u(t,x)\Big)\Big[\frac{\partial u}{\partial t} + a(u)\cdot\nabla_x u - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(u)\frac{\partial u}{\partial x_j}\Big)\Big] = 0,$$

which is equivalent to the parabolic conservation law (64) for smooth solutions.

Another way to see the equivalence is to multiply (72) by $S'(\xi)$ (a test function) and integrate $d\xi$. Then one still recovers the entropy equalities (66) which are equivalent to parabolic equation (64).

The interesting part of this formalism appears for degenerate diffusions ($\nu = 0$) and especially in the hyperbolic case $A_{ij} = 0$. Then possible singularities of the solution (shock waves for instance) make the chain rule no longer available. The kinetic formulation however holds true, with the only difference being that the measure on the right hand side is no longer defined explicitly by formula (73). It is replaced by a bounded measure $\tilde{n}(t,x,\xi) \geq n(t,x,\xi)$. An application is to the regularity of the solution in the degenerate case. This can be proved using tools as averaging lemmas. Another application is to uniqueness of the solution [102], [35]. In order to treat degenerate diffusions,

$$a_{ij} \quad \text{is a symmetric matrix}, \qquad a_{ij}(\cdot) \geq 0, \tag{74}$$

the method consists of passing to the limit as ε vanishes in the family of solutions u_ε associated with the diffusion matrix $\varepsilon I + (a_{ij})$. It is possible to prove that this is a strong limit in any L^p spaces, $1 \leq p < \infty$, when we assume that $u^0 \in L^1 \cap L^\infty(\mathbb{R}^d)$. In order to state correctly the limit problem we need the notations $\Sigma_{ik}(u)$ and $\Sigma_{ik}^{\psi}(u)$ for $\psi \in C_0(\mathbb{R})$ with $\psi \geq 0$:

$$\begin{aligned} & a_{ij}(\cdot) = \sum_{k=1}^{d} \sigma_{ik}(\cdot)\sigma_{jk}(\cdot), \\ & \Sigma_{ik}'(\cdot) = \sigma_{ik}(\cdot), \qquad (\Sigma_{ik}^{\psi})'(\cdot) = \sqrt{\psi(\cdot)}\,\sigma_{ik}(\cdot). \end{aligned} \tag{75}$$

Then we end up with the two equivalent definitions:

$$n_{\varepsilon}^{\psi}(t,x) := \sum_{k=1}^{d}\Big(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}^{\psi}(u^\varepsilon)\Big)^2 = \sum_{k=1}^{d}\psi(u^\varepsilon)\Big(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}(u^\varepsilon)\Big)^2. \tag{76}$$

A fundamental remark is that this equality still holds in the limit $\varepsilon \to 0$, and this allows us to define, for weak solutions, the nonlinear term arising in the entropy relation. Then, we end up with

Definition 4.3. An entropy solution is a function $u(t,x) \in L^\infty\Big([0,\infty); L^\infty \cap L^1(\mathbb{R}^d)\Big)$ such that

(i) $\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\Sigma_{ik}(u) \in L^2\Big([0,\infty)\times\mathbb{R}^d\Big)$, for any $k \in \{1,\cdots,d\}$;

(ii) for any function $\psi \in C_0(\mathbb{R})$ with $\psi(u) \geq 0$ and any $k \in \{1,\cdots,d\}$, the chain rules hold:

$$\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}^{\psi}(u) = \sqrt{\psi(u)}\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}(u) \in L^2([0,\infty)\times\mathbb{R}^d), \tag{77}$$

$$n^{\psi}(t,x) := \psi(u(t,x))\sum_{k=1}^{d}\Big(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}(u(t,x))\Big)^2 = \sum_{k=1}^{d}\Big(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}^{\psi}(u(t,x))\Big)^2, \ \textit{a.e.}; \tag{78}$$

(iii) there exists an entropy dissipation measure $m(t,x,\xi)$ such that for any smooth function $S(u)$, we have in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d)$

$$\begin{aligned} &\frac{\partial}{\partial t}S(u) + \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\eta_i^S(u) - \sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i\,\partial x_j}B_{ij}^S(u) = -(m^{S''}+n^{S''}),\\ &S(u(t=0)) = S(u_0), \end{aligned} \tag{79}$$

$$m^{S''}(t,x) = \int_{\mathbb{R}} S''(\xi)\,m(t,x,\xi)\,d\xi, \quad \text{with } m(t,x,\xi) \text{ a nonnegative measure.} \tag{80}$$

The entropy dissipation measure m appears here because we have to take into account the weak limit process when passing to the limit as $\varepsilon \to 0$ in the quadratic terms. It accounts for the inequality

$$\Big(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}(u(t,x))\Big)^2 \leq \mathrm{w}-\lim_{\varepsilon\to 0}\Big(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}(u_\varepsilon(t,x))\Big)^2$$

because, weakly in L^2, we have

$$\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}(u_\varepsilon(t,x)) \to \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}(u(t,x)).$$

Notice that the following total mass control is satisfied:

$$\int_0^\infty\int_{\mathbb{R}^d}(m+n)(t,x,\xi)\,dt\,dx \leq \mu(\xi) \tag{81}$$

(recalling the definition of μ in (70)) with

$$n(t,x,\xi) = \delta(\xi-u(t,x))\sum_{k=1}^{d}\Big(\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Sigma_{ik}(u(t,x))\Big)^2.$$

A consequence of this definition is a kinetic formulation for the degenerate non-isotropic parabolic equations

$$\frac{\partial}{\partial t}\chi(\xi;u) + a(\xi)\cdot\nabla_x\chi(\xi;u) - \sum_{i,j=1}^{d}a_{ij}(\xi)\frac{\partial^2}{\partial x_i\partial x_j}\chi(\xi;u) = \frac{\partial}{\partial_\xi}(m+n)(t,x,\xi), \tag{82}$$

in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^{d+1})$ with initial data

$$\chi(\xi; u)|_{t=0} = \chi(\xi; u_0).$$

5. Asymptotic analysis

One of the early and main successes of the Boltzmann equation has been to derive macroscopic equations from the kinetic scale and thus to provide a theoretical basis to models established earlier by phenomenological laws. The derivation of compressible transport equations (thanks to the Chapman-Enskog method [113], [32], [33]) allows us not only to recover the structure of Navier-Stokes system for compressibles fluids but also to give specific coefficients, the so-called transport coefficients (in the case of hard spheres, for instance for monatomic gases). The mathematical program of proving the compressible fluid limit for global large solutions is still open and considered out of reach with the known tools. A rigorous derivation for short times is given in Caflisch [29], and Kawashima, Matsumura and Nishida [84]. Also for numerical purposes, it would be useful to find 'good' approximations of kinetic equations, which use the $2d$ phase space, by systems of d dimensional reduced problems. This is still an active area where several recent results are promising (see Levermore [86], Junk [83], Dubroca and Feugeas [52]), but the Boltzmann equation is still a challenge, especially for the mathematical basis.

It remains that many theoretical questions have progressed significantly these last years, and we present some examples in this section. We also refer to the chapter by Golse in [26] for additional references and material.

5.1. **Hyperbolic limit.** It is natural to assume that in the Boltzmann equation, the collision term dominates transport when the density becomes larger because the gas molecules undergo more collisions. A mathematical way to express this is to consider the hyperbolic scaling of (47):

$$\frac{\partial}{\partial t} f_\varepsilon(t, x, \xi) + \xi \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} Q(f_\varepsilon).$$

This can be derived either as a rescaling of the density f in f/ε (this is specific to the quadratic aspect of Boltzmann kernel) or a time-space rescaling in $1/\varepsilon$ (which turns out to be a general point of view). Then one expects that $Q(f_\varepsilon) \to 0$ as ε vanishes and thus that the limit f of f_ε satisfies $Q(f) = 0$. Therefore, as explained earlier (see formula (56)) this implies that f is a Maxwellian distribution

$$f = M[f] := \frac{\varrho}{(2\pi T)^{d/2}} e^{-\frac{|\xi - u|^2}{2T}},$$

with $\varrho(t, x)$, $u(t, x)$ and $T(t, x)$ related to f thanks to the definition of macroscopic quantities in (4)–(7). In order to study the limit as $\varepsilon \to 0$, we *firstly* integrate the Boltzmann equation against the measures $d\xi$, $\xi d\xi$ and $\frac{1}{2}|\xi|^2 d\xi$, and obtain *macroscopic* conservation laws, thanks to equations (52),

$$\begin{cases} \frac{\partial}{\partial t}\varrho + \operatorname{div}(\varrho u) = 0, \quad t \geq 0,\ x \in \mathbb{R}^d, \\ \frac{\partial}{\partial t}(\varrho u) + \operatorname{div}(\varrho u \otimes u + P) = 0, \\ \frac{\partial}{\partial t} E + \operatorname{div}(E u + P \cdot u + q) = 0. \end{cases} \tag{83}$$

To obtain this system, we have used notations (4)–(7), and the pressure tensor P in the momentum equation (second equation of (83)) is given by

$$P_{ij}(t,x) = \int_{\mathbb{R}^d} (\xi_i - u_i)\,(\xi_j - u_j)\, f(t,x,\xi) d\xi. \tag{84}$$

The energy flux is given by the formula

$$\frac{1}{2}\int_{\mathbb{R}^d} |\xi|^2 \ \xi\, f(t,x,\xi)\, d\xi = Eu + \tfrac{1}{2}\int_{\mathbb{R}^d} |\xi|^2\,(\xi - u)\, f(t,x,\xi)\, d\xi.$$

But, since we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\xi|^2\,(\xi_i - u_i)\ f(t,x,\xi)\, d\xi \\
&\quad = \int_{\mathbb{R}^d} \Big(|\xi - u|^2 + |u|^2 + 2\,(\xi - u)\cdot u\Big)\,(\xi_i - u_i)\ f(t,x,\xi) d\xi \\
&\quad = \int_{\mathbb{R}^d} \Big(|\xi - u|^2 + 2\,(\xi_j - u_j)\, u_j\Big)\,(\xi_i - u_i)\ f(t,x,\xi) d\xi \\
&\quad = 2q_i(t,x) + 2\sum_{j=1}^{d} u_j \cdot P_{ij},
\end{aligned}$$

we obtain that the heat flux q is defined by

$$q_i(t,x) = \frac{1}{2}\int_{\mathbb{R}^d} (\xi_i - u_i)|\xi - u|^2 f(t,x,\xi)\, d\xi. \tag{85}$$

We may also derive a *macroscopic entropy inequality* which complements system (83). From (54), we deduce

$$\frac{\partial}{\partial t} S(t,x) + \mathrm{div}\eta(t,x) \le 0, \tag{86}$$

with

$$S(t,x) = \int_{\mathbb{R}^d} f\ \ln(f)\, d\xi, \qquad \eta(t,x) = \int_{\mathbb{R}^d} \xi\, f \ln(f)\, d\xi.$$

System (83), together with laws (84) and (85), is always satisfied by solutions to the Boltzmann equation.

In order to go further in the study of the limit $\varepsilon \to 0$, we *secondly* identify f with its local Maxwellian limit as mentioned earlier. We end up with the system (83) completed by

$$P = \varrho\, T\ I_{d\times d}, \qquad q = 0.$$

This is the Euler system for compressible flow. As is well known, this is a nonlinear hyperbolic system, and shocks (discontinuities) appear generically in finite time. The entropy inequality (86) is therefore fundamental to selecting the right solutions, and it is completed in this limit by the relations (simplified after taking into consideration the conservation laws)

$$S = \varrho \ln\big(\frac{\varrho}{T^{d/2}}\big), \qquad \eta = u\, S.$$

This program is again open for discontinuous limits (see, however, the references in the introduction to this section). It has been solved for simplified kinetic equations which involve a single conservation law because compactness related to contraction properties helps. In such cases the hyperbolic limit is strongly related with kinetic formulations in §4.4. Other problems have also been treated, as e.g.

the collision operator of granular materials. Then the Maxwellian is no longer the right equilibrium, and this leads to different pressure laws; see Benedetto *et al.* [18]. We refer to [102] for further references on this huge subject.

5.2. **Diffusion limits.** The classical field of application of the diffusion limits is to derive macroscopic equations like the heat equation from a scattering model. The regime of interest is again when the scattering part dominates transport; see §4.2. Then a parameter $\varepsilon > 0$ is introduced to represent the ratio 'transport/scattering', and one considers the rescaled problem $f_\varepsilon = f(\varepsilon t, \frac{x}{\varepsilon}, \xi)$. We arrive at

(87)
$$\frac{\partial}{\partial t} f_\varepsilon(t,x,\xi) + \frac{\xi}{\varepsilon}\cdot\nabla_x f_\varepsilon + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^d}[K_\varepsilon(\xi,\xi')f_\varepsilon(t,x,\xi) - K_\varepsilon(\xi',\xi)f_\varepsilon(t,x,\xi')]d\xi' = 0.$$

The problem of studying the limit as ε vanishes is extremely classical and leads to a diffusion equation; see [11]. For instance, in case of the nonlinear model of *chemotaxis* where equation (87) is coupled with (45)–(46), we can prove (see [99], [34]) that $f_\varepsilon(t,x,\xi) \to \varrho(t,x)$, $S_\varepsilon(t,x) \to S(t,x)$ and that the Keller-Segel model holds in the limit

$$\frac{\partial}{\partial t}\varrho(t,x) + \mathrm{div}(\chi\varrho\nabla S) = \mathrm{div}(D\nabla\varrho), \quad t \geq 0,\ x \in \mathbb{R}^d,$$
$$-\Delta S = \varrho,$$

with transport coefficients given by

$$D(t,x) = \frac{1}{3|V|(\alpha_+ + \alpha_-)\psi(S)}\int_V |\xi|^2\, d\xi, \qquad \chi(S) = \frac{k'(S)}{3k(S)}\int_V |\xi|^2\, d\xi.$$

Notice that drift terms in diffusion equations have also been considered in Degond *et al.* [39]. Radiative transfer equations were the first nonlinear equations to be treated [10] (again compactness is required which uses averaging lemmas; see §3). Other phenomena can also lead to diffusion limits as thin films where the diffusion comes from reflection conditions on the boundary; see Babovski *et al.* [7].

The same scaling can be applied to the Boltzmann equation and raises one of the beautiful mathematical theories which has followed the notion of renormalized solutions. We present the result very roughly and restrict here to dimension $d = 3$. In a series of papers by Bardos, Golse, Levermore and Saint-Raymond [9], [68], and by Lions and Masmoudi [89], the incompressible Navier-Stokes limit has been considered. It relates, in three dimensions, the Boltzmann equation with the incompressible Navier-Stokes system. The problem is to find a velocity field $u(t,x) \in \mathbb{R}^3$ and a temperature field $T(t,x) \geq 0$ such that

(88)
$$\begin{aligned} \mathrm{div} u(t,x) &= 0,\\ \tfrac{\partial}{\partial t}u + u\cdot\nabla_x u + \nabla p &= \nu\Delta u, \quad t > 0,\ x \in \mathbb{R}^3,\\ \tfrac{\partial}{\partial t}T + u\cdot\nabla_x T &= \kappa\Delta T. \end{aligned}$$

Since this is a diffusive regime, the Boltzmann equation (47) is rescaled as explained before:

$$\frac{\partial}{\partial t} f_\varepsilon(t,x,\xi) + \frac{\xi}{\varepsilon}\cdot\nabla_x f_\varepsilon = \frac{1}{\varepsilon^2}Q(f_\varepsilon).$$

But, in view of the hyperbolic limit in §5.1, an additional scaling has to be imposed so as to generate velocities of order ε. This is achieved by considering initial distributions close to a uniform Maxwellian (see (56)),

$$f_\varepsilon^0(x,\xi) \approx M(\xi)[1+\varepsilon\, u^0(x)\cdot\xi + T^0(x)\frac{|\xi|^2-5}{2}], \qquad M(\xi) = \frac{1}{(2\pi)^{3/2}}e^{-\frac{|\xi|^2}{2}}. \tag{89}$$

When this is written correctly, it is proved in [68] that

$$\frac{1}{\varepsilon}\int_{\mathbb{R}^3}\xi\, f_\varepsilon(t,x,\xi)d\xi \to u(t,x), \quad \frac{1}{\varepsilon}\int_{\mathbb{R}^3}(\frac{1}{3}|\xi|^2-1)\, f_\varepsilon(t,x,\xi)d\xi \to T(t,x),$$

and (u,T) is a solution to (88) where the viscosity ν and heat conductivity κ depend on the collision kernel B. In fact, for strong enough assumptions on f_ε^0, it is proved that some expansion like (89) holds true also for the solution f_ε.

5.3. **Strong field limits.** The possibility to scale Vlasov equations (see §4.1) appeared only recently in the mathematical literature and raises a fascinating class of problems which is far from being closed, both in terms of models of interest and mathematical theory. It is again a hyperbolic scaling when the force term is dominating. Typically, one obtains

$$\frac{\partial}{\partial t}f_\varepsilon(t,x,\xi) + \xi\cdot\nabla_x f_\varepsilon + \frac{1}{\varepsilon}\mathrm{div}_\xi[F(x,\xi)\, f_\varepsilon] = Q(f_\varepsilon), \qquad t\geq 0,\ x,\ \xi\in\mathbb{R}^d.$$

The difference with the hyperbolic or diffusive limits (then the kernel of the collision operator gives the formal limit) is that for strong field limits we cannot guess (up to a finite number of macroscopic parameters) the limiting distribution f as $\varepsilon\to 0$ from the formal limit

$$\mathrm{div}_\xi[F(x,\xi)\, f(t,x,\xi)] = 0.$$

Typical examples are

(i) combinations of strong collisions and strong field can appear. Then, the collision operator and the force term are scaled together with the same parameter. This appeared first for semiconductor models in Poupaud [106] (see also Nieto, Poupaud and Soler [98] and the references therein), and this class of hydrodynamic limits has attracted much interest; see for instance Degond and Jüngel [40], Arnold *et al.* [6] and the references therein.

(ii) strong magnetic fields in stratospheric plasmas ($F=\xi\wedge B$, with B the magnetic field plus possible Poisson forces); see Frénod and Sonnendrücker [55], Golse and Saint-Raymond [67], Saint-Raymond [109]. This is a situation of homogenization type; a fast variable appears.

(iii) strong electric field in Vlasov-Poisson system with a background; see Grenier [71] and Brenier [28] for a method based on modulated hamiltonians.

(iv) strong friction for particles in a fluid ($F=U-\xi$); see Jabin [79]. Here the distribution does not remain bounded in L^∞, and it concentrates on a Dirac mass in ξ. See also [69].

The main mathematical difficulties can come from the properties of the kinetic kernel itself (typically $\mathrm{div}_\xi[F(x,\xi)\, f] = Q(f)$ in case (i)) which can require specific mathematical treatment. But they can also come from the limits themselves, which can be nonlinear hyperbolic and thus lack regularity for justifying the asymptotics.

Acknowledgment

I wish to thank C. Villani for carefully reading this paper and for many valuable suggestions.

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