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Fixed point theory, by Andrzej Granas and James Dugundji, Springer-Verlag, New York, 2003, xv + 690 pp., \$84.95, ISBN 0-387-00173-5

If you asked your colleagues to tell you the first fact that comes to mind concerning fixed points, I expect some would say "contractions have fixed points" and others would say "maps of balls have fixed points." There might also be some showoffs who would give you a fancier answer, and we'll discuss a couple of those fancy answers later in this review. On the other hand, some of your colleagues might tell you that they have better things to do than answer annoying questions.

1. Banach's theorem

Your colleagues who mentioned contractions were remembering the Banach contraction principle [Ba]. Given metric spaces X with metric d, and Y with metric ρ, a *Lipschitzian* function $F\colon X \to Y$ is one for which there exists a constant $L > 0$ such that $\rho(F(x_1), F(x_2)) \leq Ld(x_1, x_2)$ for all $x_1, x_2 \in X$. A function $F\colon X \to X$ on a metric space is a *contraction* if it is Lipschitzian for some $L < 1$. Banach proved that, if X is complete, such a (continuous) map has a fixed point: a solution to $F(x) = x$. Moreover, it has only one solution and it is the limit of the sequence $\{F^n(x)\}$ obtained by iteration of F; that is, $F^n(x) = F(F^{n-1}(x))$, starting with any x in X.

There are better reasons for learning about fixed points than to be prepared to answer annoying questions. Given a map $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$, the initial value problem

$$\frac{du}{ds} = f(s,u) \qquad u(0) = 0 \tag{1}$$

is equivalent to the integral equation

$$u(t) = \int_0^t f(s, u(s))\, ds.$$

Obviously a solution to that equation is a fixed point $F(u) = u$ where F is the function on the space X of continuous real-valued functions on $[0,T]$ defined by

$$F(u)(t) = \int_0^t f(s, u(s))\, ds. \tag{2}$$

If f is Lipschitzian for any constant L, then with the metric on X induced by the norm

$$\|g\| = \max_{0 \leq t \leq T} e^{-Lt}|g(t)|,$$

the function F is Lipschitzian with respect to the constant $1 - e^{-LT}$. Thus F is a contraction and Banach's principle implies that there is a unique global solution to the initial-value problem in this case, that is, a map $u\colon [0,T] \to \mathbb{R}$ that satisfies the differential equation for all $s \in [0,T]$.

There is much more fixed point theory, and many applications, that can be obtained by pursuing the direction initiated by Banach. The subject has its own name, metric fixed point theory, and a vast literature. Since the book under review

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offers only a relatively brief, though well-chosen, sampling of this sort of fixed point theory and there is a recently published *Handbook of Metric Fixed Point Theory* [KS] that will tell you much, much more, it's time we moved on to the other part of the subject: topological fixed point theory.

2. Brouwer's theorem

Your colleagues who said that maps of balls have fixed points were thinking of the Brouwer fixed point theorem [Br]: if $f\colon X \to X$ is a map and X is homeomorphic to the (closed) unit ball B^n in a euclidean space $\mathbb{R}^n$, then $f(x) = x$ has a solution. The Brouwer theorem is easily shown to imply, and to be implied by, some striking facts about the unit sphere S^n, the boundary of B^{n+1}. One is the Lusternik-Schnirelmann theorem, which says that if $\{C_1, C_2, \ldots, C_{n+1}\}$ is a covering of S^n by closed sets, then one of the C_j must contain a pair of antipodal points, that is both x and $-x$. Another is the Borsuk-Ulam theorem: for any map of S^n to $\mathbb{R}^n$ there must be a pair of antipodal points sent to the same point. The Brouwer theorem usually turns up early in an introductory algebraic topology course because all you need to know to prove it is that the homology group $H_n(S^n)$ isn't trivial, as the n-th homology group of a point is. A map of B^{n+1} to itself that had no fixed points could be used to construct a contraction of S^n through itself to a point, and that would imply that $H_n(S^n) = 0$.

Although the Brouwer theorem tells us only about finite-dimensional spaces, it has a consequence in an infinite-dimensional setting that is a powerful tool in nonlinear analysis. Let K be a compact subset of a normed linear space X. There is a finite $\frac{1}{m}$-net N_m for K, that means every point of K is within $\frac{1}{m}$ of the finite set N_m. The convex hull $con(N_m)$ of N_m is homeomorphic to a closed ball, and it lies in the finite-dimensional subspace X_m of X spanned by the elements of N_m. Identifying X_m with a euclidean space, it's not difficult to show that the Brouwer theorem implies that every map of $con(N_m)$ to itself must have a fixed point. The Schauder projection P_m maps K to $con(N_m)$, moving no point of K more than $\frac{1}{m}$. Suppose C is a closed, convex subset of a normed linear space X and $f\colon C \to C$ is a compact map; that is, $K = \overline{f(C)}$, the closure of its image, is compact. Restricting f to $con(N_m) \subseteq C$ and then projecting by P_m defines a map f_m of $con(N_m)$ to itself; choose one of its fixed points x_m. The limit of a convergent subsequence of $\{x_m\}$ turns out to be a fixed point of f and that proves the Schauder fixed point theorem: every compact map of a closed convex subset of a normed linear space to itself has a fixed point.

Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function that is continuous in a neighborhood of the origin in $\mathbb{R}^2$. The Cauchy-Peano existence theorem states that the initial-value problem (1) above, for this function f, has a local solution, that is, a map $u\colon [-\alpha.\alpha] \to \mathbb{R}$ that satisfies the differential equation. The reason is that, for α small enough, there is a closed, bounded and convex subset C of the space of such maps (with the supremum norm) mapped to itself by the function F defined by equation (2) above with $-\alpha \le t \le \alpha$. The Ascoli-Arzela theorem can be used to show that C is compact, so the map F restricted to C is compact and thus the Cauchy-Peano theorem is a consequence of the Schauder theorem.

For many problems in analysis concerning a map F on a normed linear space X, it is not possible to find a closed, convex subset of X that F will map back to itself. A useful extension of the Schauder theorem, due to Leray and Schauder, states that if a compact map F defined on the closure of an open subset U of X containing the origin has the property that $F(x) \neq \lambda x$ for all $\lambda > 1$ and all x on the boundary of U, then F must have a fixed point in $\overline{U}$. For instance, suppose $F\colon X \to X$ is completely continuous (that is, compact on bounded subsets) and, for $r > 0$ large enough, $\|x\| = r$ implies $\|F(x)\| \leq r$; then the Leray-Schauder condition is satisfied with respect to U, the open ball of radius r centered at the origin. This is just the simplest instance of the sort of fixed point results that can be obtained from *a priori* estimates.

3. Lefschetz's theorem

A fancier answer to your question about fixed points would be "maps with nonzero Lefschetz number have fixed points." A colleague who gave that answer remembered that a map $f\colon X \to X$ on a compact polyhedron induces linear transformations $\{f_{*k}\}$ of the finite-dimensional vector spaces that are the rational homology groups $H_k(X)$ of X. Their traces can be used to define the *Lefschetz number*

$$L(f) = \sum_{k \geq 0} (-1)^k tr(f_{*k})$$

which is a finite sum because the homology $H_*(X)$ is a graded vector space of *finite type*: all the $H_k(X)$ are finite-dimensional, and they are nontrivial for only finitely many k. The Lefschetz fixed point theorem states that if $L(f) \neq 0$, then f has a fixed point. It is a generalization of Brouwer's theorem because $L(f) = 1$ for any map on a ball. The Lefschetz theorem has many pleasant consequences: for instance, if some iterate f^n of the map is homotopic to the constant map, then $L(f) \neq 0$ so f has a fixed point.

In order to extend the range of applications of Lefschetz's result, it is necessary to eliminate the restriction to compact polyhedra or other spaces with rational homology of finite type. In Schauder's theorem, the hypothesis of compact map replaces the requirement of compactness of spaces that is common in topology, with a hypothesis that is appropriate for analytic applications. Analogously, it is possible to obtain a general Lefschetz theory by allowing arbitrary vector spaces as the rational homology and restricting the linear transformations $\{f_{*k}\}$. For an endomorphism $\phi\colon V \to V$ of a vector space, let K_ϕ denote the union of the kernels of all iterates of ϕ; then ϕ induces $\tilde{\phi}\colon \widetilde{V} \to \widetilde{V} = V/K_\phi$. Now we suppose only that $f\colon X \to X$ is a map of a connected space. If factoring out the union of the kernels of the induced homomorphisms of all iterates from each $H_k(X)$ produces a graded vector space of finite type, then f has a Lefschetz number: use the traces of the $\{\tilde{f}_{*k}\}$ in the formula above. This generalized Lefschetz number is useful in global analysis because, for f a compact map on a Banach manifold (an infinite-dimensional manifold modeled on a Banach space), it is defined and, if it is nonzero, then f has a fixed point.

4. Nielsen's theorem

Many fixed point questions are local in nature. Instead of a self-map of a space, they concern a map defined only on a subset of the space. To choose a specific setting, suppose X is a compact polyhedron, U is an open subset of X and $f\colon \overline{U} \to X$ is a map with no fixed points on the boundary of U. Using simplicial techniques, we need consider only the case that f is a map with a finite number of fixed points, each lying in a subset homeomorphic to a euclidean space. A fixed point of f is a zero of the map defined in a neighborhood of the fixed point by $g(x) = x - f(x)$. The *index* of f at the fixed point is the Brouwer degree of g, which can be calculated by approximating g by a map taking simplices to simplices and then adding up a $+1$ for each simplex whose image retains its orientation and a -1 for each that reverses orientation. The *fixed point index* of f on U is then the sum of the indices of f at all the fixed points. The Lefschetz-Hopf theorem relates this subject to one we have already discussed: if $U = X$, then the fixed point index of f is its Lefschetz number, so the fixed point index may be thought of as a local version of the Lefschetz number.

An equivalence relation on the set of fixed points of a map f from a compact polyhedron to itself is defined by calling fixed points x, x' equivalent if there is a path p from x to x' such that p is homotopic to $f(p)$, which is also a path from x to x', keeping x and x' fixed throughout the homotopy. The finite number of equivalence classes can be enclosed in disjoint closed neighborhoods and $N(f)$, the *Nielsen number* of f, is defined to be the number of equivalence classes with the property that the fixed point index of f on the neighborhood is nonzero. A colleague who wanted to give you a quite fancy answer to your annoying question could tell you that "every map homotopic to f has at least $N(f)$ fixed points," which is the statement of Nielsen's theorem.

The fixed point index becomes a tool for analytic applications when it is extended to the setting of compact maps of *absolute neighborhood retracts* (ANRs). A subset S of a normed linear space is a *neighborhood retract* if there is a map from an open neighborhood of S to S whose restriction to S is the identity. An ANR is a metric space that is homeomorphic to a neighborhood retract of some normed linear space. Obviously a normed linear space is an ANR and, for completely continuous maps in this setting, the fixed point index is called the *Leray-Schauder degree*, which is used to obtained such celebrated results of nonlinear analysis as the Krasnoselskii - Rabinowitz bifurcation theorem. Manifolds modeled on normed linear spaces are also ANRs, so the fixed point index is available to study the fixed points of compact maps of such manifolds. Nielsen's theorem is true for compact maps of ANRs, provided that the homotopy is a compact map, and this leads to results about the existence of multiple solutions to analytic problems [F].

5. About the book

The entire book was planned in 1978, and a Volume 1 was published in Poland in 1982. It contained roughly one-third of the contents of the present work, and, finding it a valuable resource, both the producers and the consumers of fixed point theory looked forward to the publication of the complete work. After the death of James Dugundji in 1985, Granas continued the project. The brief sketch of fixed point theory above does little justice to the ambitious scope of their plans. For instance, most of Volume 1, corresponding to about the first 200 pages of the new book, presented a wealth of fixed point material that makes no use of algebraic topology; instead it draws upon techniques from combinatorics and general topology. Also, that portion of the book, and later parts as well, includes considerable material on the fixed point theory of multivalued functions. If a function on a space X takes as its values subsets of X, then x a *fixed point* means $x \in f(x)$. The fixed point theory of these functions is very extensive, as the recent book of Gorniewicz [G] demonstrates.

However, the outline above does indicate what the new book of Granas and Dugundji is about. It presents information about fixed points, now emphasizing topics that make use of algebraic topology. The book reflects the fact that fixed point theory is an area of topology that aims for a wide range of applications. Some of the most attractive applications lie within topology itself, but the interaction between topology and nonlinear analysis is a persistent theme of the book.

Nevertheless, this is a *topology* book. The authors' main concern is to present a thorough and rigorous exposition of the topology of fixed points. A notable feature is the careful treatment of the fixed point index, one of the most technically demanding of the fixed point theorist's tools. But the book goes well beyond the basics of the subject in a wide-ranging presentation of topological fixed point theory. Of course it cannot cover everything, and, in particular, Nielsen theory receives only a brief mention, perhaps because when the book was planned, that topic was much less prominent than it is now [GW].

The reader may be surprised by the relative brevity of most of the proofs. This is the consequence of the authors' very efficient style: we are told just the essential steps that make the statement of the result believable. Thus researchers will readily find the information they seek, while students can develop their skills by filling in details of proofs, as well as by using the problem sets that end each chapter.

Fixed Point Theory is deeply concerned with the history of its subject. Photographs of many of the major contributors to fixed point theory are scattered throughout the book, and each chapter ends with an extensive section of Notes and Comments about the history of the material just presented. There are many literature references as a result, but the referencing system is not sufficiently detailed. For instance, the reference Hopf [1928] does not tell the reader in which of the three 1928 papers listed the result can be found. The authors' historical sense prompted them to rename what I, following tradition, have called the Lefschetz fixed point theorem as the Lefschetz-Hopf theorem, whereas the result with the latter name has become the Hopf index theorem. It may be too late to make the terminology of fixed point theory entirely historically accurate, but there is no doubt that, in other respects, this fine book will have a profound influence on its subject.

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