BOOK REVIEWS

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Galois theory of linear differential equations, by Marius van der Put and Michael Singer, Springer-Verlag, Berlin, 2003, xviii+438 pp., $99.00, ISBN 3-540-44228-6

Differential Galois theory is basically Galois theory for differential equations. This theory has its origins at the end of the nineteenth century in the works of Picard ([20], [21], [22]) and Vessiot [28] dealing with linear differential equations. For this reason the Galois theory of linear differential equations is also called the Picard-Vessiot theory.

The Picard-Vessiot theory was translated into the modern language of extensions of differential fields by Kolchin in the middle of the twentieth century (see [14], [11], and references therein). In a manner similar to the classical Galois theory of polynomials, in this approach we start with a differential field K containing the coefficients of the linear differential equation

$$y' = Ay, \tag{1}$$

with $A \in Mat(n, K)$. Then we consider the smallest differential field L that contains K and all of the solutions of this linear differential equation. Kolchin proved the existence and uniqueness of the extension L/K, provided that the characteristic of K is zero and has an algebraically closed field of constants. Kolchin called this extension the Picard-Vessiot extension associated to the linear differential equation. As in the Galois theory of polynomials, the Galois group of equation (1), $G = Gal(L/K)$, is defined as the set of (differential) automorphisms of L that leaves the coefficient field K fixed; furthermore, G is a linear algebraic group. An important result is that the Galois group gives a characterization of the linear equations that can be integrated in "closed form": a closed form solution is one for which the general solution is obtained by a combination of algebraic functions, quadratures and exponential of quadratures. This is similar to the Galois theorem about the solvability of a polynomial equation by radicals. We call these kinds of equations *integrable*.

Kolchin also extended the differential Galois theory to some special nonlinear differential equations in such a way that the associated differential extensions have nice normality properties; these extensions are called the strongly normal extensions [14]. From the complex algebraic-geometric point of view, the structure of the strongly normal extensions was studied by Buium [8]. Roughly speaking, strongly

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normal extensions of the complex field are given by solutions of linear differential equations with coefficients in a field of abelian functions.

From the 1940s to the 1970s, differential Galois theory was essentially studied only by Kolchin's school, and it seems that, during these years, the small but nice book [11] was the only available monograph on the subject. Nonetheless this book contributed in an essential way to development of the field. In the late 1970s the situation changed somewhat as an increasing number of mathematicians around the world turned their attention to the differential Galois theory.

For linear differential equations with meromorphic coefficients in some domain of the complex plane, recall that the singular points are given by the poles of the matrix A. The singular points are classified as regular and irregular. A singular point is regular if the growth of any solution around it is polynomial when the singularity is placed at infinity; otherwise it is irregular. A Fuchsian equation is an equation with only regular singular points. Then, motivated by the asymptotic analysis of the solutions around an irregular singular point [29], Ramis proved that the Stokes matrices associated to such a point belong to the Galois group of the linear differential equation. To be more precise, Ramis proved that the group generated by the Stokes matrices and other "formal" matrices is Zariski-dense in the Galois group of the equation. This is the Ramis density theorem ([15], [18]); see also [4]. This result can be viewed as a generalization of the classical Schlesinger density theorem for Fuchsian differential equations.

On the other hand, the Picard-Vessiot theory for a linear differential equation with meromorphic coefficients was formulated from the linear connection point of view – or its algebraic counterpart of D-modules – by Katz [12] and complemented by Deligne [10]. At the heart of this formulation is the Chevalley characterization of a linear algebraic group G by means of the representations of G on suitable tensorial constructions [9]. Usually this geometrical (or D-module) approach to the Galois theory of linear differential equations is called the Tannakian approach. Bertrand and André in Paris later became interested in this approach to the Picard-Vessiot theory ([3], [1]); for more information and references about the Tannakian approach I recommend Bertrand's review of the Magid book in this *Bulletin* [5]. A recent additional reference not included in the above is the paper of André [2].

Another reason for the revival of interest in differential Galois theory of linear differential equations comes from applications. I mention only two of these applications that arose in the late 1980s. Motivated by a theorem of Ziglin about the structure of the monodromy group of the Poincaré variational equation along a particular solution of a completely integrable complex analytic Hamiltonian system, Churchill, Rod and this reviewer introduced differential Galois methods in the study of the non-integrability of these systems. This type of result led to a theorem of Ramis and the reviewer (see [19] and references therein). On the other hand, Beukers, Brownawell and Heckman showed that a classical result of Siegel (with some improvements by Sidlovskii and others) about the algebraic independence of some set of numbers related to values of solutions of linear differential equations, can be read in the context of the differential Galois group of the equation. Two of the main ingredients of this approach include a result by Kolchin that the degree of transcendence of the Picard-Vessiot extension is equal to the dimension of the Galois group, and the consideration of some symmetric powers of the equation typical of the Tannakian approach [7].

From the 1970s forward the differential Galois theory began to develop in several different directions. Some of these new directions are:

- *The local theory of linear differential equations around an irregular singular point.* The objective of the local theory involving equation (1) is to try to copy the method of reduction to normal form when the matrix A is a constant matrix. This process consists of two steps: the formal theory and the analytical theory. The formal theory is essentially algebraic and can be generalized to other fields of constants different from $\mathbf{C}$. The analytical theory is transcendental, and it is here where the Stokes matrices enter the picture. The local theory goes back to Poincaré, Fabry, Birkhoff, Hukuhara and Turritin, and it was studied more recently by Deligne, Malgrange, Ramis, Sibuya, Babit and Varadarajan, Loday-Richaud, Barkatou and others. The connection with the Picard-Vessiot theory is given by the Ramis density theorem. This topic can be considered to be more or less complete, at least when we are dealing with a fixed linear differential equation, but unfortunately in the applications we are usually confronted with a family of differential equations; i.e., the equation depends on parameters. For precise references and more information about this aspect, I recommend the nice survey of Varadarajan [27].

- *The Riemann-Hilbert and inverse Galois problems for linear differential equations.* The Riemann-Hilbert problem is Hilbert's 21st problem: this problem asks if a given linear representation of the fundamental group of the Riemann sphere $P^1_{\mathbf{C}}$ with a finite set of points deleted is the monodromy representation of a Fuchsian linear differential equation. Of course, it is possible to give several formulations and generalizations of this problem: for example, to ask for a constructive solution, to fix the dimension of the space of solutions and the class of the representation, to consider other compact Riemann surfaces different from $P^1_{\mathbf{C}}$, to consider non-Fuchsian equations (in this case we must also include other objects like Stokes matrices as part of the monodromy data), what kind of fiber bundles are allowed, etc.... This problem goes back to Riemann's P-equation, to Hilbert himself, and to Schlesinger, Birkhoff and Lappo-Danilevsky. It was believed in the 1960s that Plemelj had given a definitive answer to the constructive classical problem over $P^1_{\mathbf{C}}$ [23], but about 35 years later Bolibrukh found a gap in the Plemelj proof [6]. The non-constructive existence theorem for a solution of the Riemann-Hilbert problem was obtained in a very general setting by Deligne and Rörlh. The Riemann-Hilbert problem continues today as an important area of research with many ramifications in the study of the isomonodromic deformations of linear differential equations. For more references and details, see the already cited survey of Varadarajan.

Another related problem is the inverse problem of the Picard-Vessiot theory. This question asks if an algebraic linear group G over a characteristic zero and algebraically closed constant field C is the Galois group of a linear differential equation. As for the Riemann-Hilbert problem, it can be formulated in several ways, but one of the differences between these two problems is that the inverse Picard-Vessiot problem admits also a purely algebraic formulation, without any mention of complex analysis; i.e., the constant field C and the coefficient field are not necessarily the complex field and a field of meromorphic functions, respectively. Early work on this inverse problem, for a broad class of differential fields of coefficients and a connected solvable G, was given by Biaclynicki-Birula and by Kovacic in the 1960s. At the end of the 1970s, C. Tretkoff and M. Tretkoff solved the existence problem in the category of Fuchsian differential equations over the Riemann sphere. Their method relies on the Schlesinger density theorem and on the Riemann-Hilbert problem. Later Ramis solved the problem in full generality for equations over compact Riemann surfaces; i.e., he obtained a complete characterization of the Galois groups of linear differential equations with meromorphic coefficients over compact Riemann surfaces. The Ramis approach was used by Mitschi and Singer to solve the inverse problem for connected Galois groups over C and linear differential equations with coefficient field $C(x)$, where C is any algebraically closed field of characteristic zero. The inverse problem was also studied by Magid for some specific Galois groups. For more information and precise references, see the survey by Singer [25] and van der Put's Bourbaki seminar [24].

- *The algorithmic and computational aspects for linear differential equations.* Very important in certain applications is the computation of the Galois group of specific linear differential equations, or at least the description of some of their properties. For instance, a typical problem is to ask for the values of the parameters for which members of a family of differential equations is integrable; it is important to remark that this problem includes as a special case the very difficult problem of linear equations that are integrated by algebraic functions. These types of problems have a long history that goes back to the nineteenth century with Schwartz's characterization of the hypergeometric equations with only algebraic solutions. Later in that century, Hermite and others studied the integrability of the Lamé equation. At the end of 1960s Kimura gave a complete characterization of the (classical) hypergeometric equations which are integrable. By the end of the 1970s Baldassarri and Dwork presented an algorithmic procedure for studying the second order equations with algebraic solutions, and their method goes back to classical works of the nineteenth century by Klein and Fuchs.

From the differential Galois perspective, a new era started with an important paper of Kovacic in 1986. This paper gives an efficient, purely algebraic algorithm that decides if a second order equation with coefficients in $\mathbf{C}(z)$ is integrable or not by computing one of the solutions in the integrable case. Previously Singer had published an algorithm for differential equations of order n. More or less at the same time Ramis' density theorem and the Tannakian approach were also used: Martinet and Ramis computed the Galois groups of classical confluent hypergeometric equations by means of Ramis' density theorem [18], and Katz used the Tannakian approach joint with some results of Levelt and Gabber [13]. Some other people that worked on this topic are Duval, Loday-Richaud, van der Put, Bronstein, J.-A. Weil, Ulmer, van Hoej and others. For more information see the survey by Singer [25].

- *The nonlinear differential Galois theory.* In recent years a new Galois theory for nonlinear differential equations has arisen. In 1996 Umemura presented a rigorous formulation of the old Drach-Vessiot theory using modern methods [26]. More recently Malgrange was able to give a definition of the Galois groupoid associated to a foliation with meromorphic singularities ([16], [17]). For linear differential equations, using the Tannakian approach, Malgrange showed that this groupoid coincides with the Galois group of the Picard-Vessiot theory. Although Malgrange's definition of Galois groupoid was given for foliations, a similar definition can be given for vector fields, i.e., for systems of autonomous ordinary differential equations. Today this nonlinear differential theory is a very active area of research.

With the above in mind, the Picard-Vessiot theory can now be formulated from at least three different points of view:

1) the purely algebraic approach of Kolchin,

2) the Tannakian approach, and

3) the topological-analytic approach given by the Schlesinger and Ramis density theorems.

All of the three approaches are useful and in some sense complement one another. It is really a question of the specific problem at hand, or even the taste of the user, as to which approach to use.

The book under review is devoted to the Picard-Vessiot theory. It is an extensive monograph that covers for the first time the three points of view above. The book is divided into two parts. The first part is entitled "Algebraic Theory". It includes the main algebraic definitions and results, the Tannakian approach, the formal local theory and the algorithmic aspects of the computation of the Galois group. In my opinion the Tannakian approach is stated in an abstract categorial way that is not suitable for beginners. The second part is entitled "Analytical Theory". This part is essentially devoted to the topological–analytical approaches and the inverse Galois problems. It also contains a final chapter on the case of positive characteristic. The book also has four appendices and a complete bibliography.

The book will be very useful for researchers in differential Galois theory. Some parts are also suitable for a first introduction to this theory; for instance, Chapter 1, devoted to the algebraic Picard-Vessiot theory, together with Section 5.1, devoted to a first study of the monodromy group and to a statement of Schlesinger's density theorem, can be useful as a brief introductory course on the algebraic and analytical aspects, although there may be some technical aspects therein that are difficult to follow for the non-specialist. This would include the proof that, in the Picard-Vessiot extensions there are no new constants as well as the terminology involving torsors.

In conclusion, I recommend this book for anyone interested in the differential Galois theory, and I think that it will become a standard reference book in the field.

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When I was writing this review I received the obituary notice that on the 11th of November 2003 Andrei Andreevich Bolibrukh passed away. I recall here his memory.

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