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Finite structures with few types, by Gregory Cherlin and Ehud Hrushovki, Annals of Math Studies, Princeton University Press, Princeton, NJ, 2003, vi + 193 pp., \$49.95, ISBN 0-691-11331-9

This book explores the deep connections, discovered in the last half century, between model theory and the study of group actions. These connections arose from some extremely basic questions about the expressive powers of formal languages and developed into a classification scheme for certain families of finite structures.

Logic is that branch of mathematics that takes care to specify the formal language (or vocabulary) for the study of a structure. A vocabulary τ is a finite or countable set of relation symbols with various finite numbers of arguments. A τ-structure is a set A along with an interpretation of each n-ary relation symbol as a subset of A^n. A first order sentence is an expression built up from these basic relations by Boolean operations and quantification over individuals. Truth of a sentence ϕ in any τ-structure is naturally defined.

What is the correct formal vocabulary to describe a given mathematical situation, e.g. vector spaces over a finite non-prime field? That is, what is the automorphism group of a vector space over a finite non-prime field? Usually, this group is taken as the general linear group of the appropriate dimension. But for some purposes the automorphisms of the field must be considered. The usual choice is taken by including unary functions for scalar multiplication in the vocabulary of vector spaces; the more complicated formalization of the other case involves binary functions for scalar multiplication. This distinction plays a background role here ([3], page 56).

This book deals with a certain class of $\aleph_0$-categorical structures: The Lowenheim-Skolem theorem asserts that any sentence with an infinite model has one in every infinite cardinality. A theory is a possibly infinite collection of sentences. A theory T is κ-categorical if all models of T with cardinality κ are isomorphic. A structure is $\aleph_0$-categorical if and only if the set of sentences true in it forms an $\aleph_0$-categorical theory. For example, the theory of dense linear order without endpoints has exactly one countable model – the rational order – so is $\aleph_0$-categorical. Similarly, the theory of those infinite Abelian groups such that every element has order 2 (more generally p^n for fixed prime p and exponent n) is $\aleph_0$-categorical. These examples differ in two important ways: the first is not categorical in any cardinal except $\aleph_0$ and is described by a single first order sentence. The second requires infinitely many axioms to insist the universe is infinite; it is categorical in all cardinalities. The analysis here shows these are fundamental distinctions.

The connections between model theory and the study of group actions are first seen in the theorem proved independently by Engeler, Ryll-Nardzewski and Svenonius in 1959. Permutation group theorists say a countable structure M is oligomorphic ([2]) if the automorphism group of M has only finitely many orbits of n-tuples for each n. The theorem asserts that M is $\aleph_0$-categorical if and only if

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it is oligomorphic. Hidden in this result is the fact that an $\aleph_0$-categorical structure is ω-homogeneous; i.e. two finite sequences which satisfy the same formulas without parameters are in the same orbit. (Model theorists say two k-tuples which satisfy the same formulas have the same k-type.) From a permutation group standpoint, the orbits of the group action are the natural objects. They provide a 'canonical language': include an n-ary relation symbol for each orbit of n-tuples. The model theorist asks, 'When does this canonical language have a finite basis?" $\aleph_0$-categoricity does not suffice [10]; sufficient conditions are described below.

Rosenstein [11] showed that for each n there is a structure which has finitely many k-types only for $k < n$. But consider the following more uniform condition: M is k-quasifinite if there is an integer m such that for any sentence ϕ true in M, there is a finite model N of ϕ which realizes at most m, k-types. Now one of the conclusions of the book is that if M is 4-quasifinite, it is $\aleph_0$-categorical and k-quasifinite for all k.

Much of the model theoretic interest in this area arose from two problems. One was simply stated by Morley [9]: Can a first order sentence have exactly one model in every infinite cardinality? The negative answer was obtained in the early 80's by Zilber [12] and Cherlin, Harrington, and Lachlan [4]. Key to the analysis was the proof that a certain building block of such a structure – a strictly minimal set– has the structure of a geometry. (This result can be deduced from the classification of two-transitive groups (Cherlin and Mills); slightly later (although earlier partial proofs were known) Zilber and Evans gave direct arguments.) The second, somewhat vaguer, motivation was Lachlan's program to classify stable finitely homogeneous structures. Lachlan introduced the following notions. The finite structure N is k-homogeneous in M if all definable (without parameters) relations on M induce definable relations (without parameters) on N and each pair of k-tuples in N has the same type in N if and only if they do so in M. The structure M is smoothly approximable if it is $\aleph_0$-categorical and every finite subset of M is contained in a finite $|N|$-homogeneous substructure N. Smooth approximation guarantees that every finite subset of M is contained in a finite 'envelope' that witnesses that every sentence true in M is true in a finite homogeneous submodel. Thus the proof in [4] that an $\aleph_0$-categorical $\aleph_0$-stable model is smoothly approximable provides a very strong answer to Morley's question.

The present book carries out Lachlan's program [8]: classify the smoothly approximable structures. The solution is a collection of six equivalent conditions. One is 'strongly 4-quasifinite' (even stronger than the condition summarized above). In a different direction M is smoothly approximable if and only if it is Lie coordinatizable. There are two components to this notion. Roughly, it means that the structure can be constructed in a nice way from a list of finite geometries. The geometries are: a pure set, a pure vector space, a polar space, an inner product space, an orthogonal space, and a quadratic geometry. The coordinatization is a rather technical process, foreshadowed by Shelah's structure theory for models of stable theories and by the analysis of models of totally categorical theories, of decomposing the model as a tree of geometries. The authors show (by induction on the complexity of the decomposition) that any Lie coordinatizable structure can be presented in a finite language and that the structure is model complete in that language. Higman's theorem is a crucial tool here. The proof ([4], [12]) that there are no totally categorical sentences raised the question of whether every totally categorical structure could be axiomatized by a single sentence plus a schema asserting the universe is infinite. This conjecture was affirmed in some cases by Ahlbrandt-Ziegler [1] and for $\aleph_0$-categorical ω-stable structures by Hrushovski [5]. The result is extended to Lie coordinatizable (i.e. smoothly approximable) structures in the present book. Namely, any Lie coordinatizable structure is determined by the cardinality of each of a finite set of dimension invariants. Although these dimensions are infinite in the given model, the model induces a class of finite structures each determined by an appropriate finite sequence of dimensions. Thus, the book's title is justified by viewing the classification of smoothly approximable structures as the classification of certain classes of finite structures. In particular, for any finite vocabulary L and natural number k, the collection of finite L-structures which realize less than k 4-types can be effectively partitioned into a finite number of classes each of which can be axiomatized (in extensions of first order logic by some generalized quantifiers).

The last two paragraphs made a subtle switch from Lachlan's program to classify stable finitely homogenous structures to the classification of smoothly approximable structures. This extends the analysis to a properly larger class. The work done by Cherlin and Hrushovski (following Kantor, Liebeck, and MacPherson [6] for the primitive case) was one influence on the recognition that stability theory could be fruitfully generalized to simplicity theory [7]. With rather gross inaccuracy, a structure is stable if it imbeds neither a linear order nor a random graph; simple structures are the best-behaved structures among those that do not imbed a linear order. In the last 10 years, it has been discovered that the independence theory which is the hallmark of a stable theory extends well to a simple theory. The analysis in this book is both one of the origins of this insight and the most delicately worked out exemplar of it. Consider the 'amalgamation of types theorem' (as it is called here; more frequently the name is the 'independence theorem'). Over models, the independence theorem characterizes simple theories; in the more special situation here it holds over algebraically closed sets. The conjectured extension of this result to arbitrary simple theories is one of the main problems of simplicity theory.

We have given an account of some of the motivations for the study in this book and a few of the major consequences. This says little about the actual content. Written in a 'take no prisoners style', which may be needed to compress such a detailed analysis to several hundred pages, stability theoretic techniques of rank and the orthogonality calculus are combined with the permutation group technology and a bit of cohomology to list all structures satisfying certain fairly simple conditions. At one stage, [3], there was a clear division of labor: permutation group theory handled the primitive case; the pasting together of the primitive components was model theory. This distinction became blurred in the final version. From this analysis, results of many kinds can be read off. One last example: there is an effective procedure to decide whether a first order sentence in vocabulary τ has a stable homogeneous model (i.e. axiomatizes an $\aleph_0$-categorical theory with elimination of quantifiers in τ).

The book at hand provides a deep and penetrating analysis of a family of structures that answers many questions from model theory and finite model theory, permutation group theory and combinatorics. What remains to be done? The following basic question regarding $\aleph_0$-categorical structures remains open. We have essentially two easily understood axioms of infinity. One says each element has a successor and so clearly cannot be $\aleph_0$-categorical. The other is dense linear order. Are there any other $\aleph_0$-categorical axioms of infinity? In more precise terms, must every finitely axiomatizable $\aleph_0$-categorical theory with no finite models have the strict order property?

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