BOOK REVIEWS

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Painlevé differential equations in the complex plane, by V. I. Gromak, I. Laine, and S. Shimomura, de Gruyter Studies in Mathematics, volume 28, Walter de Gruyter, Berlin, 2002, viii + 303 pp., \$89.95, ISBN 3-11-017379-4

The Painlevé differential equations originate from a question by E. Picard in 1887, which may be phrased as follows: *What are the necessary and sufficient conditions that guarantee that the second order differential equation*

$$w'' = R(z, w, w') = \frac{P(z, w, w')}{Q(z, w, w')} \tag{1}$$

has no movable singularities other than poles? Here the right hand side is assumed to be rational with respect to the variables w and w', and analytic in the first variable in some domain of the z-plane. We note that *freedom of movable singularities* is a property of the differential equation itself rather than a property of particular solutions. It should also be noted that in those days–and even in modern textbooks–the notion *movable singularity* lacked a precise definition, although it seems that Painlevé and his contemporaries had an intuitive idea of what it should be.

Following Bieberbach [B], a precise definition, which deserves to be better known, may look as follows: Consider any differential equation (D) in the complex domain, and let S be the set of those points z_0, such that z_0 is a singularity other than a pole of some solution. Then (D) is said to be *free of movable singularities* if S has empty interior. The isolated points in S are called *fixed singularities.* Instead of considering singularities of every type, one may also restrict to certain types of singularities–algebraic, logarithmic, essential, etc. For example, the equation

$$w'' = \frac{2w-1}{w^2+1}(w')^2$$

has solutions $w = \tan\log(z + z_0)$, and thus has movable logarithmic singularities and also movable poles, while the *Schwarzian differential equation*

$$\frac{w'''}{w'} - \frac{3}{2}\Big(\frac{w''}{w'}\Big)^2 = q(z),$$

q any analytic function, has only movable poles. Linear differential equations have only fixed singularities, and algebraic first order differential equations $P(z, w, w') = 0$ have only movable algebraic singularities and movable poles–also a theorem due to Painlevé.

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Picard's question was quite natural, since the corresponding problem for *even implicit* first order equations had been then completely solved by L. Fuchs: *The differential equation*

$$P(z,w,w') = \sum_{\nu=0}^{n} P_\nu(z,w)(w')^\nu = 0, \tag{2}$$

where $P(z,w,w')$ *is analytic in* z *and a polynomial with respect to* w *and* w', *assumed to be irreducible, is free of movable singularities other than poles, if and only if* $P_\nu(z,w)$ *has degree at most* $2(n-\nu)$ *with respect to* w. In the explicit case this leads to the *Riccati* differential equation

$$w' = a_0(z) + a_1(z)w + a_2(z)w^2.$$

The answer to Picard's question, given by P. Painlevé [P1], [P2] and completed by R. Gambier and R. Fuchs (the son of L. Fuchs), was twofold: *Necessary for equation* (1) *to be free of movable singularities is that it may be transformed by an analytic change of variables into some equation within a list of* 50(!). *Conversely, every equation in this list has what is nowadays called the Painlevé property.* This can be seen almost immediately by inspection of 44 cases, each of which may be reduced to either a linear or Riccati equation, an equation for elliptic functions, or else to one of the remaining six equations:

$$\text{(I)}\quad w'' = z + 6w^2$$

$$\text{(II)}\quad w'' = \alpha + zw + 2w^3$$

$$\text{(III)}\quad w'' = \frac{(w')^2}{w} - \frac{1}{z}w' + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}$$

$$\text{(IV)}\quad w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w}$$

$$\text{(V)}\quad w'' = \Big(\frac{1}{2w} + \frac{1}{w-1}\Big)(w')^2 - \frac{1}{z}w' + \frac{(w-1)^2}{z^2}(\alpha w + \beta) + \frac{\gamma w}{z} + \frac{\delta w(w-1)}{w-1}$$

$$\text{(VI)}\quad w'' = \frac{1}{2}\Big(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\Big)(w')^2 - \Big(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\Big)w' + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\Big(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\Big),$$

afterwards called *Painlevé's differential equations.*

Paul Painlevé was a mathematician and a politician. He received his doctorate in mathematics in 1887 and held different academic positions at Lille and École Polytechnique. About 1910 he started his second, political, career and served several times as minister and even as Président du Conseil (Prime Minister) of the French Republic. He was Wilbur Wright's first passenger, making a record 1 hour 10 minute flight, and immediately after that in 1909 he created the first course in aeronautical mechanics at École Polytechnique. He obtained his major breakthrough in the field of analytic differential equations during his stay at Stockholm (1895), where he wrote his *leçons de Stockholm*–almost 600 calligraphed pages.

Painlevé's main tool was the so-called α-method, which relies on Poincaré's theorem on analytic dependence on parameters of (analytic continuations of) solutions.

The implementation was very elaborate and voluminous, but straightforward. On the other hand, however, new ideas and a great amount of work were required to show that the remaining six equations are actually free of movable singularities. The ordering of these equations has historical reasons; they occur as items 4, 9, 13, 31, 39, and 50 in the complete list, which can be found in Ince's book [I]. Since equations (I), (II) and (IV) have no fixed singularities and only movable poles, all solutions are meromorphic functions in the complex plane by the monodromy theorem. The solutions of (III) and (V) are meromorphic functions in the $\log z$-plane, while the solutions of (VI) have fixed singularities $z = 0$, $z = 1$, and, of course, $z = \infty$.

The Fuchsian conditions $\deg_w P_\nu(z,w) \le 2(n-\nu)$, $0 \le \nu \le n$, are also necessary for (2)–with R rational in all variables–to have a transcendental meromorphic solution in the complex plane; in particular, the differential equation $w' = R(z,w)$, R rational, admits transcendental meromorphic solutions in the whole plane only in the Riccati case. This is the contents of the *first and second Malmquist theorem.* It would be an interesting and challenging problem to find and prove a Malmquist's theorem for second order algebraic differential equations (1): *Give a complete list, up to a change of variable*

$$w \mapsto \frac{a(z) + b(z)w}{c(z) + d(z)w},$$

of those equations (1)*, where now R is rational with respect to all variables, having at least one transcendental meromorphic solution in $\mathbb{C}$, which is* not *a solution of some equation* (2)*.*

In two voluminous papers [B1], [B2] P. Boutroux made a great step forward in investigating the transcendental solutions of (I), (II) and (IV), nowadays called *Painlevé transcendents.* He developed a kind of value distribution theory for logarithmic derivatives of entire functions and also a kind of asymptotic integration of non-linear differential equations, leading him to far-reaching statements about the asymptotic distribution of poles and the orders of growth of the first, second and fourth transcendents. Although his proofs were not quite satisfactory from today's point of view, his work had great impact and was a source of inspiration for his successors.

Starting in the 1980's, a great number of papers appeared dealing with different and rather disjoint topics related to Painlevé's equations and, in particular, to Boutroux's work. Without aiming at completeness we mention the following topics, where emphasis is laid on those which are also dealt with in the book under review.

• *Bäcklund transformations.* Generally speaking, a Bäcklund transformation transforms a given differential equation into another, maybe of the same type with possibly different parameters. To give an example, the change of variable

$$w \mapsto w_1 = -w - \frac{\alpha + 1/2}{w' + w^2 - z/2}$$

transforms equation (II) into itself, but with a new parameter $\alpha + 1$.

• *Special functions.* In certain cases, equations (II)-(VI) admit special solutions which are related to special functions from mathematical physics or are rational functions. These particular solutions (and equations) are obtained by applying Bäcklund transformations to a special situation; the corresponding parameters form a discrete set in parameter space.

• *Nevanlinna theory* was used earlier to investigate the transcendental solutions of equations (I), (II), and (IV); pioneering work in this field was done by H. Wittich [W]. Only recently estimates of the orders of growth and also the asymptotic distribution of poles, already stated by Boutroux, were confirmed (see [Sh] and [St2]).

We also mention several topics which are not or only marginally discussed in the present book, but nevertheless play a major role in the recent literature:

• *Asymptotic representations.*
• *Integrable systems, Painlevé-Kovalevskaya property.*
• *Integrability of partial differential equations from mathematical physics.*
• *Discrete Painlevé equations.*
• *Higher order Painlevé equations.*
• *Riemann-Hilbert problems.*
• *Isomonodromic deformations.*
• *Inverse scattering theory.*

The book under review is divided into ten chapters and two **appendices**, which, for the convenience of the reader, deal with *existence and uniqueness* of analytic solutions of differential equations and *Nevanlinna theory.*

Chapter 1 contains recent proofs, mainly due to A. Hinkkanen and the second author ([HL1], [HL2], [HL3]; see also [St1]) of the *meromorphic nature* of solutions, in particular of equations (I), (II) and (IV). Equations (III), (V) and (VI) are only touched upon briefly.

Chapter 2 is devoted to the study of *growth* of the transcendental solutions of equations (I), (II) and (IV) in the sense of Nevanlinna theory. It reports on recent results achieved by the third author [Sh] and the reviewer [St2], independently.

Chapter 3 contains a rather complete collection of results about *value distribution* (Nevanlinna theory) of the Painlevé transcendents in the spirit of Wittich. Here, too, emphasis lies on equations (I), (II) and (IV).

Each of Chapters 4 to 9 (details see below) is devoted to one of the Painlevé equations (I) – (VI), while **Chapter 10** contains several remarks on *discrete Painlevé equations* and relations between Painlevé equations and particular partial differential equations from mathematical physics, such as Korteweg–de Vries and sine–Gordon, in an informal way. Discrete Painlevé equations are special difference equations, in some sense associated with Painlevé's differential equations. There is, however, no unique correspondence between difference and differential equations, and also no unique way of constructing these equations. This field is still in its infancy, and little is known about existence and uniqueness of solutions and their analytic properties.

Equation (I), studied in **Chapter 4**, is singular in some sense. It contains no parameter, hence admits no Bäcklund transformations, and its solutions behave more or less uniformly. The first Painlevé transcendents are *new* transcendental functions; i.e., they do not solve algebraic first order equations. Higher order Painlevé-I equations are just mentioned. An interesting open problem is *whether their solutions are meromorphic in the plane* (single-valued).

Chapters 5 and **6** deal with equations (II) and (IV), which, in some sense, are much more interesting than equation (I). We restrict ourselves to Chapter 5, since it is quite similar to Chapter 6. The section about canonical representations $w = u/v$ as quotients of entire functions is followed by a discussion of the sequence of poles of transcendental solutions and a section about Bäcklund transformations.

These transformations are used to identify the rational solutions (which occur only for parameters $\alpha \in \mathbb{Z}$) and those exceptional solutions which also satisfy some algebraic first order equation (these occurring only for $\alpha + \frac{1}{2} \in \mathbb{Z}$). The latter solutions are closely related to the solutions of Airy's equation $u'' + zu = 0$ and are called *Airy solutions.* Here an open problem arises in a natural way: *Is it true that all transcendental solutions of (II) have order of growth $\varrho = 3$, the only exceptions being the Airy solutions, with order of growth $\varrho = \frac{3}{2}$?* The same question arises in case (IV), where the number 3 has to replaced by 4, and *Airy* by *Weber-Hermite.*

Chapters 7, 8 and **9** are also quite similar to each other, dealing with poles, Bäcklund transformations, rational and classical transcendental solutions and relations between equations (III), (V) and (VI). In some sense, equation (VI) is the master type. By a procedure called the *method of coalescence*, all Painlevé equations may be derived from (VI). Introducing a new parameter ϵ into some equation (A) in the right way results in an equation (A_ϵ), whose limit as $\epsilon \to 0$ turns out to be some equation (B). In this way the following hierarchy is obtained:

$$\begin{array}{ccccccc} (\mathrm{VI}) & \to & (\mathrm{V}) & \to & (\mathrm{IV}) & & \\ & & \downarrow & & \downarrow & & \\ & & (\mathrm{III}) & \to & (\mathrm{II}) & \to & (\mathrm{I}). \end{array}$$

The emphasis of the book lies on global aspects of solutions of Painlevé's equations, although several other topics are also touched upon. The book is therefore of interest and recommended to all mathematicians who work within the triangle *differential/difference equations – Nevanlinna theory – special functions.* Unfortunately, the notation (and style) are not standardized (probably an inevitable consequence of co-production), which sometimes may mislead the reader; this, however, seems to be the only point giving rise to criticism.

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Norbert Steinmetz
University of Dortmund
E-mail address: stein@math.uni-dortmund.de