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Discrete convex analysis, by Kazuo Murota, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003, xxii+389 pp., \$111.00, ISBN 0-89871-540-7

The author writes in the preface: "*Discrete Convex Analysis* is aimed at establishing a novel theoretical framework for solvable discrete optimization problems by means of a combination of the ideas in continuous optimization and combinatorial optimization." Thus the reader may conclude that the book presents a new theory (the name "discrete convex analysis" was, apparently, coined by the author). The reader may also notice the adjective "solvable" attached to "discrete optimization problems" and hence ask whether "solvable" means solvable in principle, solvable by the new theory, solvable by all other known approaches, or solvable by some of the known approaches for which the theory provides a unified framework. I think that the best fit is given by the last option. Thus, I would like to describe the general area of discrete convexity and the contribution of the book.

A set $A \subset \mathbb{R}^n$ is called *convex* if, for any two points $x, y \in A$, the interval $[x, y] = \{\alpha x + (1-\alpha)y : \ 0 \leq \alpha \leq 1\}$ also lies in A. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called *convex* provided that for every $\lambda \in \mathbb{R}$, the set $\{x : \ f(x) \leq \lambda\}$ is convex. Equivalently, f is convex if and only if $f\big(\alpha x + (1-\alpha)y\big) \leq \alpha f(x) + (1-\alpha)f(y)$ for all $x, y \in \mathbb{R}^n$ and all $0 \leq \alpha \leq 1$. These remarkably simple definitions lead to a remarkably rich and useful theory with a great many applications. Here we are interested in optimization problems with convex objective functions.

Convex functions and convex sets have some nice properties as far as optimization is concerned. A local minimum of a convex function f on a convex set A is necessarily a global minimum. There is also a powerful *duality theory*, which we sketch below, having optimization in mind.

Let $\langle \cdot, \cdot \rangle$ be the standard scalar product in $\mathbb{R}^n$. For a non-empty set $A \subset \mathbb{R}^n$, let $A^\circ = \{c \in \mathbb{R}^n : \ \langle c, x \rangle \leq 1 \text{ for all } x \in A\}$ be the *polar* of A. Then, A° is a closed convex set containing the origin, and if A is itself a closed convex set containing the origin, then $(A^\circ)^\circ = A$ (the Bipolar Theorem). For a set $A \subset \mathbb{R}^n$, let $[A] : \mathbb{R}^n \longrightarrow \mathbb{R}$ be its indicator: $[A](x) = 1$ if $x \in A$ and $[A](x) = 0$ if $x \notin A$. The polarity correspondence $A \longmapsto A^\circ$ preserves linear dependencies among indicators of closed convex sets: if $\sum_{i=1}^m \alpha_i [A_i] = 0$ for real numbers α_i and non-empty closed convex sets A_i, then $\sum_{i=1}^m \alpha_i [A_i^\circ] = 0$.

With every set $A \subset \mathbb{R}^n$, we associate two problems. First, the *Membership Problem*: given a point $x \in \mathbb{R}^n$, decide whether $x \in A$. Second, the *Optimization Problem*: given a vector $c \in \mathbb{R}^n$, compute the minimum (maximum) value of the linear function $\langle c, x \rangle$ for $x \in A$. The polarity correspondence $A \longleftrightarrow A^\circ$ naturally gives rise to the correspondence between the Optimization Problem for A and the Membership Problem for A°. The Bipolar Theorem implies a certain symmetry between the Membership and Optimization problems.

Suppose now that a convex body (a convex compact set with a non-empty interior) $A \subset \mathbb{R}^n$ is *defined* by its *Membership Oracle*, that is, a black box which, given

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a point $x \in \mathbb{R}^n$, tells us whether $x \in A$. Then one can solve the Optimization Problem on A using a reasonably small number of arithmetic operations and calls to the Membership Oracle. To make the statement precise, one needs to introduce some numerical guarantees, such as an upper bound R on the radius of a ball containing A, a point $a \in A$ and a lower bound $r > 0$ on the radius of a ball centered at a and contained in A, and an error $\epsilon > 0$ with which we perform computations. Then the Optimization Problem on A can be solved in time polynomial in n, $\log R$, $\log r^{-1}$, and $\log \epsilon^{-1}$. This relation between the Membership and Optimization Problems for convex bodies is far from trivial: it is a corollary of the powerful Ellipsoid Method; see [GLS93].

A similar theory can be built for convex functions $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$. The role of the polar is played by the convex conjugate, a.k.a the Legendre-Fenchel transform $f^\circ : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by $f^\circ(c) = \sup\limits_{x \in \mathbb{R}^n} \Big(\langle c, x \rangle - f(x) \Big)$.

Various *discrete* versions of convexity have been of interest since long ago. Let $\mathbb{Z}^n \subset \mathbb{R}^n$ be the standard integer lattice. For a convex set $A \subset \mathbb{R}^n$, let $A_{\mathbb{Z}} = A \cap \mathbb{Z}^n$ be the set of integer points in A. Wouldn't it be nice to have as rich a theory of "discrete convex sets" $A_{\mathbb{Z}}$ as we have of "continuous" convex sets A ? Wouldn't it be nice to have a similar equivalence of the local and global optimality (for some reasonably defined neighborhood structure) and a useful duality theory? Of course it would be nice, and, in some sense, such a theory would be too good to exist. One argument why a comprehensive theory of discrete convexity parallel to that of continuous convexity is unlikely to exist comes from the computational complexity side of things. Here is an example of a discrete convex set X or, rather, a family of discrete convex sets X_n, where the Membership Problem (the problem of testing whether a given point lies in X) is trivial, whereas the Optimization Problem (the problem of optimizing a given linear function on X) is quite hard. Let us choose an integer $n \geq 3$. For a permutation σ of the set $\{1, \ldots, n\}$, let a^σ be the $n \times n$ permutation matrix, so that $a^\sigma_{ij} = 1$ if $\sigma(i) = j$ and $a^\sigma_{ij} = 0$ if $\sigma(i) \neq j$. Finally, let $X_n \subset \mathbb{Z}^{n \times n}$ be the set of matrices a^σ, where σ ranges over all $(n-1)!$ permutations consisting of a single cycle $1 \mapsto i_1 \mapsto \ldots \mapsto i_{n-1} \mapsto 1$. Clearly, X_n is the set of integer points in a convex set. The Membership Problem for X_n is trivial, while the Optimization Problem, called the Traveling Salesman Problem, is a classical example of an NP-hard problem ("NP-hard" just means "hard" in this context).

The convex hull of X_n is called the *Traveling Salesman Polytope*; it has a complicated facial structure, which is not well understood, and unless something unexpected happens in the computational complexity theory (unless *NP=co-NP*, for example), *cannot* be understood too well; cf. Chapter 58 of [Sc03]. Thus, Papadimitriou [P78] showed that it is an NP-hard problem to determine whether two given points in X_n are the endpoints of some edge of the Traveling Salesman Polytope, which indicates that there is no simple natural neighborhood structure on X_n.

Although an all-powerful discrete convexity theory is unlikely to exist, some general arguments geared towards a particular set of problems in discrete convexity have been used. We start with some examples, which are not discussed in the book under review.

Perhaps the first example of a discrete convexity argument goes back to Minkowski (cf. Section V.8.2 of [C97]) and proceeds as follows. If $A \subset \mathbb{R}^n$ is a convex set, then for every two points $x, y \in A$ the midpoint $(x+y)/2$ also lies in A. If $S \subset A \cap \mathbb{Z}^n$ is a set of at least $2^n + 1$ integer points in A, then some two points $x, y \in S$, $x \neq y$, lie in the same coset $\mathbb{Z}^n/(2\mathbb{Z}^n)$ and hence $(x+y)/2$ is an integer point from A. This simple argument leads to interesting insights in the intersection theory of discrete convex sets. The famous Helly's Theorem states that if $A_1, \ldots, A_m$, $m \geq n+1$, is a finite family of convex sets in $\mathbb{R}^n$ such that every collection of $n+1$ sets $A_{i_1}, \ldots, A_{i_{n+1}}$ has a point in common, then all the sets $A_1, \ldots, A_m$ have a point in common. A discrete version of Helly's Theorem, proved first by Doignon [Do73], states that if $m \geq 2^n$ and every collection of 2^n sets $A_{i_1}, \ldots, A_{i_{2^n}}$ has an integer point in common, then all the sets $A_1, \ldots, A_m$ have an integer point in common. Similar arguments are used to bound the complexity of maximal lattice-free convex bodies in connection with problems of integer programming; see [S97].

The fractional Helly Theorem of Katchalski and Liu asserts that for every $0 < \alpha \leq 1$ there exists a $\beta = \beta(\alpha, n) > 0$ with the following property: if among all $\binom{m}{n+1}$ tuples $A_{i_1}, \ldots, A_{i_{n+1}}$ at least $\alpha\binom{m}{n+1}$ tuples have a point in common, then there is a point common to at least βm of the sets $A_1, \ldots, A_m$. The best possible value $\beta(\alpha, n) = 1 - (1-\alpha)^{1/(n+1)}$ is due to Kalai; see [M02] for these and related topics. In a quite surprising recent development [BM03], Bárány and Matoušek proved the following discrete fractional Helly Theorem: for every $0 < \alpha \leq 1$ there exists a $\beta = \beta_{\mathbb{Z}}(\alpha, n) > 0$ with the following property: if among all $\binom{m}{n+1}$ tuples $A_{i_1}, \ldots, A_{i_{n+1}}$ at least $\alpha\binom{m}{n+1}$ tuples have an integer point in common, then there is an integer point common to at least βm of the sets $A_1, \ldots, A_m$. Unlike $\beta(\alpha, n)$, the number $\beta_{\mathbb{Z}}(\alpha, n)$ does not approach 1 as α approaches 1 (note that the bound 2^n in Doignon's Theorem is the best possible).

In another direction of discrete convexity (not discussed in the book under review), we seek to extend the analogy between the two quantities: the volume of a convex body A and the number of integer points in A. We get an especially close analogy if we restrict ourselves to the class of polyhedra. A *polyhedron* P is a subset of $\mathbb{R}^n$ defined by a finite set of linear inequalities: $P = \{x \in \mathbb{R}^n : \langle c_i, x\rangle \leq \alpha_i, \ i = 1, \ldots, m\}$, where $c_i \in \mathbb{R}^n$ and $\alpha_i \in \mathbb{R}$. If we can choose $c_i \in \mathbb{Z}^n$ and $\alpha_i \in \mathbb{Z}$, the polyhedron P is called *rational*.

Let $\mathcal{P}(\mathbb{R}^n)$ be the (complex) vector space spanned by the indicators $[P]$ of polyhedra in $\mathbb{R}^n$ and let $\mathcal{P}(\mathbb{Q}^n)$ be the subspace spanned by the indicators of rational polyhedra. Let $\mathbb{C}(x)$ be the field of rational functions in n complex variables $x = (x_1, \ldots, x_n)$, considered as a complex vector space.

Let $P \subset \mathbb{R}^n$ be a non-empty polyhedron without lines and hence, necessarily, with a vertex. Then, the integral $\displaystyle\int_P e^{\langle x,y\rangle}\, dy$ converges absolutely for all x in some non-empty open subset $U \subset \mathbb{C}^n$ to a rational function $\phi_P(x)$ on $\mathbb{C}^n$. A remarkable result obtained by Khovanskii and Pukhlikov [PK92], and, independently, by Lawrence [L91], states that the correspondence $P \longmapsto \phi_P$ extends to a linear transformation $\Phi : \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathbb{C}(x)$ such that $\Phi([P]) \equiv 0$ if P contains a line. We note that if P is a bounded polyhedron (polytope), then $\phi_P(0)$ is the volume of P.

Now let $P \subset \mathbb{R}^n$ be a rational polyhedron without lines. Then the sum (generating function) $\displaystyle\sum_{m \in P \cap \mathbb{Z}^n} x^m$ converges absolutely for all x in some non-empty subset $U \subset \mathbb{C}^n$ to a rational function $f_P(x)$, where $x^m = x_1^{m_1} \cdots x_n^{m_n}$ for $x = (x_1, \ldots, x_n)$ and $m = (m_1, \ldots, m_n)$. Lawrence [L91] and, independently, Khovanskii and Pukhlikov [PK92] proved that the correspondence $P \longmapsto f_P$ extends to a linear transformation $F : \mathcal{P}(\mathbb{Q}^n) \longrightarrow \mathbb{C}(x)$ such that $F([P]) \equiv 0$ if P contains a line. We note that if P is a bounded polyhedron (polytope), then $f_P(1, \ldots, 1)$ is the number of integer points in P. The usefulness of F is that it transforms various identities in the quotient of $\mathcal{P}(\mathbb{Q}^n)$ modulo the subspace spanned by the indicators of polyhedra with straight lines into identities for generating functions f_P for integer points in polyhedra. Here is an example: let us define the tangent cone K_v of a polyhedron P at a vertex $v \in P$ in the most natural way (thus K_v is a polyhedral cone with the vertex at v). Then one can prove that $[P] \equiv \sum_v [K_v]$ modulo indicators of polyhedra with straight lines, where the sum is taken over all vertices v of P. The corresponding identity $f_P = \sum_v f_{K_v}$ was first obtained by Brion via algebraic geometry methods [Br88]. These ideas lead to practically efficient algorithms for solving discrete optimization and counting problems as demonstrated by the LattE project; see http://www.math.ucdavis.edu/~latte.

Yet another direction of discrete convexity explores the *integrality phenomenon*, that is, explores various situations when a rational polyhedron $P \subset \mathbb{R}^n$ for some reason has integer vertices only (and here we come closer to the topic of the book under review). If this is the case, to find the minimum (maximum) of a linear function on $P \cap \mathbb{Z}^n$ is the same as to find the minimum (maximum) of the function on P, so the discrete and continuous linear optimization problems are equivalent. A typical example is provided by the family of *transportation polytopes*. Let us fix two sets of non-negative integers $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_n)$. Let us consider the set $P(a, b)$ of all $m \times n$ non-negative matrices $x = (x_{ij})$ such that $\sum_{j=1}^n x_{ij} = a_i$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m x_{ij} = b_j$ for $j = 1, \ldots, n$. If $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, then $P(a, b)$ is a non-empty polytope and every vertex of $P(a, b)$ is an integer matrix. In particular, if $m = n$ and all row sums a_i and column sums b_j are equal to 1, the vertices of $P(a, b)$ are the permutation matrices. Thus the convex hull of all permutation matrices has a very simple description, in contrast with the convex hull of all n-cycles; cf. the Traveling Salesman Polytope mentioned above.

For an excellent exposition of this integrality phenomenon (and a related phenomenon of the *dual integrality*), see [Sc03]. Here we note that integrality is rather fragile: consider, for example, the polytope P_n of all $n \times n \times n$ tensors $x = (x_{ijk})$ with $x_{ijk} \geq 0$ and the sums over each of the $3n$ "layers" (one index is fixed, the other two vary) equal to 1. Then, as n grows, the arithmetic of the vertices of P_n can get arbitrarily complicated [G74].

The book under review builds on a particular source of integrality. Let I be a finite set and let $f : 2^I \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function. The function f is called *submodular* provided $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$ for all subsets $X, Y \subset I$. Here are some examples of submodular functions. Let $a_i : i \in I$ be a finite set of vectors in a vector space and let $f(X)$ be the dimension of the span of $\{a_i : i \in X\}$. Then f is submodular. Let $A_i : i \in I$ be a family of subsets of a finite set U. Suppose that to every element $u \in U$ a real weight $w(u)$ is assigned. Let the weight of a subset of U be the sum of the weights of the elements of the subset. Finally, for $X \subset I$, let $f(X)$ be the weight of $\bigcup_{x \in X} A_x$. Then f is submodular.

Let $I = \{1, \ldots, n\}$. With a submodular function $f : 2^I \longrightarrow \mathbb{R}$ we associate a polyhedron $P_f \subset \mathbb{R}^n$, called the *(extended) polymatroid* of f and defined by $P_f = \left\{x \in \mathbb{R}^n : \sum_{i \in A} x_i \leq f(A) \text{ for all subsets } A \subset I\right\}$. An important theorem of Edmonds (see Chapter 46 of [Sc03]) states that if $f, g : 2^I \longrightarrow \mathbb{Z}$ are submodular, then the polyhedron $P_f \cap P_g$ has integer vertices only, and, moreover, the intersection of $P_f \cap P_g$ with every integer box $a_i \leq x_i \leq b_i$, $a_i, b_i \in \mathbb{Z}$ for $i = 1, \ldots, n$

has integer vertices only. This result can be considered as a generalization of the integrality result for transportation polytopes. It does not extend to intersections of three polymatroids, and, moreover, one can show that polymatroids form a maximal class of polyhedra with integer vertices such that the intersection of every two polyhedra from the class is a polyhedron with integer vertices, although there exist other classes of polyhedra with similar properties; see [DK03].

Geometrically, we can think of submodular functions as follows. Let $n = |I|$ and let $\{0,1\}^n$ be the Boolean cube in $\mathbb{R}^n$, that is, the set of all 0-1 vectors in $\mathbb{R}^n$. For two vectors x and y from $\mathbb{R}^n$, let $x \vee y \in \mathbb{R}^n$ be the coordinate-wise maximum of x and y and let $x \wedge y \in \mathbb{R}^n$ be the coordinate-wise minimum of x and y. Then, a function $f : \{0,1\}^n \longrightarrow \mathbb{R}$ is submodular if $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ for all x and y. Using this as a motivation, the author defines certain classes ("L-convex", "M-convex") of functions $\mathbb{Z}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ which combine useful properties of convex and submodular functions. For example, a function $f : \mathbb{Z}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is called L-convex if $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ and $f(x + e) - f(x)$ is a constant, where $e = (1, \ldots, 1)$.

The book under review presents a duality theory for these classes of functions and relates their global and local properties. The functions are nice and easy to optimize, and the author shows that the objective functions in some problems of mathematical economics belong to one of the classes.

While building a sufficiently general discrete convexity theory, we would like to avoid falling into one of the two extremes. One extreme is that in trying not to leave any interesting problem behind, we develop methods that are too general and hence too weak. The other extreme is that in trying to impose some nice structure in the discrete chaos, we severely restrict the class of sets and functions we agree to deal with and hence develop methods that are too special to be useful. This caution applies to any general theory, but since the field of discrete convexity is so precariously close to the $P \stackrel{?}{=} NP$ issue, the two extremes make the field for such a theory particularly narrow. Thus the main question is how many interesting new problems, previously intractable, can be solved within the new framework suggested in the book.

I think the book represents an interesting development. Let me add that in view of [DK03], it appears that the theories of L-convexity and M-convexity developed in the book are tied up somehow to the root system A_{n-1} and that there may be similar theories for other root systems.

References

[BM03] I. Bárány and J. Matoušek, *A fractional Helly theorem for convex lattice sets*, Adv. Math. **174** (2003), 227–235. MR **2003m:**52006

[Br88] M. Brion, *Points entiers dans les polyèdres convexes* (French), Ann. Sci. École Norm. Sup. (4) **21** (1988), 653–663. MR **90d:**52020

[C97] J.W.S. Cassels, *An Introduction to the Geometry of Numbers*, Corrected reprint of the 1971 edition. Classics in Mathematics, Springer-Verlag, Berlin, 1997. MR **97i:**11074

[DK03] V.I. Danilov and A. Koshevoy, *Discrete convexity and unimodularity. I*, preprint arXiv:math.CO/0304291, 2003.

[Do73] J.-P. Doignon, *Convexity in cristallographical lattices*, J. Geometry **3** (1973), 71–85. MR **52:**7937

[G74] M.B. Gromova, *The Birkhoff-von Neumann theorem for polystochastic matrices* (Russian), Operations Research and Statistical Simulation, No. 2, Izdat. Leningrad. Univ., Leningrad, 1974, pp. 3–15; Selected translations. Selecta Math. Soviet. **11** (1992), no. 2, 145–158. MR **92m:**00062, MR **52:**448

[GLS93] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization* (Second edition), Algorithms and Combinatorics, 2, Springer-Verlag, Berlin, 1993. MR **95e:**90001

[L91] J. Lawrence, *Rational-function-valued valuations on polyhedra*, Discrete and computational geometry (New Brunswick, NJ, 1989/1990), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 6, Amer. Math. Soc., Providence, RI, 1991, pp. 199–208. MR **92m:**52025

[M02] J. Matoušek, *Lectures on Discrete Geometry*, Graduate Texts in Mathematics, 212, Springer-Verlag, New York, 2002. MR **2003f:**52011

[P78] C.H. Papadimitriou, *The adjacency relation on the traveling salesman polytope is NP-complete*, Math. Programming **14** (1978), 312–324. MR **58:**20422

[PK92] A.V. Pukhlikov and A.G. Khovanskii, *The Riemann-Roch theorem for integrals and sums of quasipolynomials on virtual polytopes* (Russian), Algebra i Analiz **4** (1992), 188–216; translation in St. Petersburg Math. J. **4** (1993), no. 4, 789–812. MR **94c:**14044

[S97] H.E. Scarf, *Test sets for integer programs. Lectures on mathematical programming (ismp97)* (Lausanne, 1997), Math. Programming, Ser. B **79** (1997), 355–368. MR **98e:**90098

[Sc03] A. Schrijver, *Combinatorial Optimization. Polyhedra and efficiency. Vols. A-C*, Algorithms and Combinatorics, 24, A-C, Springer-Verlag, Berlin, 2003. MR **2004b:**90004a, MR **2004b:**90004b, MR **2004b:**90004c

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