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THE IMPACT OF THOM'S COBORDISM THEORY

MICHAEL ATIYAH

1. Introduction

At the 1958 International Congress of Mathematicians in Edinburgh, René Thom received one of two Field Medals for his development of cobordism. In his citation [11] Heinz Hopf described the definition of cobordism as one of those elementary and apparently trivial constructions which can hardly be expected to yield significant results. He compares this with Hurewicz's definition of homotopy groups, a very simple idea which has turned out to be extremely fruitful. Hopf then points out that there is in fact a close link between cobordism and homotopy which Thom exploits.

In fact the basic idea linking homotopy theory to differentiable manifolds goes back to a construction of Pontrjagin [13]. Given a smooth map $f : Y \to X$ between two compact, connected and oriented differentiable manifolds, the inverse image $f^{-1}(p)$ of a regular value $p \in X$ is an oriented submanifold F of Y with $\dim F = \dim Y - \dim X$. It is easy to see that the homology class of F in Y is independent of p and is a homotopy invariant of $f :$ in fact it is the Poincaré dual of the cohomology class $f^*(u)$ where u is the fundamental class of X in top dimension.

But the geometry of F contains more information about f than just this homology class. For example, when $Y = S^3$, $X = S^2$ are spheres and f is the Hopf fibration, then F is a circle and any two such circles have linking number 1. Applied to any map $f : S^3 \to S^2$, we get in this way a homotopy invariant known as the Hopf invariant.

In another direction the homological equivalence between two fibres F_p and F_q (for $p, q \in X$) can be strengthened to a more precise geometrical relation, namely that there is an oriented manifold W with boundary F_p and $-F_q$ (i.e. F_q with the opposite orientation). This manifold W appears naturally as a submanifold of $Y \times I$, where I is the unit interval defined by a generic path in X from p to q. In Thom's terminology W is a **cobordism** between F_p and F_q.

Pontrjagin's idea was to use the geometry of F to deduce information about the homotopy of f, in the spirit of the Hopf invariant. In the early days of homotopy theory this geometric approach paid some dividends, but it was delicate to use (and could lead to mistakes). But in the early fifties powerful new algebraic methods were introduced into homotopy theory, notably the Leray-Serre spectral sequence for fibrations, and Pontrjagin's method then became obsolete, but Thom turned the tables and used homotopy theory to attack the geometry of manifolds. Specifically he showed [15] that the abstractly defined **cobordism groups** could

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be interpreted as homotopy groups of certain spaces $MSO(n)$. These spaces, as the notation implies, are constructed from the orthogonal groups $SO(n)$ which enter because a smooth manifold has infinitesimally a linear structure. This is a crucial observation and it has major consequences.

Remarkably the homotopy groups of the complicated spaces $MSO(n)$ are much easier to compute than the homotopy groups of spheres. This led rapidly to the complete determination of all cobordism groups, whereas the homotopy groups of spheres have still not yielded their final secrets.

Over the rationals the cobordism ring turned out to be a polynomial algebra with one generator in each dimension of the form $4k$, and a representative generator was the complex projective space $P_{2k}(C)$. Moreover, a complete set of (rational) invariants was given by the Pontrjagin members. These are constructed from the Pontrjagin classes which ultimately come from the cohomology of the orthogonal groups, emphasizing the importance of the tangent bundle of a manifold, given by its infinitesimal linear structure.

2. Relation with K-theory

One of the first major applications was made by Hirzebruch [9], who derived his famous formula for the signature of the quadratic form on $H_{2k}(X, R)$ of a $4k$-manifold. This formula given by the "L-genus" was then used by Hirzebruch with great virtuosity in his proof [10] of the general Riemann-Roch theorem (HRR) for complex projective algebraic manifolds. This proof was a synthesis of cobordism theory with various new techniques, notably the Cartan-Serre theory of coherent analytic sheaves.

Shortly afterwards Grothendieck introduced his K-groups and formulated and proved the Grothendieck-Riemann-Roch Theorem (GRR) for an algebraic morphism $f : Y \to X$, reducing to HRR when X is a point [6].

This in turn, together with the advent of Bott's periodicity theorems [7] for the homotopy groups of the classical groups, led Atiyah and Hirzebruch to develop topological K-theory as a periodic "generalized cohomology theory" [2]. This had intimate relations with cobordism theory, and it suggested a parallel definition of a generalized cobordism cohomology theory.

In due course Quillen [14] took this a step further by showing that (a $U(n)$-analogue of cobordism) was a "universal theory" and used this to deduce its algebraic structure by a totally new method related to the theory of "formal groups".

3. Relation to the index theorem

Motivated by HRR, Atiyah and Singer were led to their general index theorem for elliptic differential operators on compact manifolds [5]. The first proof of this [4] mimicked Hirzebruch's approach to the signature by showing that the index of a certain basic operator was a cobordism invariant and then using Thom's work.

Subsequent proofs went along different lines. One proof [5] copied the proof of GRR, while another one [3] relied on an invariant theory approach that characterized the local expression of Pontrjagin forms in terms of curvature. In a sense this invariant theory (derived from the classical representation theory of $SO(n)$) replaced Thom's global geometric methods.

4. Relation to quantum field theory

Undoubtedly the most surprising descendant of Thom's cobordism theory arose from new ideas in quantum field theory (QFT). Since physics is formulated in terms of differential equations, it has always been the case that physics is closely related to differential geometry, as in Maxwell's equations (which led to Hodge theory) and in Einstein's General Relativity (GR). In the past thirty years there has been a serious attempt to unify GR and quantum theory, and this has led physicists into deeper territory in differential geometry. In particular the Atiyah-Singer index theorem has found a natural place in modern physics, and so cobordism is seen lurking in the background.

Certain parts of QFT have a topological character, and this leads to the notion of a topological QFT, introduced by Witten and formally developed by Atiyah in [1]. According to this a topological QFT (in dimension n) is a functor from the category of n-manifolds to the category of complex vector spaces, where the morphisms of manifolds are **cobordisms**. There are a small number of rather obvious axioms, so this definition is very much in the spirit of Thom. What is surprising, and highly non-trivial, is that there are a number of extremely deep examples of such theories in dimensions $n = 2, 3, 4$. These are numerous for $n = 2$ and lead in particular to the notion of the "elliptic genus" [16]. For $n = 3$ we have the Jones polynomial invariants of knots [12],[17], and for $n = 4$, the Donaldson invariants of 4-manifolds [8],[18]. This whole area has been of intense interest to geometers and physicists over the past decades, and it demonstrates the long life of a really fundamental idea such as cobordism.

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Department of Mathematics and Statistics, James Clerk Maxwell Building, Kings' Buildings, University of Edinburgh, Edinburgh EH9 3JZ, Scotland