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Introduction to the Langlands program, by J. Bernstein and S. Gelbart (Editors), with contributions by D. Bump, J. W. Cogdell, D. Gaitsgory, E. de Shalit, E. Kowalski, S. S. Kudla, Birkhäuser, Boston, 2003, x + 281 pp., \$39.95, ISBN 0-8176-3211-5

In the late 1960's, R. P. Langlands gradually became convinced of the existence of a deep and systematic relation between number theory, specifically the Galois theory of number fields, and the theory of automorphic forms, which can be described as harmonic analysis on a certain class of locally compact groups and their homogeneous spaces, also defined in terms of number theory.[1] The nature of this relation formed the basis of a network of interrelated and increasingly precise conjectures known at first (and still, to some extent) as the "Langlands philosophy", more recently as the "Langlands program".

Since Langlands began to formulate his conjectures, the influence of his insights has grown spectacularly. The Langlands program has by now assimilated the greater part of the traditional theory of automorphic forms. Many scattered and curious phenomena in the classical theory of automorphic forms were "explained" *a posteriori* in terms of Langlands' framework and the expected relation to number theory. More strikingly, Langlands' conjectures, when specialized, give rise to an endless stream of predictions regarding properties of automorphic forms on specific groups which have proved astonishingly accurate. Indeed, the expressions "automorphic forms" and "Langlands program" are now often used interchangeably, although this is an error. Conjectures analogous to those of Langlands have become prominent in algebraic geometry and mathematical physics and in nearly every branch of representation theory and harmonic analysis on groups. Finally, and most importantly from the standpoint of the book under review, the Langlands perspective has become pervasive in number theory; no one contemplating a career in number theory can reasonably hope to escape.

Any attempt to give a brief account of the Langlands program faces a simple but fundamental pedagogical paradox: one cannot state accurately any of Langlands' conjectures, much less explain such proofs as are known, in the space of a readable article. The statements of the conjectures combine the full theory of reductive algebraic groups with the framework of modern number theory. The former is well known to be heavy on notation and combinatorics, whereas the latter makes essential use of the language of adèles, little known outside the specialty. To make matters worse, the only known statement of the full Langlands conjectures for number fields is in terms of the representation theory of a conjectural generalization of the absolute Galois group known as the Langlands group, whose properties are

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[1]There were of course precedents, especially the work of Eichler and Shimura on the zeta functions of modular curves, as well as the conjecture formulated by Taniyama and Shimura on the modularity of elliptic curves over $\mathbb{Q}$, which first appeared in print, in a more precise form, in Weil's celebrated article. Historians may some day have the chance to sort this out. In this review I want to stress the *systematic* nature of the relations conjectured by Langlands, based on his principle of functoriality, as well as the central role of representation theory.

next to impossible to explain without circularity. As for proofs, they typically require the complete theory of representations of reductive groups over local fields, including Harish-Chandra's theory of representations of real groups; the theory of arithmetic groups as presented at the AMS Boulder conference (*Proc. Symp. Pure Math.* **9**); a good dose of spectral theory and complex analysis; class field theory and the full strength of Shimura's theory of canonical models of hermitian symmetric domains ("Shimura varieties"); in the same vein, any available techniques from arithmetic algebraic geometry; and, increasingly, analytic number theory.

With this list of prerequisites it's not surprising that automorphic forms is reputed to be a difficult field, particularly but not exclusively by graduate students seeking thesis projects. Moreover, while the Langlands conjectures themselves, though highly technical, can with some effort be presented concisely and systematically, available methods are mostly *ad hoc* and can treat only very special cases, and in practice the theory of automorphic forms appears to some people as

"a diffuse, disordered subject driven as much by the availability of techniques as by any high esthetic purpose." (Langlands, Where Stands Functoriality Today?, *Proc. Symp. Pure Math.* **61** (1997) 457-471).

In other words, if, like Joseph Bernstein and Stephen Gelbart and the six contributors to the book under review, you have set yourself the goal of producing a short, aesthetically compelling but realistic introduction to the Langlands program – or if you are reviewing such a book – it's hard to know where to start. Bernstein and Gelbart start with number theory, postponing the harsh encounter with reductive group theory until the end of the book. I will begin my review by trying to explain what an automorphic form is.

If $\mathcal{G}$ is a locally compact group, say a Lie group, or more specifically $SL(n,\mathbb{R})$, and if $\Gamma \subset \mathcal{G}$ is a discrete subgroup, then the space $L^2(\Gamma\backslash\mathcal{G})$ affords a Hilbert space representation of the group $\mathcal{G}$ that can be decomposed as a direct integral of unitary representations of $\mathcal{G}$. The study of this decomposition is what we understand loosely as "harmonic analysis" on $\Gamma\backslash\mathcal{G}$ and includes as a special case the Plancherel formula on $\mathcal{G}$ itself when Γ is the trivial group. When $\mathcal{G}$ is a reductive Lie group and $\Gamma\backslash\mathcal{G}$ has finite volume – for example, $\mathcal{G} = SL(n,\mathbb{R})$ and Γ is a subgroup of finite index in $SL(n,\mathbb{Z})$ – Harish-Chandra singled out a special class of C^∞ functions $f : \Gamma\backslash\mathcal{G} \to \mathbb{C}$ to qualify as *automorphic forms*: (a) the translations of f under a (chosen) maximal compact subgroup $K \subset \mathcal{G}$ (e.g., $SO(n) \subset SL(n,\mathbb{R})$) must be contained in a finite-dimensional space of functions; (b) f must satisfy a large collection of $\mathcal{G}$-invariant differential equations (more precisely, f must be annihilated by an ideal of finite codimension in the center of the universal enveloping algebra of the complexified Lie algebra of $\mathcal{G}$); (c) finally, if $\Gamma\backslash\mathcal{G}$ is non-compact, f must satisfy a condition of moderate growth at the boundary. The L^2-condition is optional but will be assumed for the sake of exposition.

In the arithmetic theory of automorphic forms one can begin with a group $\mathcal{G}$ as above of the form $G(\mathbb{R})$, where G is a reductive linear algebraic group over the field $\mathbb{Q}$ or, more generally, a finite extension F of $\mathbb{Q}$. Thus if G is the group $SL(n)$ viewed as an algebraic group over $\mathbb{Q}$, then $\mathcal{G}$ is $SL(n,\mathbb{R})$ as above. If instead G is $SL(n)$ viewed as an algebraic group over F, then $\mathcal{G} = SL(n, F \otimes_{\mathbb{Q}} \mathbb{R}) = \prod SL(n,\mathbb{R})^{r_1} \times SL(n,\mathbb{C})^{r_2}$, where r_1 is the number of field embeddings of F in $\mathbb{R}$ and r_2 is the number of pairs of complex conjugate embeddings of F in $\mathbb{C}$, so that $r_1 + 2r_2 = [F : \mathbb{Q}]$. One can choose a faithful rational matrix representation

$\rho : G \to GL(N)$ for some large N, defined over $\mathbb{Q}$ (resp. F), and take Γ to be the group Γ_ρ of elements $g \in \mathcal{G}$ such that the matrix coefficients of $\rho(g)$ lie in $\mathbb{Z}$ (resp. in the ring $\mathcal{O}_F$ of algebraic integers in F) or more generally a subgroup of Γ_ρ of finite index, defined by a congruence condition on the matrix coefficients. One can develop the entire theory in this setting, but the relations between the automorphic forms for different choices of ρ and Γ are by no means obvious. It is more canonical to take $\Gamma = G(F)$, the group of all points of G with values in F (if this is defined as before via a faithful rational matrix representation ρ, it is independent of the choices) and to find a locally compact group $\mathcal{G}$ that contains $G(F)$ as a discrete subgroup.

The natural choice is to take $\mathcal{G}$ to be the group of points of G with values in the *adèles* of F. To construct the adèles of $\mathbb{Q}$ one first takes the ring $\mathbb{R} \times \prod_p \mathbb{Q}_p$ (direct product) where if p is a prime number, $\mathbb{Z}_p = \varprojlim_N \mathbb{Z}/p^N\mathbb{Z}$ is the compact totally disconnected ring of p-adic integers, $\mathbb{Q}_p = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ its fraction field, which contains $\mathbb{Q}$ as a dense subring, and the product runs over all prime numbers. This direct product contains $\mathbb{Q}$ as a discrete subring but is too big to be locally compact, so it is replaced by the subring $\mathbf{A}_{\mathbb{Q}}$ of sequences $(x_\infty, (x_p))$ where $x_p \in \mathbb{Z}_p$ for all but finitely many p. This has a natural locally compact topology and contains $\mathbb{Q}$ as a discrete subring with compact quotient. More generally we define $\mathbf{A}_F = \mathbf{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} F$. Now if G is a semisimple algebraic group over F, then the quotient $G(F)\backslash G(\mathbf{A}_F)$, though in general not compact, has finite invariant volume, and one defines an automorphic form on G to be a function $f : G(F)\backslash G(\mathbf{A}_F) \to \mathbb{C}$ that is locally constant in the p-adic variables for all p, C^∞ in the real and complex variables, where it also satisfies the analogues of conditions (a), (b), and (c) above. In fact, when G is semisimple the quotient $G(F)\backslash G(\mathbf{A}_F)$ can be interpreted as a projective limit of C^∞ manifolds, each of which can be identified with a finite union of the homogeneous spaces of the form $\Gamma\backslash G(\mathbb{R})$ considered above. But the adèlic perspective has a precious advantage: the group $G(\mathbf{A}_F)$ acts by right translation[2] on the space $\mathcal{C}(G)$ of automorphic forms on G and stabilizes the subspace of L^2-forms, which decomposes as a direct integral over irreducible unitary representations of $G(\mathbf{A}_F)$. An irreducible $G(\mathbf{A}_F)$-invariant *direct summand* of $\mathcal{C}(G)$ we will call an *automorphic representation* (this is more restrictive than the standard definition).

The set of automorphic representations of G is denoted $\mathcal{A}(G)$. Note that the object of interest is no longer the individual automorphic form f but the automorphic representation it generates. In the classical case $G = SL(2)_{\mathbb{Q}}$, one can identify (certain) automorphic forms f in this sense with holomorphic elliptic modular forms. This is carried out in the second of Kudla's contributions to the volume under review. The property of belonging to an irreducible automorphic representation corresponds in the classical language to being an eigenvector for almost all the classical Hecke operators.

This definition has the additional advantage of carrying over unchanged when F is the field of rational functions $k(X)$ on a smooth proper algebraic curve X over a finite field k. In defining the adèles of $k(X)$ one takes instead of $\mathbb{R}$ or $\mathbb{Q}_p$ the fields $\widehat{k(X)}_x$ of formal power series in a neighborhood of the point x in $X(\bar{k})$,

[2] Not quite; condition (a) is incompatible with an action of the group $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ of real points of G, which has to be replaced by the enveloping algebra of its complexified Lie algebra. This is just one of the seemingly endless list of details that makes exposition of the theory so difficult, and we will just wish them away for the sake of this review.

where two points are identified if they belong to the same orbit under $Gal(\bar{k}/k)$. In what follows, "function fields" will be fields of the form $k(X)$, necessarily of positive characteristic, in contrast to "number fields" which are finite extensions of $\mathbb{Q}$.

When G is merely reductive rather than semisimple – e.g., $G = GL(n)$, in principle the most important example – then $G(F)\backslash G(\mathbf{A}_F)$ does not have finite volume, and to speak of automorphic representations one has to amend the theory in one of several more or less inelegant ways. It's enough to add the condition that the center of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ acts by a character. The group $GL(1, \mathbf{A}_F) = \mathbf{A}_F^{\times}$ is the *idèle group* of F, and an automorphic form in this setting is just a complex-valued character of the locally compact abelian group $F^{\times}\backslash\mathbf{A}_F^{\times}$, the *idèle class group*. Here is where number theory enters the picture. Let $\mathcal{A}(GL(1))_{fin} \subset \mathcal{A}(GL(1))$ denote the set of (complex) characters of the idèle class group of finite order. The main theorem of (abelian) class field theory, valid for function fields as well as for number fields, can be interpreted to assert a canonical identification between $\mathcal{A}(GL(1))_{fin}$ and the set $\mathcal{G}(GL(1))$ of characters $Gal(\bar{F}/F) \to \mathbb{C}^{\times}$ of the absolute Galois group of F, or equivalently of the Galois group of the maximal abelian extension of F.

It would be wonderful to be able to say at this point that the Langlands conjectures predict that the elements of $\mathcal{A}(G)$ correspond to something Galois-theoretic for general reductive groups G, but this would be wrong on several grounds. For general G there is a distinguished subset $\mathcal{A}(G)_{alg} \subset \mathcal{A}(G)$ ("alg" for "algebraic"), extending the set $\mathcal{A}(GL(1))_{fin}$ introduced above, but more general even for $G = GL(1)$, whose members should have Galois-theoretic meaning. A bit more precisely, the elements of $\mathcal{A}(G)_{alg}$ are (conjecturally) *parametrized* by certain kinds of representations of $Gal(\bar{F}/F)$ in a way to be described in a bit more detail below. When F is a function field there is essentially no difference between $\mathcal{A}(G)_{alg}$ (see below) and $\mathcal{A}(G)$, but for number fields non-algebraic automorphic representations, including the classical Maass forms, arise from many sources, including the spectral theory of locally symmetric spaces, and are of great importance in analytic number theory.

The only way proposed to account for non-algebraic members of $\mathcal{A}(G)$ is to parametrize them by representations of the conjectural Langlands group mentioned above. I will not try to explain what this means but instead describe what is meant by a parametrization. To each (connected) reductive group G over F Langlands associates a complex (connected) reductive group $\hat{G}$, the (Langlands) dual group, generalizing the classical duality of tori. The formula is explicit but a bit tricky. Langlands duality switches groups of type B with those of type C, otherwise leaving the types alone; it also switches simply connected and adjoint groups. Fortunately $\widehat{GL(n)} = GL(n)$, and generally $\hat{G}$ depends only on G over $\bar{F}$. The L-group ${}^LG$ is (in one version) a semi-direct product $\hat{G} \rtimes Gal(\bar{F}/F)$, where the action depends on the specific form of G over F. A (Galois-theoretic) Langlands parameter is then a homomorphism $\phi : Gal(\bar{F}/F) \to {}^LG$ satisfying certain axioms. One would now like to say that such ϕ parametrize elements of $\mathcal{A}(G)_{alg}$ at least, but there are additional complications. In the first place, the same ϕ can parametrize an *L-packet* consisting of several (even infinitely many) distinct elements of $\mathcal{A}(G)_{alg}$; this is the phenomenon of *endoscopy* or *L-indistinguishability*. This complication is fortunately absent when $G = GL(n)$. In the second place, for general automorphic representations Langlands parameters have to be supplemented by the more general Arthur parameters. I will not even pretend to define these but simply mention that here again, the situation is under control for $GL(n)$, thanks to work of Moeglin and Waldspurger.

At this point one might incautiously predict that $\mathcal{A}(GL(n)_F)_{alg}$, at least, is in bijection with the set of Langlands (or Arthur) parameters $\phi : Gal(\bar{F}/F) \to {}^L GL(n) = GL(n,\mathbb{C}) \times Gal(\bar{F}/F)$, and in fact it's enough to consider the first factor $\phi_0 : Gal(\bar{F}/F) \to GL(n,\mathbb{C})$. But one would still be wrong; only the analogues of $\mathcal{A}(GL(1))_{fin}$ can be parametrized in this way, and more general algebraic automorphic representations of $GL(n)$ are best parametrized by something more involved – either by compatible families of ℓ-adic representations, or by a single ℓ-adic representation that is *geometric* in the sense of Fontaine-Mazur, or else by reference once again to the conjectural Langlands group. The problem is that ϕ as above is always assumed continuous; hence the image of ϕ_0 in $GL(n,\mathbb{C})$ is a finite group. However, the conjecture that such a ϕ_0 actually parametrizes an automorphic representation of $GL(n)_F$ subsumes the Artin conjecture, by general consensus one of the deepest open questions in algebraic number theory.

It is natural to hope that the parametrization I have avoided defining precisely is not merely a set-theoretic bijection but is so devised as to permit translation of important properties of the Galois side to the automorphic side and vice versa. As with abelian class field theory, this is indeed the case. The best way to characterize the bijection is to postulate that it identifies the *L-functions* of the terms on both sides. For our purposes, an L-function is a Dirichlet series – i.e., an expression of the form $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ for $Re(s)$ sufficiently large, with (i) an Euler product factorization $L(s) = \prod_p L_p(s)$, where p runs through primes and $L_p(s)$ is the reciprocal of a polynomial in p^{-s} with constant term 1, and (ii) meromorphic continuation to $\mathbb{C}$, satisfying a functional equation analogous to that of the Riemann zeta function. The historical starting point of the Langlands conjectures was Langlands' discovery, using his theory of Eisenstein series, that (in general, several) L-functions with these properties could be attached to most automorphic representations. This approach, refined and extended by Shahidi, is now known as the *Langlands-Shahidi method* of analytic continuation of L-functions and has been shown to attach entire L-functions to large classes of automorphic forms on many different groups. A second approach to the analytic continuation of L-functions, generalizing constructions first introduced by Hecke, Rankin, Selberg, and Shimura in the setting of elliptic modular forms, is usually called the *method of integral representations* or, in somewhat more specialized circumstances, the *Rankin-Selberg method*.

On the other hand, Artin showed how to attach an Euler product $L(s,\phi_0)$ to a continuous homomorphism $\phi_0 : Gal(\bar{F}/F) \to GL(n,\mathbb{C})$ as above, and Brauer proved the meromorphic continuation and functional equation. When $n = 1$ and ϕ_0 is trivial, one obtains the Riemann zeta function. The Artin conjecture is that $L(s,\phi_0)$ is *entire* if ϕ_0 is irreducible and non-trivial. The Artin conjecture for ϕ_0 would follow from the existence of a cuspidal[3] automorphic representation π of $GL(n)_F$ with $L(s,\pi) = L(s,\phi_0)$ (as Euler products, not merely as analytic functions).

[3]Roughly, an automorphic representation of G is *cuspidal* if it does not come by parabolic induction from an automorphic representation of a Levi factor of a proper parabolic subgroup of G.

When $n = 2$, Langlands, Tunnell, and quite recently Taylor and his collaborators (for $F = \mathbb{Q}$) have established the Artin conjecture for a large class (most?) of ϕ_0. This is practically all that is known regarding the Artin conjecture for number fields. The situation for function fields is much better. Readers are undoubtedly aware that Lafforgue received the Fields Medal in 2002 for his proof of the Langlands conjecture for $GL(n)_F$ when F is a function field of characteristic $p > 0$. Fortunately, it is reasonably simple to give a precise statement of the theorem Lafforgue proved [L]. We can let $\mathcal{A}^0(GL(n)_F)_{alg}$ denote the set of cuspidal automorphic representations of $GL(n)_F$ for which the center – the idèles of F – acts via a character of finite order. Let ℓ denote a prime different from p. In the function field case automorphic forms on $GL(n)_F$ are locally constant functions on $GL(n, \mathbf{A}_F)$, so we lose no generality by assuming they take values in the algebraic closure $\overline{\mathbb{Q}_\ell}$ of $\mathbb{Q}_\ell$. We let $\mathcal{G}^0_{n,alg}(F)$ denote the set of equivalence classes of irreducible n-dimensional representations of $Gal(\bar{F}/F)$ with coefficients in $\overline{\mathbb{Q}_\ell}$, which are unramified outside a finite set S of places, with determinant a character of finite order. Viewing F as the function field over k of the smooth projective curve X and S as a finite set of ($Gal(\bar{F}/F)$-orbits of) points on X, one sees that elements of $\mathcal{G}^0_{n,alg}(F)$ correspond to ℓ-adic representations of the fundamental group over k of the complement of S in X. The L-function $L(s, \phi)$ associated to an element ϕ of $\mathcal{G}^0_{n,alg}(F)$ is known to be entire by Grothendieck's theory. Lafforgue's theorem is that there is a bijection between the sets $\mathcal{G}^0_{n,alg}(F)$ and $\mathcal{A}^0(GL(n)_F)_{alg}$ that preserves a number of arithmetic invariants, notably the L-functions of the two sides.

The general Langlands conjectures for number fields are much more difficult to state correctly. But once one admits the notion of parametrization of automorphic representations by some sort of representations with values in the L-group, one is naturally led to Langlands' functoriality conjectures, which lie at the heart of Langlands' approach to automorphic forms. Suppose G and H are two reductive groups over F, and suppose $r : {}^LH \to {}^LG$ is an L-homomorphism, meaning a homomorphism of complex algebraic groups satisfying some additional axioms I will not specify. Composing r with a Langlands parameter for H yields a Langlands parameter for G and thus conjecturally a *functorial transfer* of (L-packets of) automorphic representations of H to (packets of) automorphic representations of G. This is by no means simpler than the global conjectures sketched above. For example, the general Artin conjecture is roughly the special case of the functoriality conjecture with $G = GL(n)$ and $H = \{1\}$ the trivial group. Nevertheless, there has recently been significant progress if $G = GL(n)$ is a general linear group (and so is ${}^LG$), notably when H is a classical group (orthogonal or symplectic) with the standard L-homomorphism [CKPSS], [GRS], or when $H = GL(2)$ (resp. $H = GL(2) \times GL(3)$) and $n \leq 5$ (resp. $n = 6$) (this is due to Kim and Shahidi; cf. [S] for a general account).

The results mentioned in the last two paragraphs, together with the final steps in the proof of the local Langlands conjecture for $GL(n)$ [LRS], [HT], [He], and the proof of the modularity of elliptic curves over $\mathbb{Q}$, begun in [W], [TW] – with Wiles' proof of Fermat's Last Theorem as a corollary – and completed in [BCDT], are a striking vindication of Langlands' insight and form the background for the book under review. They also form the principal subject matter of three lectures by Cogdell, which come near the end but represent the core of the book. Cogdell's first chapter contains a concise and readable sketch of the analytic theory of the tensor product

L-functions $L(s, \Pi_1 \times \Pi_2)$ where Π_i is a cuspidal automorphic representation of the group $GL(n_i)$, $i = 1, 2$. In this he follows the method of integral representations due to Jacquet, Piatetski-Shapiro, and Shalika, leading to his own results with Piatetski-Shapiro on the generalized converse theorems[4] for $GL(n)$, which, combined with the Langlands-Shahidi method, form one of the most powerful methods presently known for proving special cases of the functoriality conjecture. (The Arthur-Selberg trace formula, its principal rival, gets equal time below.) Cogdell includes just enough details to make the material plausible without losing sight of the goal. His second and third chapters treat respectively the Langlands conjectures for $GL(n)$ and the framework of functoriality for general reductive groups. Taken together, Cogdell's chapters provide the best introductory account currently available of the state of Langlands' functoriality conjectures – the state as of early 2002, when the lectures were submitted for publication.

Most of the rest of the book consists of a gradual introduction to the theory of automorphic forms, concentrating on the groups $GL(1)$, $SL(2)$, and $GL(2)$, as preparation for Cogdell's articles. It may be too much to expect that a reader who encounters the material in these preliminary chapters for the first time will actually be able to make the leap to the generality required for the Langlands conjectures. This is all right, because the material stands on its own as an introduction to the analytic theory of automorphic forms on adèle groups, with applications to number theory. There are very few proofs, but the exposition is at a uniformly high level.

The first two chapters, by Emmanuel Kowalski, contain a thorough presentation of elementary algebraic number theory from the standpoint of L-functions and their analytic properties. Given the centrality of L-functions to the Langlands program, nothing would seem more natural, but in fact the properties of L-functions traditionally of interest to analytic number theorists – for example, the location of zeroes in the critical strip (the Generalized Riemann Hypothesis) – have historically had little to do with the preoccupations of the Langlands program. Thanks largely to the efforts of a few charismatic and determined individuals, this is beginning to change, and Langlands himself has in recent years turned to methods from analytic number theory in an attempt to get beyond the visible limits – more about this below – of the techniques developed over the last few decades. In this sense Kowalski's articles are a welcome sign of the times. They are also written in a polished and appealing discursive style, with frequent references to the history of the subject. In a few short pages Kowalski manages to present the major results of a standard introductory course in algebraic number theory, including a statement of (one form of) the main theorem of class field theory. Kowalski's articles also include coherent and convincing sketches of Hecke's proof of the functional equations of Hecke L-series, the explicit formulas relating the zeroes of the Riemann zeta function and Dirichlet L-functions to prime numbers, and the convexity bounds for the magnitude of L-functions in the critical strip – a particularly active point of contact between automorphic forms and analytic number theory.

Most of this material has been presented elsewhere, but rarely with such clarity and efficiency. The same can be said of Kowalski's third chapter, devoted to

[4]The method of integral representations shows that an Euler product admits an analytic continuation and functional equation if it is attached to an automorphic representation. A *converse theorem* shows, conversely, that if the Euler product admits an analytic continuation that satisfies enough functional equations, then it comes from an automorphic representation. For $GL(2)$ there are celebrated converse theorems due to Hecke, Maass, Weil, and Jacquet-Langlands.

the classical theory of automorphic forms on $SL(2,\mathbb{R})$. The Langlands conjectures make no distinction in principle between holomorphic modular forms and their non-holomorphic analogues, the Maass forms, which typically arise as eigenfunctions of the invariant Laplacian on arithmetic quotients of the Poincaré upper half-plan. However, only the former are amenable to methods of arithmetic algebraic geometry, whereas Maass forms are more sparse and can only be handled using analytic techniques. As a result, most introductions concentrate on holomorphic modular forms. Kowalski treats both cases on an equal footing.

Ehud de Shalit's contribution consists of one chapter each on Artin L-functions and on the relations between elliptic curves over $\mathbb{Q}$ and elliptic modular forms. As far as arithmetic algebraic geometers are concerned, this is the payoff of the Langlands program, and one can hardly imagine a book of this kind omitting this material. This also means that there is nothing really novel in de Shalit's presentation, but it is comprehensive – there are good sketches of the theory of local constants of Artin L-functions and of Hasse's theory of the zeta function of an elliptic curve over a finite field – and should be useful to non-specialists.

The two chapters by Stephen S. Kudla introduce adelic methods on $GL(1)$ and $GL(2)$, respectively. The chapter on $GL(1)$ is devoted to Tate's thesis, in which the functional equation of Hecke L-functions was proved by Fourier analysis on the adèles. Tate's thesis remains the model for all subsequent proofs of analytic continuation and functional equations of L-functions by the method of integral representations, and Kudla's presentation is written with this in mind. This is one of two chapters to contain detailed proofs. The exposition is limpid and, by choosing to follow a Bourbaki report of Weil emphasizing the role of equivariant distributions on groups over local fields, Kudla's article provides an excellent introduction to the higher-dimensional theory. Kudla's second article is a standard account of the dictionary between classical and adelic modular forms and contains no surprises.

Daniel Bump's chapter on spectral theory and the trace formula on $SL(2,\mathbb{R})$ is the longest in the book. It is also the only chapter that gives any indication that the recent progress in the Langlands program comes at a considerable technical cost. Viewing $L^2(\Gamma\backslash\mathcal{G})$ as a direct integral over the set (or space) $\hat{\mathcal{G}}$ of unitary representations of $\mathcal{G}$, the problem of spectral theory is to describe this direct integral explicitly. The technique of choice is to study the traces of a good class of integral operators – usually the smooth compactly supported functions ϕ on $\mathcal{G}$ when $\mathcal{G} = G(\mathbb{R})$ or $G(\mathbf{A})$ – in the regular representation R on $L^2(\Gamma\backslash\mathcal{G})$.[5] The tool of choice is the Selberg trace formula in one of its modern variants, which expresses the trace of $R(\phi)$ as a sum of explicit geometrically defined distributions on ϕ, the *orbital integrals* and their weighted variants. For Langlands and his closest collaborators, the trace formula has always been the royal road to at least the simplest cases of functoriality, because the orbital integrals are geometrically explicit and therefore can in principle be compared for different groups H and G. This gives rise to comparisons of $\mathcal{A}(H)$ and $\mathcal{A}(G)$, in keeping with the predictions of functoriality. The first successful comparison of this type was contained in the final chapter of the massive treatise of Jacquet and Langlands, relating automorphic forms on the multiplicative group of a quaternion algebra over F to automorphic forms on $GL(2)_F$. A simplified variant of this example is sketched, convincingly, at the end of Bump's chapter. Bump's discussion of the trace formula for the most part reflects

[5] Adjustments are necessary when G is reductive but not semisimple.

Selberg's original viewpoint, however, emphasizing Weyl's law on the asymptotics of the spectrum of Maass forms (presented without proof) and the analytic continuation of the Selberg zeta function, described as a geometric analogue of the explicit formulas in the theory of the Riemann zeta function and proved in some detail when $\mathcal{G} = SL(2, \mathbb{R})$ and $\Gamma\backslash\mathcal{G}$ is compact. In this approach, the trace formula for integral operators is a way to get at eigenvalues of the $SL(2, \mathbb{R})$-invariant Laplacian on the upper half-plane. As a warmup, Bump develops the representation theory of spherical functions on $SL(2, \mathbb{R})$ and gives a completely explicit proof of the Plancherel formula in this setting, purely in terms of differential calculus and the residue theorem. No Lie theory is involved at all, as Bump admits, and the techniques do not generalize to higher dimensions, but the exposition has the great merit of making representation theory concrete. Bump also proves analytic continuation of the Eisenstein series, again by explicit methods, in the simplest case where the discrete group Γ has a single cusp.

The book concludes with a short coda in the form of Gaitsgory's introduction to the geometric Langlands conjecture. This is an analogue of the Langlands correspondence described above, in which Galois representations are replaced by n-dimensional ℓ-adic local systems on a curve X over a field k of characterstic different from ℓ, and automorphic representations are replaced by certain kinds of perverse sheaves on the moduli stack classifying n-dimensional vector bundles over X. The case $k = \mathbb{C}$ is specifically included, and the correspondence is highly non-trivial in that case as well. This is one area in which there has been considerable progress since 2001, when the lectures on which this book was based were presented: the geometric Langlands conjecture for $GL(n)$ is now a theorem, thanks to work of Gaitsgory with Frenkel and Vilonen.

The book is not without flaws. The different contributors occasionally refer to one another's articles, but for the most part they do not; notation is not consistent from one author to the next (and occasionally within a single article), and there is some duplication of effort (why are there two essentially identical proofs of the explicit formulas?).

The book's chief weakness, however, lies in what it omits. I have already indicated that the book is written with the approach to functoriality via the converse theorem in mind, and that the alternative route, via the stable Arthur-Selberg trace formula, is considered only briefly. For over two decades the problem of *(twisted) endoscopy* has occupied the attention of a good number of people, under Langlands' guidance, and it is now known that a large number of special cases of functoriality will follow simultaneously from a specific assertion about harmonic analysis on p-adic groups known misleadingly as the *Fundamental Lemma*. There has been important recent progress toward proof of the fundamental lemma, though it is still too soon to know how much remains to be done. Moreover, the trace formula is the principal analytic tool in Lafforgue's work on $GL(n)$, where stabilization is not a problem, and in the successful resolution of the local Langlands conjectures for $GL(n)$. This is duly mentioned in Cogdell's final article, and one can hardly be faulted for failing to treat this excruciatingly technical material in an introductory text. Nevertheless, endoscopy is a central issue in the Langlands program, and the editors could have found a way to make this clear.

At the same time, it is all too clear that the combined results to be expected from endoscopy and from (currently accessible versions of the) converse theorem fall far short of the scope of Langlands' functoriality conjectures, even in the simplest case of Artin L-functions for $GL(2)$. Langlands, always attentive to the limitations of our present knowledge, has made this point forcefully in his most recent writings. Again, the editors could have found room to stress the immense disproportion between what our current techniques can provide and the full ambitions of the Langlands program.

Finally, the editors apparently made a conscious decision to say as little as possible about Lie theory. This is perfectly understandable, since introducing the necessary notation alone would probably have doubled the size of the book. But the theory of automorphic forms gets much, much harder once one leaves the case of $GL(2)$, where almost everything can be done by hand, so to speak. Readers finishing this book may well not suspect just how hard it is.

The book lacks an introduction where all these issues could easily have been addressed. But these complaints[6] should not obscure the book's considerable merits. I have already recommended it to graduate students as a remarkably well-written introduction to the Langlands program and a gentle introduction to the techniques of the modern theory of automorphic forms more generally. It is not, nor does it pretend to be, *the* introduction to the Langlands program. There is room for many more introductions as successful as this one.

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[6]Which the reader shouldn't take too seriously. I have known and worked closely with most of the authors for upwards of two decades. A too chummy approach would diminish the reviewer's credibility.