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Homotopy theoretic methods in group cohomology, by W. G. Dwyer and H.-W. Henn, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, Basel, 2001, x + 98 pp., $29.95, ISBN 3-7643-6605-2

1. Groups in homotopy theory

All over mathematics we find the benefits of embodying the classification of a class of objects in a universal example. The universal example can then be studied, dissected and manipulated to learn about the original class. The classifying space BG of a group G is a prime example of this, but its real importance, and the reason for the existence of the book under review, is that BG is the homotopy theoretic embodiment of the group itself. It is rather surprising that the soft mathematics of homotopy theory can say anything very much about a strict algebraic object like a group. Early efforts gave the sort of soft conclusions that one might expect, but it is one of the surprises of the last 20 years that by imposing the right sort of finiteness hypotheses, one may understand a great deal about groups using purely homotopy theoretic methods: groups are much more rigid than expected. The technical advances that made this possible go back to work of Carlsson [3], Miller [13] and Lannes [10] on the Segal and Sullivan conjectures in the early 1980s, but the apparatus for processing and exploiting it has been steadily developed since then. The book under review provides introductions to two of the main ingredients for studying groups from a homotopy theoretic point of view.

2. Classifying spaces

We describe two rather different reasons for considering the classifying space. The first is more familiar and lets us give some examples, but the second explains the importance of BG in studying groups.

The space BG is called a classifying space since it classifies free G-spaces (or equivalently principal G bundles), in the sense that homotopy classes of maps $X \longrightarrow BG$ correspond to homotopy types of free G-spaces $\tilde{X}$ with $\tilde{X}/G = X$ (or equivalently to isomorphism classes of principal G-bundles $\tilde{X} \longrightarrow X$). This rather indirect description characterizes BG up to homotopy equivalence and shows that it may be constructed as $(EG)/G$ for any contractible free G-space EG. Thus for example if G is a subgroup of the circle, then we may construct EG as follows. First note that G acts by scalar multiplication on $\mathbb{C}^n$ and that it acts freely on the unit sphere $S(\mathbb{C}^n)$. Taking the union over all n we obtain

$$EG = S(\mathbb{C}^\infty) = \bigcup_n S(\mathbb{C}^n).$$

Hence $BG = S(\mathbb{C}^\infty)/G$; for example the classifying space BC_2 for the group of order 2 is infinite real projective space $\mathbb{R}P^\infty$ and the classifying space for the circle group BS^1 is infinite complex projective space $\mathbb{C}P^\infty$. Equally, it is clear that $E(G\times H) = EG\times EH$ so that $B(G\times H) = BG\times BH$, so this gives explicit models for the classifying space of any abelian compact Lie group.

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For a finite group the classifying space also embodies features of the representation theory of the group. Indeed, the fact that EG is G-free and contractible shows that its chains give a $\mathbb{Z}G$-free resolution of $\mathbb{Z}$ and hence the cohomology of BG is the algebraic group cohomology $H^*(BG) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$.

The classifying space plays another rather different role: indeed the space G and the group multiplication on it can both be recovered up to homotopy from BG so that it simultaneously captures both aspects of the object. In fact the space ΩBG of based loops in BG is equivalent to G and the concatenation of loops corresponds to the group multiplication. This gives a very flexible way of studying the interaction between the underlying space of a topological group and its multiplication. These interact in interesting ways. For example most spaces do not admit a multiplication making them a topological group at all (the first condition being that the fundamental group is abelian). We can then vary the constraints on the multiplication map. When a smooth manifold admits the structure of a topological group, Hilbert's 5th problem asks whether the multiplication can be taken to be smooth, and the solution of Gleason and Montgomery-Zippin answers that it can. We thus reach the class of Lie groups, and the structure is so rigid that compact connected Lie groups can be classified. In the other direction we can ask: if the multiplication satisfies the group axioms only up to homotopy, can it be rigidified to satisfy the condition strictly? This time the answer is no: there are uncountably many distinct homotopy multiplications on S^3 Rector [15], whereas only one of them gives a group structure.

Having decided to study spaces like BG, it is natural to consider them one prime at a time by completion. This has the usual benefit that the situation is simplified and the interactions between various primes are easier to understand. It is also a much more rigid object: quite often multiplications are unique at each individual prime. This leads to the Dwyer-Wilkerson notion of a p-compact group [6], which is the homotopical analogue of a compact Lie group with group of components a p-group. It turns out that much of the classification of connected compact Lie groups can be carried out using only the methods of homotopy theory: connected p-compact groups at odd primes p have been completely classified [1], and although there are several classes of additional examples, they share many features with Lie groups.

This process of concentrating at a prime is embodied in a space. The properties of BG at p can be captured in the Bousfield-Kan p-completion $BG^\wedge_p$, and BG itself can be reassembled from the local pieces using the Hasse square. The p-completion $BG \longrightarrow BG^\wedge_p$ has the property that it induces an isomorphism of mod p cohomology. If G is a p-group, p-completion has no effect on BG. On the other hand p-completion is quite drastic in general. For example, if G is a finite group, $\pi_1(BG^\wedge_p) = G/O^p(G)$, the largest p-quotient of G. Furthermore, passage to loop spaces does not commute with completion, so that although ΩBG is the compact manifold G, it is no longer necessarily true that $\Omega(BG^\wedge_p)$ is finite, even as a p-adic complex. This means that even from the point of view of homotopy theory, compact Lie groups whose group of components is a p-group are much simpler than in general. In fact there is a hierarchy of behaviour of groups at a prime depending on the size of the mod p cohomology of $\Omega(BG^\wedge_p)$: it is finite for p-groups, and R. Levi [12] has shown that it is either polynomial or semi-exponential for other finite groups. In a precise sense [4], this corresponds to the hierarchy of commutative local rings which are regular, complete intersection or Gorenstein (remarkably, all groups have the latter duality property).

For simplicity, in this review we will generally restrict our attention to finite groups G. However, the fact that the methods extend (to compact Lie groups, to p-compact groups, to p-local generalizations of finite groups [2], to arithmetic groups, to mapping class groups and beyond) is one of their main attractions and explicit in much of the book. Since we are studying groups at a fixed prime p, all cohomology has coefficients in $\mathbb{F}_p$.

3. Quillen's theorem

We described the classifying spaces of abelian compact Lie groups above, but in general it is much harder to give a small and illuminating construction, even at a single prime p. We would like to explain how BG is built from simpler spaces, either precisely or just up to cohomology. This is the purpose of the book under review.

The simplest groups of all are the elementary abelian 2-groups $V \cong C_2 \times \cdots \times C_2$. Since $BC_2 = \mathbb{R}P^\infty$, we find $H^*(BC_2; \mathbb{F}_2) = \mathbb{F}_2[y]$, and it follows that the cohomology of V is polynomial. Similarly, the mod p cohomology of an elementary abelian p-group $V = C_p \times \cdots \times C_p$ is polynomial tensor exterior.

Quillen's theorem [14] describes the cohomology of a compact Lie group in terms of elementary abelian groups and thereby determines its cruder features, such as its Krull dimension. He considered the category $\mathcal{E}$ of elementary abelian subgroups of G with morphisms $E \longrightarrow E'$ consisting of group homomorphisms effected by conjugation in G. Since conjugation by an element of G induces the identity in $H^*(BG)$, there is a map

$$q_G : H^*(BG) \longrightarrow \lim_{\substack{\leftarrow \\ E}} H^*(BE).$$

Quillen showed that for a large class of groups (including finite groups, compact Lie groups, arithmetic groups and mapping class groups) the map q_G is an isomorphism *up to nilpotents*, in the strong sense that elements of the kernel are nilpotent and any element of the codomain has a p^nth power in the image for some n. In particular q_G induces an isomorphism of varieties, and so the variety of $H^*(BG)$ is completely described by the category $\mathcal{E}$. On the other hand, q_G is not usually an exact isomorphism. For instance, q_G is injective if and only if the cohomology of G is detected on elementary abelian subgroups. This holds for dihedral 2-groups, but not for the cyclic group of order 4.

Furthermore, Quillen also showed that the category $\mathcal{E}$ does encode how to construct the mod p homotopy type of BG from the classifying spaces of elementary abelian subgroups. This makes it natural to look for variations and improvements. One might seek to use a little more algebraic structure from $\mathcal{E}$ in attempting to describe the cohomology, hoping to get a better approximation. One might also consider different ways of building categories from subgroups of G, and one might then play the whole game over again for each one. The two parts of the book take these two points of view.

4. Homology decompositions

We begin at the geometric level: we want a way to build a space approximating BG using classifying spaces of subgroups of G. The simpler the subgroups, the less complicated their classifying spaces will be, but we also need to ensure that they are assembled in a simple way; in practice we must balance these two aims. This is the subject of the half of the book written by Dwyer.

Before descending into abstraction, it is worth describing the very early example of the group $G = Sp(1)$ of unit quaternions. Dwyer, Miller and Wilkerson [5] proved that the homotopy type of $BSp(1)$ is determined by its cohomology, and a major input was the fact that the space $BSp(1)$ is 2-adically equivalent to the homotopy pushout of a diagram $BG_{48} \longleftarrow BQ_{16} \longrightarrow BQ$ arising from the diagram $G_{48} \longleftarrow Q_{16} \longrightarrow Q$ of subgroups of $Sp(1)$, where G_{48} is the binary octahedral group of order 48, Q_{16} is the quaternion group of order 16 and Q is the normalizer of a circle in $Sp(1)$.

The pushout is a special case of constructing a space from a diagram of spaces, encoded as a functor $F : \mathbf{D} \longrightarrow$ Spaces. We may form its direct limit (or colimit), $\underrightarrow{\lim} F$, which has the universal property that compatible maps $F(d) \longrightarrow T$ assemble to a map $\underrightarrow{\lim} F \longrightarrow T$. However, it may be hard to calculate the cohomology of the direct limit because it depends on the actual values of F and not just on their homotopy type. The solution is to form the *homotopy* direct limit $\underrightarrow{\operatorname{holim}} F$, which is homotopy invariant and has the desired property; since there is a map $\underrightarrow{\operatorname{holim}} F \longrightarrow \underrightarrow{\lim} F$, we obtain a comparison map $\underrightarrow{\operatorname{holim}} F \longrightarrow T$.

To understand BG we want to construct the diagram $\mathbf{D}$ and the functor F out of the group theory of G, so that $F(d) \simeq BH(d)$ for a subgroup $H(d)$ of G. We say that F gives a *mod p homology decomposition* of G if the map

$$\underrightarrow{\operatorname{holim}} F \longrightarrow BG$$

is a mod p homology isomorphism. This gives a spectral sequence

$$R^i \underleftarrow{\lim}_d H^*(BH(d)) \Rightarrow H^*(BG),$$

where $R^i \underleftarrow{\lim}$ refers to the derived functors of the inverse limit functor. To use this we certainly need to know the cohomology of $H(d)$, but if the diagram is complicated, the calculation of the derived functors may be hard, so it is not enough by itself to know $H^*(BH(d))$. We would really like it if $R^i \underleftarrow{\lim} = 0$ for $i > 0$. When this happens the homology decomposition is said to be *sharp*, and in that case

$$\underleftarrow{\lim}_d H^*(BH(d)) = H^*(BG).$$

This is the ideal situation, and we can think of a sharp decomposition as reducing the calculation of the cohomology of G to the cohomology of the smaller groups $H(d)$.

In practice we construct a diagram D and a functor F from a collection $\mathcal{C}$ of subgroups. There are three different ways to do this: the subgroups $H(d)$ with $F(d) = BH(d)$ are either (i) *subgroups* in $\mathcal{C}$ or (ii) an intersection of *normalizers* of subgroups in $\mathcal{C}$ or (iii) *centralizers* of subgroups in $\mathcal{C}$. These give the subgroup approximation, the normalizer approximation and the centralizer approximation to BG, associated to $\mathcal{C}$, and we seek collections $\mathcal{C}$ so that the approximations are actually homology decompositions.

There are two key ideas. Firstly, we may form the finite simplicial complex $K_{\mathcal{C}}$ whose simplices are the chains $H_0 \subset H_1 \subset \cdots \subset H_n$ in $\mathcal{C}$. This complex with the natural action of G controls the situation and is used to organize the study. If the Borel homology of $K_{\mathcal{C}}$ is that of a point, the collection $\mathcal{C}$ is said to be *ample*, and all three of the functors give homology decompositions. Secondly, if any one of the three functors associated to $\mathcal{C}$ give a homology decomposition, so do the other two, and $K_{\mathcal{C}}$ is ample. On the other hand, if the decompositions are ample, then one can be sharp without the others being sharp.

To tie this to the discussion above, it turns out that the collection $\mathcal{C}$ of non-trivial elementary abelian subgroups is ample. It is centralizer and normalizer sharp, but not generally subgroup sharp. Another important ample collection is that of the subgroups which are both p-centric and p-radical (a p-subgroup P is *p-centric* if its centre is a Sylow p-subgroup of the centralizer $C_G(P)$; it is *p-radical* if $N_G(P)/P$ has no normal p-subgroup). This is useful because it provides a fairly small class $\mathcal{C}$ which is subgroup sharp, and has been important in studying maps between classifying spaces and decompositions of p-compact groups [8], [9].

5. Algebraic approximations

Let us return to the inspiration of Quillen's theorem and move on in a different direction. This time we concentrate on the fact that it is based on the collection of elementary abelian groups. The collection of elementary abelian groups is ample, so the homotopy type of BG at p is completely determined by the diagram of elementary abelian subgroups. However, the collection is rarely subgroup sharp, so the action of the isotropy groups on the cohomology of elementary abelian subgroups contributes to $H^*(BG)$. We would like to add this information in an algebraically accessible way and improve the algebraic approximation. It turns out that if we include information from cohomology operations, we need only a little extra information from the cohomology of centralizers of elementary abelian groups to give the whole answer up to a finite error. This is the subject of the half of the book by Henn.

For definiteness we restrict attention to $p = 2$ for the rest of the review. The Steenrod squares $Sq^i : H^n(X) \longrightarrow H^{n+i}(X)$ give natural operations on the mod 2 cohomology of a space. These operations satisfy certain universal identities (the Adem relations) and generate the mod 2 Steenrod algebra $\mathcal{A}$. However, the cohomology of a space is not a typical module over $\mathcal{A}$ since it is *unstable* in the sense that $Sq^i(x) = 0$ if $i > \deg(x)$. The category $\mathcal{U}$ of modules satisfying this is called the category of unstable modules over $\mathcal{A}$. Actually, $H^*(X)$ is also an algebra with the Steenrod squares acting on products via the Cartan formula, and with $Sq^i(x) = x^2$ if x is of degree i; the category $\mathcal{K}$ of algebras which are unstable as modules and satisfy these additional properties is called the category of unstable algebras. These categories were studied in the 1960s by Steenrod and Massey-Peterson, but they became fundamental in understanding maps between classifying spaces because of the special algebraic property of $H^*(BV)$: Carlsson proved [3] in early work on the Segal conjecture that for any elementary abelian 2-group V the cohomology $H^*(BV)$ is injective in the category $\mathcal{U}$. Using this, the work of Lannes [10], [11] since the 1980s transformed them into a powerful tool.

The other sign that this might be relevant to Quillen's theorem arises since one may detect the fact that the kernel and cokernel of Quillen's map are nilpotent in the category of unstable *modules*. One says that an unstable module M is nilpotent if for any element x we have $Sq^{2^n|x|}\cdots Sq^{2|x|}Sq^{|x|}x = 0$ for sufficiently large n (which coincides with the usual notion if M is an unstable algebra). It turns out that M is nilpotent if and only if $\mathrm{Hom}_{\mathcal{U}}(M, H^*(BV)) = 0$ for all V. The fact that the kernel and cokernel of Quillen's map q_G are nilpotent is thus equivalent to requiring that $\mathrm{Hom}_{\mathcal{U}}(q_G, H^*(BV))$ is an isomorphism for each V. We are thus led to consider $\mathrm{Hom}_{\mathcal{U}}(M, N \otimes H^*(BV))$.

For apparently different reasons, Lannes introduced a functorial algebraic approximation to the cohomology of the function space $\mathrm{map}(BV, X)$ using only $H^*(X)$ as an unstable module. The function space is detected by maps into it: in topology, any map $f : A \longrightarrow \mathrm{map}(BV, X)$ is adjoint to a map $\tilde{f} : A \times BV \longrightarrow X$. The map $\tilde{f}$ induces a map $\tilde{f}^* : H^*(X) \longrightarrow H^*(BV) \otimes H^*(A)$ of unstable modules over the Steenrod algebra. It is a formality that tensoring with $H^*(BV)$ has a left adjoint T_V so that $\tilde{f}^*$ is adjoint to a map $T_V H^*(X) \longrightarrow H^*(A)$. Taking $A = \mathrm{map}(BV, X)$ and f to be the identity, we obtain a comparison map $T_V H^*(X) \longrightarrow H^*(\mathrm{map}(BV, X))$. Lannes has shown that under very mild hypotheses this is an isomorphism. The three remarkable properties of the functor T_V which make it so powerful are (i) it is exact, (ii) it preserves tensor products, and (iii) it takes unstable algebras to unstable algebras and again behaves as the adjoint to tensor product in that context.

Applying this to a finite group, one finds that any group homomorphism $\rho : V \longrightarrow G$ gives a map $B\rho : BV \longrightarrow BG$; and evidently if two maps ρ are conjugate in G, the maps $B\rho$ are homotopic, so the maps are indexed by $\mathrm{Rep}(V, G)$. It turns out that these give all the components of $\mathrm{map}(BV, BG)$. Considering the group homomorphism $V \times C_G(\rho) \longrightarrow G$, we obtain a map $BV \times BC_G(\rho) \longrightarrow BG$ and by the formal properties of the T-functor, a map

$$l_G : T_V H^*(BG) \longrightarrow \prod_{\rho \in \mathrm{Rep}(V,G)} H^*(BC_G(\rho)).$$

An important theorem of Lannes states that this is an isomorphism for any finite group G. Thanks to the theorem of Adams-Gunawardena-Miller stating

$$\mathrm{Hom}_{\mathcal{U}}(H^*(BW), H^*(BV)) = \mathbb{F}_2[\mathrm{Hom}(V, W)],$$

the fact that l_G is an isomorphism in degree 0 is equivalent to Quillen's theorem, so Lannes' theorem is already a considerable strengthening.

The way to exploit the information in higher degrees is to note that there are two maps $V \times V \times C_G(V) \longrightarrow V \times C_G(V)$ which become equal in $C_G(E)$, namely those sending (u, v, g) to (uv, g) or to (u, vg). Extending this to allow for inclusions and conjugations, there is a map from $H^*(BG)$ into the equalizer of the fork

$$\prod_V H^*(BV) \times H^*(BC_G(V))^{<n} \rightrightarrows \prod_{V_1 \longrightarrow V_2} H^*(BV_1) \times H^*(BV_2) \times H^*(BC_G(V_2))^{<n},$$

where $M^{<n}$ means the part of M in degrees less than n. When $n = 1$ this equalizer is Quillen's inverse limit, but for higher n it takes into account a little information about $C_G(V)$. Henn, Lannes and Schwartz [7] use the machinery to state precisely the sense in which this becomes a steadily better approximation to $H^*(BG)$ as n

increases. The book goes on to discuss other approximations using centralizers and their uses in calculations, such as his exact calculation of $H^*(BSL(3,\mathbb{Z}[1/2]);\mathbb{F}_2)$.

6. The book

The book is in two parts; both parts are written by authors who were intricately involved in the development of the theory. They fit together because of their subject matter, but they have different styles.

The first part of the book, by Dwyer, considers homology decompositions. On the way it provides a beautifully economical and elegant introduction to simplicial methods, homotopy direct limits and many other useful techniques. The route is so carefully chosen that it appears almost effortless, and almost all proofs are given, at least in outline. It is elementary in a certain sense and could be highly recommended as an introduction. It does not attempt to do things in great generality and refers to the many developments and applications of the methods only in passing.

The second, by Henn, has more of the character of a survey. Because of the substantial literature, this is very valuable. By the nature of the subject he is obliged to omit more details, but he takes care to summarize the salient facts and outlines enough of proofs to give the essential ideas. He measures the abstract theory he describes against its success in calculations of cohomology groups relevant to stable homotopy theory and algebraic K-theory.

Both parts of the book provide valuable introductions to important techniques that should be known to all algebraic topologists and group cohomologists.

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