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Trigonometric series, Vols. I, II, by Antoni Zygmund, third edition, with a foreword by Robert Fefferman, Cambridge Mathematical Library, Cambridge University Press, 2002, (Vol. I) xiv + 383 pp., (Vol. II) viii + 364 pp., \$60.00, ISBN 0-521-89053-5

J.E. Littlewood called it the Bible. After so many years, it is more the Bible than ever: it is a message of permanent value, an absolute chef-d'oeuvre, and a reference book for now and for years to come.

This "third edition", apart from an illuminating foreword by Robert Fefferman as an heir of Zygmund's "Chicago school", is just a reproduction of the second edition of the book, printed first in 1959 and again in 1968 and 1977. The first edition was published in 1935 as volume V of the Polish series Monografje Matematyczne and was entitled *Trigonometrical Series*. *Trigonometrical Series* was a much smaller book than *Trigonometric Series*, but it was already full of content, methods, results, and ideas, all expressed in a pure and rigorous style. Let us compare these two books, the Old Bible and the New One. It gives us an opportunity to enter the history of the subject.

Trigonometrical Series was the work of a young man vigorously involved in the renewal of the subject that took place in the years 1900–1930. In 1900, Fejér's theorem gave a new look to the theory of Fourier series; indeed, there were strange phenomena about convergence, namely, the example of du Bois Reymond of a continuous function whose Fourier series diverges at a given point, but the simplest summability method, by arithmetical means, avoided all difficulties of that kind. Summability methods, convolutions, positive kernels, and regularization of functions reestablished Fourier series as one of the central subjects in mathematics. The main step in the renewal of the theory, however, was the new concept of an integral by Lebesgue in 1901. The Riemann integral had been introduced in order to give a precise meaning to Fourier's formulas, namely, the computation of coefficients by means of integrals. The Lebesgue integral soon appeared as a much better tool, so that the use of the term "Fourier series" became reserved for trigonometric series whose coefficients are obtained through Fourier formulas in the sense of Lebesgue. In modern notations, $L^1(\mathbb{T})$ became the natural frame for Fourier series. However, there is no easy characterization of Fourier coefficients in this context. The simple case is $L^2(\mathbb{T})$, and the Riesz-Fischer theorem (1907) expresses that the Fourier formulas provide an isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$. Both Fischer and F. Riesz used the fact that $L^2(\mathbb{T})$ is complete (except that the sentence "L^p is complete" needed a new set of definitions and was popularized in this form only in 1930 in the book of Banach, *Théorie des opérations linéaires*, tom I, of the Monograpfje Matematiczne). The Lebesgue integral established a strong interplay between trigonometric series, integration, derivation, functions of a real variable (this already appeared with Lebesgue), functions of a complex variable (Fatou),

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functional analysis (F. Riesz and Fischer), and later probability theory (Steinhaus, Wiener, Kolmogorov).

All these connections appear in *Trigonometrical Series*. Between 1935 and 1959, many young mathematicians, including me, learned a good deal of analysis in this book, either by a systematic study or just by a random walk through statements and exercises. Other books look like beautiful gardens, with entrance gates and large avenues; *Trigonometrical Series* is more like a forest full of strange things to be explored by means of a huge assortment of tricks and devices. Some of these devices are actually part of established theories, but it is only when they are needed that they appear in the book. For example, convex functions and inequalities are introduced in the chapter on classes of functions and Fourier series. Linear spaces and complete metric spaces are defined and developed only after the reader has encountered several examples, like the Riesz-Fischer theorem. The Riesz-Fischer theorem is also a motivation for the Rademacher functions and their use in randomizing trigonometric series (if the coefficients are not in ℓ^2, the randomized series fails almost surely to be a Fourier-Lebesgue series). The theory of Hardy spaces is presented as a tool for conjugate Fourier series. One chapter is entitled "Properties of some special series", and it contains a world of interesting objects and methods, like the "Van der Corput lemma", which is now constantly used for estimating trigonometric sums. But special series, in particular, lacunary trigonometric series studied by Szidon, Banach, and Zygmund himself, appear throughout the book. Convexity comes back in the form of the convexity theorem of Marcel Riesz in order to establish the $L^p - \ell^q$ substitutes of the Riesz-Fischer theorem, called Hausdorff-Young and F. Riesz's inequalities, through the new technique of interpolation of operators. Subtle relations between derivatives and integrals take place in different parts of the book, starting from variations on Lebesque's theorem that says an integrable function is equal to the derivative of its integral almost everywhere (applied to a test of convergence of Fourier series), continuing with the generalized derivatives of de la Vallée Poussin (applied to a summability problem), and ending with one of the actual origins of the theory, the Riemann theory of trigonometric series, which consists in studying the symmetric second derivative of the function obtained in a formal double integration of the given series. Here again, convexity comes back and plays a crucial role.

Trigonometrical Series is organized in twelve chapters:

1. Trigonometrical series and Fourier series.
2. Fourier coefficients. Tests for the convergence of Fourier series.
3. Summability of Fourier series.
4. Classes of functions and Fourier series.
5. Properties of some special series.
6. The absolute convergence of trigonometrical series.
7. Conjugate series and complex methods in the theory of Fourier series.
8. Divergence of Fourier series. Gibbs phenomenon.
9. Further theorems on Fourier coefficients. Integration of fractional order.
10. Further theorems on the summability and convergence of Fourier series.
11. Riemann's theory of trigonometrical series.
12. Fourier integrals.

Here are some excerpts from the short preface, dated January 1935:

"The theory of trigonometrical series of a single variable is very extensive and is developing rapidly every year, but the space devoted to it in the existing text-books is small...."

"The object of this treatise is to give an account of the present state of the theory...."

"Except for Lebesgue integration, an acquaintance with which is assumed, the book does not presuppose any special knowledge...."

"This book owes very much to Miss Mary L. Cartwright...and Dr. S. Saks...."

These names merit a few comments.

M.L. Cartwright was one of the English continuators of Hardy and Littlewood. A. Zygmund knew her personally through his visit to Oxford and Cambridge in 1929–1930. It was on this occasion that he met and worked with R.E.A.C. Paley. Paley died in 1933, just before Zygmund wrote his book.

S. Saks wrote another bestseller of the Monografje Matematyczne, *Theory of the Integral*. Another excellent book of this series is due to the collaboration of S. Saks and A. Zygmund and was first written in Polish. It was translated into English in 1952 as *Analytic Functions*. Incidentally, Saks is the author of the proof of the Banach–Steinhaus theorem that is based on Baire's theorem. This is now the standard proof given in courses and textbooks—the role of Saks is mentioned in a footnote in *Trigonometrical Series*, p. 98. Saks died in 1942. Here is what Zygmund wrote in the preface to the English edition of their book:

"Stanislaw Saks was a man of moral as well as physical courage, of rare intelligence and wit. To his colleagues and pupils he was an inspiration not only as a mathematician but as a human being. In the period between the two world wars he exerted great influence upon a whole generation of Polish mathematicians in Warsaw and in Lwow. In November 1942, at the age of 45, Saks died in a Warsaw prison, victim of a policy of extermination."

The first lines of *Trigonometric Series* are a dedication: "to the memories of A. Rajchman and J. Marcinkiewicz, my teacher and my pupil." Both Rajchman and Marcinkiewicz died during World War II, the first as a Jew in a concentration camp and the second as a Polish officer on the Russian front "in circumstances not quite clear, time indeterminate, probably in the Spring of 1940," as Zygmund wrote in his introduction to the *Collected Papers of Marcinkiewicz*. The last paper of Zygmund (1987) is an obituary of Alexander Rajchman. About Josef Marcinkiewicz he wrote that "but for his premature death he would probably have been one of the most outstanding contemporary mathematicians."

If only because of the dedication of *Trigonometric Series*, it is justified to associate the names of Rajchman, Zygmund, and Marcinkiewicz. In 2000, The Institute of Mathematics of the Polish Academy of Sciences sponsored a Rajchman–Zygmund–Marcinkiewicz Symposium; it was a unique opportunity to appreciate the permanent influence of their ideas and work [4].

Trigonometric Series is by no means restricted to the work of Zygmund and his collaborators. Nevertheless, the flavor of the book is partly a result of these collaborations. Before 1935, Zygmund's main collaborators were Rajchman, Saks, and Paley. Between 1935 and 1941, almost all Zygmund's papers have Marcinkiewicz as coauthor. From 1945 to 1959 his closest friend and collaborator was Salem. His collaboration with A. Calderón began during this period and extended after 1959. Zygmund had a number of students in Chicago. Among them, Mary Weiss and Elias Stein wrote joint papers with him. The direct and personal influence of

Zygmund was carried by his former students and their own pupils, but extended also around the world. For young mathematicians, it was a significant honor to be introduced to Antoni Zygmund, whom they already knew as the author of either *Trigonometrical Series* or *Trigonometric Series*, depending on their age.

The exact title of the work published by Cambridge University Press in 1959 is *Trigonometric Series, Second Edition, Volumes I and II.* In the preface, Zygmund insisted that *Trigonometrical Series* was the first edition of the book: "The first edition of this book was written almost twenty five years ago. Since then the theory of trigonometric series has undergone considerable changes. It has always been one of the central parts of Analysis, but now we see its notions and methods appearing, in abstract form, in distant fields like the theory of groups, algebra, theory of numbers...." This preface is the only part of the book where Zygmund expresses his views and opinions on the subject; therefore, it deserves particular attention. First, he declares that he will not treat the abstract extensions but stay on the classical theory of Fourier series, "the meeting ground of the Real and Complex Variables." Then a few examples emphasize the role of this theory as "a source of new ideas for analysts during the last two centuries," namely, the general notion of a function, the definitions of integrals, and the theory of sets.

Concerning "the main problems of the present-day theory of trigonometric series," he considers the summability of Fourier series and the convergence at individual points as closed chapters. However, "as regards the convergence or divergence almost everywhere, much still remains to be done. For example, the problem of the existence of a continuous function with an everywhere divergent Fourier series is still open." He adds that "two other major problems of the theory also await their solution: ...the structure of the sets of uniqueness and the structure of the functions with absolutely convergent Fourier series."

Among other problems or domains, he mentions the behavior of trigonometric series on sets of positive measure, further developments of complex methods, and multiple Fourier series. He then compares the advancement of the theory and that of its applications to other areas of mathematics rather critically: He notes, for example, that "convergence in norm...bypasses earlier difficulties," which refers to the need to show some kind of pointwise convergence. He adds some advice: "More subtle results of the theory, however, if we look at them in proper perspective, can give far-reaching applications." As possible applications, he mentions partial differential equations of elliptic type and the boundary behavior of analytic functions of several complex variables. The rest of the preface describes the organization of the book and expresses thanks to a number of colleagues.

There was no third edition while Zygmund was still living. The book was reprinted in 1968 (for the first time with Volumes I and II combined) and again in 1977. Zygmund wrote a very short "note on the 1968 impression" and a few lines as a "note on the 1977 impression": "We have not attempted to deal with the remarkable transformation of perspective in the field of almost everywhere convergence of Fourier series which was brought about by Carleson through the proof of his celebrated theorem on almost everywhere convergence of Fourier Series of L^2-functions, a result subsequently extended by Hunt to $L^P(p > 1)$."

Fortunately the present "third edition" contains this preface and notes, together with the content of the "1977 impression". As I have already mentioned, this new edition is the 1977 version of the second edition, plus the foreword by Robert Fefferman. To give an idea of the content, I shall first indicate the general organization of the book and how it differs form the first edition. I shall then stroll through several topics according to my own taste.

As indicated in the preface to *Trigonometric Series*, important new material had to be added to the first edition. Instead of 12 chapters and 329 pages, the second edition has 17 chapters and 747 pages. The Paley–Littlewood theory, just mentioned in the first edition, needs two chapters for a complete exposition. Throughout the book, one finds important results by Marcinkiewicz or by Marcinkiewicz and Zygmund that were developed after the first edition. The same is true for results by Salem and by Salem and Zygmund. Not only Calderón's results but also unpublished ideas or proofs of known theorems by Calderón appear at several places. The bibliography contains about 350 references, most of which appeared in the period 1935–1959. There are even more references to work prior to 1935 than in the first edition. An index was added in 1959 and completed in 1968.

From a historical point of view, the main innovation is a series of notes at the end of each volume and a considerable extension of the "miscellaneous theorems and examples" given at the end of each chapter. The miscellaneous theorems and examples are stated in the form of problems to solve, with references instead of solutions. The notes give additional comments and references, and no reader should read a chapter without consulting the notes. They give a great amount of information in a very condensed format.

There is a change in the exposition from the first to second edition. Part of what I called tricks and devices and pieces of general theories is now organized in the first chapter, although many such items are still found in other chapters. Here are the titles of the chapters, inspired by but different from the titles of the first edition:

1. Trigonometric series and Fourier series, auxiliary results.
2. Fourier coefficients, elementary theorems on the convergence of $S(f)$ and $\tilde{S}(f)$.
3. Summability of Fourier series.
4. Classes of functions and Fourier series.
5. Special trigonometric series.
6. The absolute convergence of trigonometric series.
7. Complex methods in Fourier series.
8. Divergence of Fourier series.
9. Riemann's theory of trigonometric series.
10. Trigonometric interpolation.
11. Differentiation of series, generalized derivatives.
12. Interpolation of linear operations, more about Fourier coefficients.
13. Convergence and summability almost everywhere.
14. More about complex methods.
15. Applications of the Littlewood-Paley function to Fourier series.
16. Fourier integrals.
17. A topic in multiple Fourier series.

Volume 1 consists of Chapters 1 to 9; Volume II of Chapters 10 to 17.

I shall not try to describe the content of the book chapter by chapter. As far as it is possible, this was done in an excellent way by Raphael Salem in this *Bulletin* in 1960 [26] and also by Edwin Hewitt in the *Mathematical Reviews* (21 6498). I do, however, invite you on a promenade, an eclectic tour of *Trigonometric Series*.

Trigonometric series and their conjugates are just the real and imaginary parts of formal power series

$$\sum_{n=0}^{\infty} a_n z^n$$

where a_o is real and z is considered on the unit circle, $z = e^{it}$. They can be studied either by using purely real methods or by using Taylor series and analytic functions in the unit disc. Roughly speaking, the real-variable methods are mainly in Volume 1 (although they reappear constantly in Volume 2), and complex variables methods are mainly in Volume 2 (although they are already introduced in Chapter 7).

There are two main formulas (or points of view) in the theory of trigonometric series. The first says that a function is given by a trigonometric series. The second expresses the trigonometric coefficients when the function is given. We usually start from the second (harmonic analysis) viewpoint and then consider the first (harmonic synthesis). Synthesis was the point of view of Fourier and Dirichlet, and it is still the most important by far. It appears in the first chapters of the book and leads to a great variety of methods and results. It is, however, possible to start from the first and investigate trigonometric series as such and, in particular, functions that are given as sums of everywhere convergent trigonometric series. This kind of investigation began with Riemann, and it is the subject matter of Chapter 9.

Nothing prevents the reader from beginning with Chapter 9. It is elementary: no integration theory is needed, and complex variables are not used. On the other hand, it is ingenuous and difficult. When the coefficients tend to zero the double integration of the series gives a continuous function F that satisfies the condition $F(t+h)+F(t-h)-2F(t) = o(h)$ $(h \to 0)$ uniformly, which is a "smooth function" in the sense of Zygmund. When the trigonometric series converges, its sum is equal to the second symmetric derivative of F: it is Riemann's summation process. If the series converges to zero everywhere, then it is necessarily the null series: this is the uniqueness theorem of Cantor. The same conclusion holds when the convergence to zero is assumed on the complement of a closed countable set: this extension is due to Cantor and was the opportunity for him to introduce fundamental notions on sets of real numbers. All methods used in this chapter are purely real and anticipate some aspects of the theory of Schwartz distributions: integration, derivation, formal multiplication, and localization. The saga of sets of uniqueness, or U-sets, begins here. They are the sets of real numbers such that, if two trigonometric series converge and have the same sum on the complement of the set, they are the same. They have necessarily a vanishing Lebesgue measure, but that is not a sufficient condition (Menšov). The triadic Cantor set is a U-set (Rajchman). Countable unions of closed U-sets are U-sets (Bari, also Zygmund, though he hid his contribution and stated the result as "theorem of Nina Bari" (see [1], p. xxv)). Linear functions transform U-sets into U-sets (Marcinkiewicz-Zygmund). There are variations about U-sets. One can replace convergence by summability with respect to a method M; the corresponding U_M-sets were studied by Marcel Riesz. One can assume that the coefficients are $O(\epsilon_n)$, where $\epsilon = (\epsilon_n)$ is a given sequence tending to zero, and Zygmund proved that the corresponding $U(\epsilon)$ sets can have a positive Lebesgue measure whatever ϵ may be. The question asked in the book, whether there are $U(\epsilon)$-sets of full measure, was solved in a positive way by Kahane and Katznelson in 1973. Before leaving this chapter, let me point out that U-sets were the closest meeting point of Rajchman, Marcinkiewicz, and Zygmund. They were also the subject of a beautiful collaboration between Salem and Zygmund. Here is their theorem about Cantor sets with ratio of dissection $1/\theta$ instead of $1/3$ ($\theta > 2$): such a Cantor set is a U-set if and only if θ is an algebraic integer whose conjugates (other than θ itself) are inside the unit disk of the complex plane (in brief, $\theta \in S$). This is explained in Chapter 12. Important applications of the class S and simplifications of the proof of the Salem–Zygmund theorem were given later by Yves Meyer (see [13] and also [9], 1994, pp. 203-204).

U-sets for multiple trigonometric series (convergence meaning spherical convergence) became a field of investigation after the book was published. The first contribution came from A. Zygmund in 1972; then came work by B. Connes in 1976, J. Bourgain in 1996, and the whole story is told in two articles of M. Ash and C. Wang [17], [18].

Chapter 9, the Riemann theory, and the end of Chapter 12, the link with number theory, do not require any prerequisite. The same is also true for other chapters. Chapter 5, on special trigonometric series, can be read independently. It is a vast collection of examples, beginning with series whose coefficients decrease monotonically, then considering the case of coefficients with absolute values $n^{-\beta}$ and phases either $cn \log n$ (Hardy–Littlewood, Mary Weiss) or $n^{-\alpha}$ (Hardy) (a rather strange result is that the corresponding series is Fourier–Riemann for $\beta > 0$ and Fourier–Lebesgue for $\beta > \frac{1}{2}\alpha$). The sections that follow deal with lacunary series (Banach, Sidon, Zygmund), Riesz products of several kinds (F. Riesz, Salem), and Rademacher functions and their use in random trigonometric series (Paley–Zygmund, Salem–Zygmund). The final sections contain Ingham's method for series with small gaps and Salem's refinement of the Van der Corput lemma used to obtain continuous functions from L^2-functions with regularly decreasing coefficients by appropriate changes of the phases.

As a comment on this chapter, let me say that special trigonometric series deserve attention for many reasons. 1. The local behaviour of their sums is now investigated by new tools (wavelets) and raises interesting examples of "multifractal analysis" (Y. Meyer, S. Jaffard [23]). 2. Lacunary series, Riesz products, and random series were linked from the very beginning (say, the period 1920–1930), and one property of lacunary series became the definition of Sidon sets, a subject of great interest by itself [11], [21]. 3. Random methods were extended to the whole of analysis, with remarkable solutions of long-standing problems [12].

Chapter 6, about the absolute convergence of trigonometric series, deserves particular attention, if only because Zygmund considered the structure of their sums as an important open question. It contains two quite different parts, the first on sets and the second on functions. Some sets, in particular sets of positive Lebesgue measure, have the property that a trigonometric series converges absolutely everywhere as soon as it converges absolutely on the set. The sets with the opposite property are a kind of thin set named sets N. A way to obtain a set N is to use the theorem of Dirichlet on diophantine approximation. The subject was introduced by Denjoy and Lusin and developed by Salem, Erdös, and Marcinkiewicz. In particular, Salem established that if E is a set N, the Fourier coefficients of any probability measure carried by E have 1 as a limit point, and Zygmund asked in a note, p. 239, whether it is also a sufficient condition for E to be a set N. Actually this is the case, as proved by J. E. Björk ([3]; see also [9], 1994, pp. 205-206). The sets N are now called "weak Dirichlet" sets, while Dirichlet sets are defined by the condition that some infinite sequence of exponentials $\exp(int)$ converges to 1 uniformly on the set.

The structure of functions with an absolutely convergent Fourier series was studied by a number of authors, including Zygmund, from a descriptive point of view: some regularity conditions on the function (say, belonging to a Hölder class of order $\alpha > 1/2$ (S. Bernstein) or to a Hölder class of order $\alpha > 0$ together with the condition of bounded variation (A. Zygmund)) imply absolute convergence of the series. It is essentially the point of view expressed in the book. Boundedness suffices if the series is lacunary (Sidon). Analytic functions operate; that is, an analytic function of a continuous function with an absolutely convergent Fourier series is again a function with the same property (Wiener–Lévy). This theorem has several proofs, and the proof given in the book, due to A. Calderón, is among the best.

Although the statements and the proofs are perfect, this part of the book was already criticized in 1960 by Salem in his review [26]. The reason is that another structure deserves attention, namely the structure of the class of functions with an absolutely convergent Fourier series, called A by Wiener, as a normed ring or Banach algebra. It is the point of view suggested by the works of Wiener, Beurling, and Gelfand. The Wiener–Lévy theorem derives from the fact that the spectrum of A (the set of maximal ideals) is the circle (a maximal ideal is the set of functions vanishing at some point). The converse of the Wiener-Lévy theorem was given by Y. Katznelson in 1958: only analytic functions operate on A. The solution of the long-standing problem on "spectral synthesis" (a closed ideal is not necessarily the set of functions vanishing on some closed subset of the circle) was obtained by P. Malliavin in 1959. Both results could have been included in the text or at least mentioned in the notes.

I do not agree completely with this criticism. The scope of the book was very large already, and a continued enlargement was not desirable. Fortunately, Zygmund left some room for books by other people (see [5] to [16] as examples). However, notes on this subject in *Trigonometric Series* would have been welcome.

Although the main emphasis was on real and complex analysis, functional analysis was already present in *Trigonometrical Series*. Banach spaces methods and the interpolation of operators did appear in the first edition. In the second edition, interpolation of operators is the main subject of a new chapter, Chapter 12. It contains two main interpolation theorems: Riesz–Thorin (with an extension by E. Stein) and Marcinkiewicz. Both deal with operators from one measure space to another and express that they map L^α into L^β when the point $(1/\alpha, 1/\beta)$ belongs to some convex set. The Marcel Riesz interpolation theorem assumes that the operator is linear and continuous from L^{α_0} to L^{β_0} and from L^{α_1} to L^{β_1} with $0 \le \beta_0 \le \alpha_0 \le 1$ and $0 \le \beta_1 \le \alpha_1 \le 1$ and concludes that the same holds from L^α to L^β whenever $(1/\alpha, 1/\beta)$ belongs to the segment of line joining $(1/\alpha_0, 1/\beta_0)$ and $(1/\alpha_1, 1/\beta_1)$. This has remarkable consequences when the operation is the Fourier transformation, either from the circle $\mathbb{T}$ to $\mathbb{Z}$, the set of all integers, or from $\mathbb{T}$ to $\mathbb{Z}$, namely the Hausdorff–Young inequalities. The Riesz–Thorin theorem assumes only that the α's and β's are between 0 and 1, and the main contribution of Thorin was to introduce methods of vector-valued analytic functions in the proof. In the theorem of Marcinkiewicz the linearity condition is relaxed and replaced by "quasi-linearity", and the behaviour at (α_0, β_0) and (α_1, β_1) is expressed as a "weak type" condition. An important consequence is the series of theorems by Paley involving L^p-norms with weights and rearrangements of function or sequences. In the course of the chapter, we also see interpolation of multilinear operators with applications to H^p spaces (starting from an inequality of Hardy–Littlewood on Taylor series established in Chapter 7) and fractional integration and derivation. As I already said, the end of Chapter 12 is devoted to Cantor-like sets and the algebraic theory of numbers.

Every chapter has a different flavour. Instead of describing their content, I shall pick a few flowers.

The strong version of the Lebesgue derivation theorem is given in Chapter 2, in the form $\int_0^h |f(x+t) - f(x)|dt = o(h)$ a.e. (I, p. 65). Given a perfect set P on the line, its points of density are the Lebesgue points of its indicator function. A theorem by Marcinkiewicz in Chapter 4 is more precise: it says that

$$\int (\mathrm{dist}(t, P))^{\lambda} |t - x|^{-\lambda - 1} dt < \infty \qquad (\lambda > 0)$$

almost everywhere on P (I, p. 129). Difficult and beautiful results are given in Chapter 11 on generalized derivatives. Here is an example, established by Marcinkiewicz and Zygmund: if $f(x+h) + f(x-h) - 2f(x) = O(h^2)$ $(h \to 0)$ whenever $x \in E$, then the second Peano derivative exists a.e. on E (meaning $f(x+h) = a_0(x) + ha_1(x) + h^2 a_2(x) + o(h^2)$ $(h \to 0)$ a.e. on E) (II, p. 78). Actually, Chapters 2 and 11 are purely real analysis: derivations, integrations (not only Lebesgue, but also Denjoy), and their relations.

In the theory of numerical series, the Abel summation theorem says that $\lim_{r \uparrow 1} \sum_0^\infty a_n r^n = \sum_0^\infty a_n$ whenever the series converges. If the left-hand member exists and $a_n = o(1/n)$ $(n \to \infty)$, then the series on the right-hand side converge. This is Tauber's theorem, the ancestor of all "Tauberian" theorems. The Tauberian theorem of Littlewood says that $a_n = O(1/n)$ suffices, and it is, by far, more difficult than the theorem of Tauber. A further extension was given by Hardy and Littlewood: the condition $a_n = O(1/n)$ can be replaced by the one sided condition $a_n \leq A/n$. According to Hardy and Littlewood, their theorem "constitutes a very interesting extension of Littlewood's generalization of Tauber's theorem; but a special proof is required," and the proof is by no means simple ([2], VI, pp. 524-525). Chapter 3 gives a very clear proof of Littlewood's theorem on one page (I, p. 82), and ten lines in the notes on Chapter 3 suffice for deriving the Hardy–Littlewood theorem.

Summability comes back in Chapter 13, but now for trigonometric or Taylor series, and it is considered almost everywhere on the circle. I shall select two topics from this chapter: the "circular structure" for partial sums of Taylor series, and the "strong summability" for Fourier series and their conjugates.

Marcinkiewicz and Zygmund were interested in partial sums of a trigonometric series S and its conjugate $\widetilde{S}$ on a set E where S is summable $(C, 1)$ (the process of Fejér), and they observed that the length of the smallest interval containing all limit points of these partial sums was the same, a.e. on E, for S and $\widetilde{S}$. This looked mysterious, until they discovered the "circular structure" for Taylor series : the fact is that the set of accumulation points of the partial sums of $S + i\widetilde{S}$ is invariant under rotation around the $(C, 1)$-limit of this series, a.e. on E. The proof is quite neat, but the distribution of these partial sums is still an open question (II, p. 178).

The "strong summability" (H_q), $q > 0$, is defined by the condition

$$|s_0 - s|^q + |s_1 - s|^q + \cdots + |s_n - s|^q = o(n) \quad n \to \infty.$$

Hardy and Littlewood, extending Fejér's theorem, proved that (H_q) holds uniformly for Fourier series of continuous functions (II, p. 182). Moreover, (H_q) holds everywhere for S and $\widetilde{S}$ as soon as S is a Fourier-Lebesgue series (this is due to Marcinkiewicz for $q = 2$ and to Zygmund for any q) (II, p. 184).

The relation between S and $\widetilde{S}$, a trigonometric series and its conjugate, is a permanent subject of interaction between real and complex methods. Chapter 4 develops the real-variable approach, Hilbert transform, and a real method for Kolmogorov's theorem ($S \to \widetilde{S}$ is of weak type L^1, a basic result used in Chapter 12) (I, pp. 131-135). The original proof by Kolmogorov used complex variables methods, as indicated in the notes (I, p. 378). The complex methods for the properties of the mapping $S \to \widetilde{S}$ (Marcel Riesz, Kolmogorov, Zygmund) are described in Chapter 7. This leads to the classes H^p (Hardy) and N (Nevanlinna), and also to the Taylor series whose sum has bounded variation on the unit circle (F. and M. Riesz: the sum is absolutely continuous; Hardy and Littlewood: the series converges absolutely). A pearl is the theorem of Helson: if the partial sums of a trigonometric series are positive, the coefficients tend to zero (I, p. 286).

There is more about complex methods in Chapter 14. The starting point is the Privalov theory of nontangential limits, leading to a uniqueness theorem (II, p. 203): if two analytic functions in the open disk $|z| < 1$ have the same nontangential limit in a set of positive measure on $|z| = 1$, then they are the same. "Nontangential" refers to the behaviour inside *all* small triangles inside $|z| < 1$ with vertex at a given z, $|z| = 1$. The key lemma for Privalov is now the main theorem (II, p. 199): if a function harmonic in $|z| < 1$ satisfies a condition of local boundedness ("condition B") on a set E in $|z| = 1$, then it has a nontangential limit at almost every point on E. Here, "local boundedness" refers to the behaviour inside *some* small triangle. Privalov's proof uses conformal mapping, explained in a figure on p. 200 (*the* figure of the book). Zygmund also gives a proof by Calderón, with possible extensions to several dimensions. Next emerges the zoo of the "square-functions", with their standard names: $s(\theta), g(\theta), \mu(\theta)$. The first, $s(\theta)$, is the Lusin function: $s^2(\theta)$ is the integral of $|F'(z)|^2$ over the "triangle" defined as the convex set generated by the point $e^{i\theta}$ and some circle around the origin, therefore also the area of the image of this "triangle" by F; it is called "the area function". A remarkable theorem, to which Marcinkiewicz, Zygmund, Spencer, and Calderón contributed, says that the existence of a nontangential limit and the finiteness of the area function are equivalent almost everywhere (II, p. 207). The function $g(\theta)$, the function of Littlewood–Paley, is defined as

$$g(\theta) = \left(\int_0^1 (1-r)|F'(re^{i\theta})|^2 dr \right)^{1/2},$$

and it is a close companion of the area function. It enjoys the property that the L^p norms of $g(\theta)$ and $F(e^{i\theta})$ are equivalent when $p > 1$. The Marcinkiewicz function

$$\mu(\theta) = \left(\int_0^\pi |F(\theta+t) + F(\theta-t) - 2F(\theta)|^2 t^{-3} dt \right)^{1/2}$$

has a purely real definition, and it is attached to a 2π-periodic function f, whose F is the indefinite integral. Then again $\int |\mu|^p$ and $\int |f|^p$ are equivalent when $p > 1$.

The Littlewood–Paley theory begins in Chapter 14 and is developed in Chapter 15. The most significant part is the study of the "Littlewood–Paley decomposition" of a trigonometric series, that is, the decomposition into dyadic blocks of the form $\sum \Delta_k$, where

$$\Delta_k = \sum_{2^k < n \le 2^{k+1}} (a_n \cos nt + b_n \sin nt)\,.$$

A striking result is that $\sum \pm\Delta_k$ is the Fourier series of an L^p-function $(1 < p < \infty)$ as soon as it is the case for $\sum \Delta_k$. This can be expressed in many ways (in particular by considering the "square function" $(\sum \Delta_k^2)^{1/2}$) and has many consequences. I shall return to the Littlewood–Paley theory at the end of the promenade.

S and $\widetilde{S}$, a trigonometric series and its conjugate, appear from the beginning to the end of *Trigonometric Series*: Chapter 2 sets the definitions, Chapter 4 explains the Kolmogorov weak L^1 theorem using real variables methods, and Chapter 7 gives a series of functional properties of the mapping $S \to \widetilde{S}$ using complex methods. Chapters 11 and 13 establish the link between generalized derivatives and summability of S and $\widetilde{S}$ and introduce strong summability for both series. The final pearl is to be found in Chapter 14: whatever S may be, the sets on which S and $\widetilde{S}$ converge differ only by a set of Lebesgue measure zero (II, p. 216; a weaker version is given on p. 175).

Let me end this promenade with what was the most tantalizing problem for Zygmund: the convergence problem. It has two faces: convergence and divergence. Roughly speaking, the part of the book devoted to divergence has a permanent value, and the part devoted to almost everywhere convergence is now obsolete.

The convergence problem began with Fourier: given a function, check that it is the sum of its Fourier series. Fourier did it in a particular case and claimed that it could be done for an arbitrary function. This is not true as a statement, but has proved very valuable as a program. The main part of Zygmund's book is devoted to this program. First, one has to be more specific about the functions under consideration, and this depends of the notion of an integral. This observation was already made in 1829 by Dirichlet, and it leads to the introduction of classes of functions, or generalized functions, like measures. Then, one has to be more specific about convergence. The original question is about pointwise convergence and leads to a series of sufficient conditions or "tests" (Dirichlet, Jordan, Dini, and others). Since pointwise convergence can fail even for continuous functions (du Bois Reymond, Fejér, Lebesgue), two variations deserve to be considered: convergence almost everywhere and convergence in functional spaces, like L^2. Moreover, simpler statements appear when convergence is replaced by summability. For example, it is possible to begin a course on Fourier series with Fejér's theorem in functional spaces, as does Katznelson [10].

For Zygmund, as explained in his preface, the main problem was about convergence almost everywhere. It is the underlying theme of Chapters 13 to 15. The question arises for L^2 first. In 1959, these were the available results: 1. $s_n = o(\sqrt{\log n})$ a.e. 2. If $\sum(a_n^2 + b_n^2)\log n < \infty$, then s_n converges to f a.e. (Kolmogorov-Seliverstov). 3. If $n_{k+1}/n_k > q > 1$ $(k = 1, 2, \ldots)$, then s_{n_k} converges to f a.e. This last result, on a lacunary sequence of partial sums, derives easily from the corresponding result on Fejér's sums, via a decomposition between n_k and n_{k+1}. In order to extend the result to L^p $(1 < p < 2)$ instead of L^2, the

Littlewood–Paley decomposition was needed, and it was the original motivation of this difficult theory. Chapter 15 was written from this point of view: "one of the main results of this chapter is that the theorem holds for $f \in L^p$, $p > 1$."

The Carleson–Hunt theorem changed the perspective completely. It was a shock for Zygmund. How could it be that such a beautiful theory becomes of no use for its original purpose? What should be dropped from the book and what added? In 1966 Zygmund was involved in other questions. His decision was wise: there should not be a third edition of his book.

However, another change of perspective appeared at the same time: the Littlewood–Paley theory became much more important than the convergence problem for Fourier series. Its probabilistic version, established by Paley before Littlewood wrote their joint work, is now basic in the theory of martingales. The Littlewood–Paley decomposition appears in all parts of analysis: partial differential equations, wavelets, diffusions. Chapter 14, on the "square-functions", remains highly valuable even if the main applications are not those described in Chapter 15. A good motivation for the present "third edition" is precisely to show how interesting the exposition of the Littlewood–Paley theory by Zygmund still is.

The Carleson–Hunt theorem was extended to functions f such that

$$\int |f(t)| \log^+ |f(t)| \log^+ \log^+ \log^+ |f(t)| dt < \infty$$

(Antonov, 1966 [17]). It cannot be extended to $f \in L^1$, since there exists an integrable function whose Fourier series diverges everywhere (Kolmogorov, 1926; see Chapter 8 in *Trigonometric Series*). The situation is not yet clarified for the partial sums s_n of functions in L^1. It is known (Hardy, 1913; see Chapter II) that $s_n = o(\log n)$ a.e., and Hardy conjectured that it was a best possible result: "I have not proved rigorously that it is so, but it seems to me very probable" ([2], III, p. 124). Zygmund asked the question again (I, p. 308): "We know (Chapter II, (II.9)) that for any $f \in L$ we have $S_n(x; f) = o(\log n)$ for almost all x. It is conceivable that this result is best possible, that is, for any sequence of positive numbers $\lambda_n = o(\log n)$ there is an $f \in L$ such that at almost every point x we have $S_n(x; f) > \lambda_n$ for infinitely many n." For a long time the best answer in that direction was $\lambda_n = o(\log \log n)$. The best answer now is $\lambda_n = o(\sqrt{\log n}/\sqrt{\log \log n})$ (Konyagin, 2000 [24]). For Walsh series $\lambda_n = o(\sqrt{\log n})$ suffices [20]. The challenge is between $\sqrt{\log n}$ and $\log n$.

The estimate $s_n(t) = o(\log n)$ appears in another context, namely at a given point t for a continuous function f. Here it was already known by Lebesgue that it is a best possible estimate ([25], p. 117, I, p. 298). There is no way to solve Hardy's problem using Lebesgue's method. This warning is suggested by a wrong comment on Hardy's conjecture in ([2], III, p. 125).

Chapter 10, on trigonometric interpolation, also contains many interesting results on divergence of interpolating trigonometric polynomials of degree n with equidistant nodes, $I_n(t, f)$, when f is continuous. It may happen that $I_n(t, f)$ diverges almost everywhere even when $S_n(t, f)$ converges uniformly (Grünwald) (II, p. 40). It may even happen that $I_n(t, f)$ diverges everywhere (Marcinkiewicz) (II, p. 44). The estimate $I_n(t, f) = o(\log n)$ holds for every t, and given $\lambda_n = o(\log n)$ there exists a continuous function f such that at almost every t we have $I_n(t, f) > \lambda_n$ for infinitely many n (II, p. 42). Here the divergence problem seems to be a closed subject.

Although I already announced the end of the promenade, let me go back to the last chapters of the book, those on Fourier integrals (Chapter 16) and multiple Fourier series (Chapter 17).

There were already excellent books on Fourier integrals. The originality of Zygmund's approach is to start from Fourier series and extend their theory. The first formulas written in Chapter 16 are

$$\begin{aligned} S_\omega(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\frac{\sin\omega t}{t}dt\,, \\ \widetilde{S}_\omega(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\frac{1-\cos\omega t}{t}dt\,, \end{aligned}$$

expressing the partial sums and conjugate partial sums when f is periodic. The aim is explained on p. 246: to investigate the representability of functions by their Fourier integrals, single or repeated, and to prove that, locally at least, the problem is reduced to that of the representability of periodic functions by their Fourier series. Among all extensions of notions or results on Fourier series to Fourier integrals, let me give one example: the interpretation of the notion of U-set using Fourier integrals is the only way to prove that linear functions transform U-sets into U-sets.

The very last chapter has a different character. It is the introduction to a new topic. For multiple Fourier series the notion of partial sums disappears, since there is no natural order on the terms. "Spherical" or "rectangular" partial sums correspond to different methods of summation. Strong differentiability of multiple integrals, power series in several variables, and harmonic functions in a polydisk introduce a series of new questions. The references in II, p. 355, show that a series of works by Calderón and Zygmund are related to these questions. Their paper "On singular integrals" was published in 1952. While Zygmund prepared the second edition of his book, he was already involved in the theory of Calderón–Zygmund operators.

And now let me go back to the beginning of this review. It was an excellent decision not to change a single word of *Trigonometric Series* as it was published in 1977 for this "third edition". This book has to be considered as a piece of art as well as a source of information. It has a global structure, but its beauty has to be discovered in every chapter, page, or sentence. Everything is clear, condensed, and complete. Zygmund had very broad views, and they have a strong historical interest. Now, however, we may and should have other interests and perspectives, as shown in particular by all the books of Elias Stein and collaborators. Antoni Zygmund had also a personality as a human being: open to all problems of the world, warm with his friends, pleasant with his colleagues, and good to his students, and at the same time absolutely rigorous in all aspects of life. The style of his book, pure and simple, mirrors this personality. I am sure that it will remain as a model forever.

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