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Hankel operators and their applications, by Vladimir V. Peller, Springer-Verlag, New York, 2003, xvi+784 pp., $99.95, ISBN 9-387-95548-8

A Hankel matrix is a square complex matrix (finite or infinite) that is constant on each diagonal orthogonal to the main diagonal—its $(m, n)^{\text{th}}$ entry is a function of $m+n$. The name derives from H. Hankel's 1861 dissertation. A Hankel operator, in abstract terms, is a Hilbert space operator that is represented by a Hankel matrix with respect to some orthonormal basis.

It sounds rather special. Is there really enough of interest here to fill a book of over 700 pages (plus appendices, etc.)?

Of course, for the devout operator theorist it is interesting enough that Hankel operators arise in a familiar setting (that of Hardy spaces–see below), that they are amenable to analysis, and that the analysis involves nontrivial function theory. But more is going on. In Peller's book we encounter Hankel operators as a theme whose variations reach into unitary dilations and related scattering theory, interpolation problems of Nevanlinna–Pick type, approximation theory, prediction theory, and linear systems theory.

In what follows I'll try to fill out the preceding picture somewhat without getting overly technical. The treatment will loosely follow the chronological development of the subject. Everything to be mentioned is clearly treated with abundant detail in the book. In particular, the book contains helpful introductions to prediction theory and systems theory, plus two appendices with background on operator theory and function spaces.

Early Sightings. Interest in Hankel operators increased noticeably around 1970. Before then attention was sporadic.

Infinite Hankel matrices arose before the dawn of operator theory in an 1881 paper of L. Kronecker, where he answered the question: How does one recognize from its coefficients whether a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

represents a rational function? His answer: It does if and only if the infinite Hankel matrix with $(m, n)^{\text{th}}$ entry a_{m+n} has finite rank. Moreover, the rank of the matrix is the degree of the denominator of the rational function in case the numerator has a smaller degree, and it is one more than the degree of the numerator in the contrary case (excluding the trivial case of the zero matrix). (This would be a good exercise for an honors algebra class.)

After the birth of operator theory, infinite Hankel matrices turned up in the solvability criterion for the Hamburger moment problem, introduced in a 1920 paper of H. Hamburger. The problem is to determine when an infinite sequence $(a_n)_{n=0}^{\infty}$ of real numbers is the moment sequence of a positive measure on the real line.

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Thus, one wants a criterion for the existence of a positive measure μ on $\mathbb{R}$ such that

$$a_n = \int_{\mathbb{R}} x^n \, d\mu(x)$$

for all n. The solution: Such a μ exists if and only if the infinite Hankel matrix with $(m,n)^{\text{th}}$ entry a_{m+n} is positive semidefinite.

That the preceding condition is satisfied by moment sequences lies fairly near the surface. The converse is deeper and can be established in several ways. The operator-theoretic approach is of interest here. Assuming the Hankel matrix in question is positive semidefinite, one uses it in a now familiar way to construct an inner product space. The elements of that space can be interpreted as functions on $\mathbb{R}$, in fact polynomials, at least in the positive definite case. (In the indefinite case, which is more elementary, one must pass to equivalence classes.) Completion of the inner product space yields a Hilbert space. Multiplication by the coordinate function induces a symmetric operator on that space whose deficiency indices are either $0,0$ or $1,1$. If the deficiency indices are $0,0$, the symmetric operator has a unique self-adjoint extension. The spectral theorem for self-adjoint operators hands you a unique solution to the corresponding moment problem. If the deficiency indices are $1,1$, the symmetric operator has a family of self-adjoint extensions, each of which (including those going beyond the original Hilbert space) corresponds via the spectral theorem to a different solution of the moment problem.

The Hilbert Matrix. The most famous Hankel matrix is the Hilbert matrix, whose $(m,n)^{\text{th}}$ entry is $1/(m+n+1)$ $(m,n = 0,1,\dots)$. It is thus the Hankel matrix associated with the moment sequence of Lebesgue measure on $[0,1]$. D. Hilbert proved in his lectures that the Hilbert matrix induces a bounded operator on ℓ^2, whose norm was shown to be π by I. Schur in a 1911 paper. One finds the preceding information and various generalizations of the Hilbert–Schur inequality in the famous book *Inequalities* by G. H. Hardy, J. E. Littlewood and G. Pólya [2], originally published in 1934. The Hilbert matrix is an amusing topic in its own right [1].

Nehari's Paper. In 1957 Z. Nehari published the first paper on general Hankel operators. His theorem: The Hankel matrix with $(m,n)^{\text{th}}$ entry a_{m+n-1} $(m,n = 1,2,\dots)$ induces a bounded operator on ℓ^2 if and only if there is a bounded function φ on the unit circle whose Fourier coefficients with negative indices are given by the sequence $(a_n)_{n=1}^{\infty} : \hat{\varphi}(-n) = a_n$ $(n = 1,2,\dots)$. Moreover, the norm of the operator is the minimum of the L^{∞}-norms of all such functions φ.

Nehari's theorem can be rephrased in terms of a trigonometric moment problem, known now as the Nehari interpolation problem. The question: Given a complex sequence $(a_n)_{n=1}^{\infty}$, is there a bounded function φ on the unit circle, with $\|\varphi\|_{\infty} \le 1$, such that $\hat{\varphi}(-n) = a_n$ for all $n > 0$? Nehari's theorem tells us that such a φ exists if and only if the associated Hankel matrix has norm at most 1 as an operator on ℓ^2. The analogy with the solvability criterion for the Hamburger moment problem jumps out. As would be appreciated later, it is more than superficial.

The key step in Nehari's proof of his theorem is to handle the case where the given sequence $(a_n)_{n=1}^{\infty}$ has only finitely many nonzero terms. Nehari showed that this case can be reduced to the Carathéodory–Fejér interpolation problem, dating from 1911, which asks: Given complex numbers $c_0, c_1, \dots, c_N$, what is the least supremum norm for holomorphic functions in the unit disk having these numbers as their first $N+1$ power series coefficients at the origin? That Hankel matrices arise in interpolation problems of this kind was recognized in the second decade of the 1900's, when such problems were first intensively studied, but the central role Hankel matrices can play seems not to have been appreciated at the time.

Classical Hankel Operators. Nehari's theorem presaged the role Hankel operators were eventually to play in function theory in the unit disk and the unit circle. By now the theory has spilled over into many other settings. In his book, Peller limits his attention to what he calls classical Hankel operators, those that act on the Hardy space H^2 of the unit disk $\mathbb{D}$.

The space H^2 consists of the holomorphic functions in $\mathbb{D}$ whose power series coefficients at the origin are square summable. It has a Hilbert space structure induced by its natural bijection with ℓ^2. A function in H^2 has an associated boundary function on $\partial\mathbb{D}$, defined almost everywhere by means of nontangential limits. The boundary functions comprise the subspace of L^2, also called H^2, of functions whose Fourier coefficients with negative indices vanish. The orthogonal complement of H^2 in L^2, denoted by H^2_-, consists of the functions whose Fourier coefficients with nonnegative indices vanish.

Two kinds of Hankel operators on H^2 arise. Operators of one kind map H^2 to H^2_-; those of the other kind map H^2 to itself. To define them, let φ be a function in L^2. The operator $H_\varphi : H^2 \to H^2_-$ is defined to be the operator of multiplication by φ followed by orthogonal projection onto H^2_-. Of course, if φ is unbounded, there is no guarantee H_φ is bounded, but at least its domain contains all bounded functions in H^2, in particular, all polynomials. With respect to the standard orthonormal bases for H^2 and H^2_-, the matrix for H_φ is the Hankel matrix whose $(m,n)^{\text{th}}$ entry is $\hat{\varphi}(-m-n)$ $(m = 1, 2, \ldots,\ n = 0, 1, \ldots)$. Thus, H_φ does not determine φ uniquely; it only determines the Fourier coefficients of φ with negative indices. The function φ is called a symbol of H_φ. Nehari's theorem says that H_φ is bounded if and only if it has a bounded symbol. The unique symbol of H_φ in H^2_-, called the conjugate-analytic symbol of H_φ, need not be bounded even if H_φ is bounded, for the orthogonal projection into H^2_- of a bounded function need not be bounded.

With φ as above, the operator $\Gamma_\varphi : H^2 \to H^2$ is by definition the operator whose matrix with respect to the standard orthonormal basis is the Hankel matrix whose $(m,n)^{\text{th}}$ entry is $\hat{\varphi}(m+n)$ $(m, n = 0, 1, \ldots)$. (Equivalently, Γ_φ sends the function f in H^2 to the orthogonal projection onto H^2 of the function $\varphi\tilde{f}$, where $\tilde{f}$ is given by $\tilde{f}(e^{i\theta}) = f(e^{-i\theta})$.) The terminology used here is the analogue of that used with the other kind of Hankel operator: the function φ is called a symbol of Γ_φ, and the unique symbol in H^2 is called the analytic symbol. Again, Nehari's theorem says that Γ_φ is bounded if and only if it has a bounded symbol.

The preceding notions can be generalized to the context of vector-valued H^2 spaces, giving rise to so-called block Hankel operators, something important for certain applications. For simplicity, that aspect of the subject will not be discussed in this review.

Toeplitz Operators. Since around 1970 Hankel operators have been a prominent part of operator theory. Interest in them was partly inspired by earlier progress in analyzing their first cousins, Toeplitz operators. For φ a function in L^∞ of $\partial\mathbb{D}$, the Toeplitz operator $T_\varphi : H^2 \to H^2$ is the operator of multiplication by φ followed by orthogonal projection onto H^2. The matrix for T_φ with respect to the standard orthonormal basis is a Toeplitz matrix; i.e., it is constant on each diagonal parallel to the main diagonal: the $(m, n)^{\text{th}}$ entry of the matrix equals $\hat{\varphi}(m - n)$. While Hankel operators and Toeplitz operators behave very differently, they are closely related, a relationship typified by the identity $T_{\varphi\psi} - T_\varphi T_\psi = H^*_{\overline{\varphi}} H_\psi$.

Given the success people had had by 1970 in understanding Toeplitz operators, it is no wonder that Hankel operators would be an inviting target. A prominent contributor in the 1970's was S. C. Power, who was able to use some of the techniques developed in connection with Toeplitz operators to make progress in the spectral analysis of the operators Γ_φ. His survey article [3] and book [4] brought home the many roles played by Hankel operators and helped spark further interest.

AAK. Beginning in 1968, V. M. Adamyan, D. Z. Arov and M. G. Kreĭn (aka AAK) published a series of influential papers that brought new prominence to Hankel operators. Interestingly, these authors were not aware of Nehari's paper when they began their work; a reference to Nehari was added to their first paper during the correction of proofs.

One of AAK's accomplishments was to find an approach to the Nehari interpolation problem that parallels, but is technically rather more subtle than, the operator-theoretic treatment of the Hamburger moment problem. Suppose $A : H^2 \to H^2_-$ is the Hankel operator corresponding to a given Nehari interpolation problem. The main step in the proof of Nehari's theorem is to show that, if $\|A\| \leq 1$, then there is a function φ in L^∞ such that $\|\varphi\|_\infty \leq 1$ and $A = H_\varphi$. To do this, AAK use the Hankel operator A to create a certain inner product space, which they complete to obtain a Hilbert space. In that Hilbert space they define a certain isometry, whose deficiency indices are either $0, 0$ or $1, 1$. As with the Hamburger moment problem, the case of deficiency indices $0, 0$ corresponds to a determinate problem and the case of indices $1, 1$ to an indeterminate one. In the indeterminate case, each unitary extension of the isometry, perhaps going beyond the original Hilbert space, produces a different solution of the interpolation problem. In fact, by using the theory of unitary extensions of isometries, AAK were able to describe the general solution in the indeterminate case, generalizing results of G. Pick and R. Nevanlinna for the interpolation problem that bears their name.

The AAK treatment of the Nehari problem can be expressed in the language of scattering theory; presumably it was inspired by earlier work on scattering theory of Adamyan and Arov. From this perspective, the solutions of an indeterminate Nehari problem can be interpreted as scattering operators.

The theory of operator models and unitary dilations, created by B. Sz.-Nagy and C. Foiaş, enters the picture here, because that theory is mathematically equivalent to the version of scattering theory associated with the Nehari problem. Nehari's theorem is in fact a special case of one of the centerpieces of the Sz.-Nagy–Foiaş theory, their commutant lifting theorem. In the other direction, R. Arocena showed in 1989 that the AAK method can be adapted to give a proof of the commutant lifting theorem.

AAK also studied the singular values of Hankel operators. The n^{th} singular value of a Hilbert space operator is by definition the distance of the operator from the set of operators whose ranks are at most n. AAK proved that the n^{th} singular value of a Hankel operator equals its distance from the set of Hankel operators whose ranks are at most n. In conjunction with Kronecker's theorem, this gives information about uniform approximation of functions in L^∞ by functions that are holomorphic in $\mathbb{D}$ except for finitely many poles (more accurately, by their boundary functions).

Enter Peller. Thanks to C. Fefferman's famous theorem on BMO (the space of functions of bounded mean oscillation), Nehari's theorem, at least its qualitative version, can be reexpressed as follows: a Hankel operator from H^2 to itself is bounded if and only if its analytic symbol belongs to BMO.

In 1958 P. Hartman provided a companion to Nehari's theorem, proving that a Hankel operator on H^2 is compact if and only if it has a continuous symbol. The analogous reexpression of this is: a Hankel operator on H^2 is compact if and only if its analytic symbol is in VMO (the space of functions of vanishing mean oscillation).

Can one similarly characterize Hankel operators of trace class? The question had been around at least since 1960 when Peller settled it in 1979. His theorem states that a Hankel operator on H^2 belongs to the trace class if and only if its analytic symbol belongs to a certain Besov space. He quickly extended this, obtaining a similar characterization of the Hankel operators in the Schatten classes $\mathcal{S}_p$, $p > 1$. A few years later he and, independently, S. Semmes handled the cases $0 < p < 1$. An interesting application of these results involves characterizing the Besov spaces that arise in terms of rational approximation in the BMO norm.

Best Approximation. If φ is in BMO, then Nehari's theorem tells us that the Hankel operator H_φ is bounded and that there is a function ψ in L^∞ such that $H_\varphi = H_\psi$ and $\|\psi\|_\infty = \|H_\varphi\|$. The difference $\varphi - \psi$ is then in the space BMOA $=$ BMO $\cap H^2$ and differs from φ in L^∞-norm by $\|H_\varphi\|$. No function in BMOA can be closer to φ in L^∞-norm, so $\varphi - \psi$ is a best BMOA-approximation to φ with respect to the L^∞-norm. If φ is bounded, such a best approximation is of course bounded; i.e., it belongs to H^∞, the space of (boundary functions of) bounded holomorphic functions in $\mathbb{D}$.

A function φ in BMO can have more than one best approximation in the sense above, but one can show that if φ is in VMO, then its best approximation is unique. One can thus define on VMO an operator $\mathcal{A}$ of best approximation, which assigns to each function in VMO the unique function in BMOA that differs from it the least in L^∞-norm.

Various questions about the operator $\mathcal{A}$ now arise. For example, although $\mathcal{A}$ does not preserve continuity, does it preserve smoothness of one kind or another? To answer such questions, an elaborate theory has been constructed by Peller and S. V. Khruschëv. Sample result: If φ belongs to one of the Hölder classes λ_α or Λ_α, $0 < \alpha < 1$, then $\mathcal{A}\varphi$ belongs to the same class.

Prediction Theory. A stationary Gaussian process with discrete time has an associated spectral measure, a positive measure on $\partial\mathbb{D}$. An interesting chapter in prediction theory involves relating properties of the process to properties of its spectral measure. There is a vast theory here occupying two chapters of Peller's book. Only one outgrowth will be touched upon in this review.

If the process satisfies a condition called regularity, its spectral measure is absolutely continuous with respect to Lebesgue measure on $\partial\mathbb{D}$, with a density of the form $|h|^2$ where h is a so-called outer function in the space H^2. The Hankel operator

$H_{\bar{h}/h}$ carries information about the relation between the past and the future of the process. Namely, if P_- and P_+ are the orthogonal projections onto the subspaces representing the past and the future in the complexification of the process, then the restriction of $P_+P_-P_+$ to the future is unitarily equivalent to $H^*_{\bar{h}/h}H_{\bar{h}/h}$. The question of which pairs of subspaces of a Gaussian Hilbert space can serve as the past and future of a Gaussian process suggests the problem of characterizing those positive Hilbert space operators that are unitarily equivalent to $|H_\varphi| = (H^*_\varphi H_\varphi)^{1/2}$ for some φ. The preceding problem, the so-called inverse problem for the moduli of Hankel operators, was posed by Peller and Khruschëv in 1982. S. G. Treil obtained substantial partial results in 1985, and again in 1989 in collaboration with V. I. Vasyunin.

Systems Theory. The symbiosis between operator theory and linear systems theory is well established. Large parts of both theories are identical except for motivation and language, and both have benefited from the other's viewpoint. The story of the Peller–Khruschëv problem is a case in point. In papers published in 1987 and 1990, R. Ober proposed a systems-theoretic approach, featuring the notion of a balanced realization, to the problem. With this approach he succeeded in recapturing a large part of what Treil and Vasyunin had obtained. Ober's insight proved fruitful, for in 1990 Treil, by building on his ideas, arrived at a complete solution.

It is easily proved that a Hankel operator is noninvertible and that its kernel is either trivial or infinite dimensional. The modulus of a Hankel operator therefore has the same properties. Treil's beautiful theorem says that the converse holds: any positive operator on a separable infinite-dimensional Hilbert space that is noninvertible and whose kernel is either trivial or infinite dimensional is unitarily equivalent to the modulus of a Hankel operator.

There is more to the story. The inverse problem for the moduli of Hankel operators suggests the analogous problem for self-adjoint operators: when is a self-adjoint operator unitarily equivalent to a self-adjoint Hankel operator? The problem was solved in a remarkable 1995 paper of Peller, Treil and A. V. Megretskii, and again the notion from systems theory of a balanced realization played a key role. The theorem of Megretskii–Peller–Treil states that a self-adjoint operator is unitarily equivalent to a Hankel operator if and only if it is noninvertible, its kernel is either trivial or infinite dimensional, and, very roughly speaking, its spectral multiplicity function can be unsymmetric only in a very restricted way (a condition I won't try to make precise here). A sample consequence: any noninvertible cyclic self-adjoint operator with a trivial kernel is unitarily equivalent to a Hankel operator. Treil's earlier theorem on the moduli of Hankel operators is an easy corollary of the Megretskii–Peller–Treil theorem.

The reader will find in Peller's book much beyond what is touched on above. The book is the work of a master, a treasure trove of operator theory in its myriad aspects.

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