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Perfect lattices in Euclidean spaces, by Jacques Martinet, Springer-Verlag, Berlin, 2003, xxii + 523 pp., $99.00, ISBN 3-540-44236-7

Probably Minkowski was the first person who developed systematic and very powerful methods to translate arithmetic questions into the framework of Euclidean spaces; cf. [11]. One such example is the arithmetic theory of positive definite integral quadratic forms, which has its roots in the 18th century (cf. [6] for a detailed historical introduction). This theory can be reformulated in terms of lattices in Euclidean space, a language also chosen in the present book. Also the many cross connections with other branches of mathematics such as number theory, integral representation theory of finite groups, algebraic geometry, modular forms, and coding theory become more self-evident in this 20th century formulation of the theory. Its long history on the one hand, the modern applications (for example in communication technology) on the other hand, as well as concrete results such as the discovery of many interesting lattices that usually have various additional remarkable properties often reflecting the connections above, are mainly responsible for the attractiveness of the theory of lattices.

Let $E := (\mathbb{R}^n, (,))$ denote the n-dimensional Euclidean space with standard scalar product

$$(x, y) := \sum_{i=1}^{n} x_i y_i = xy^{tr}.$$

A *lattice* $L \subset E$ is the set of all integral linear combinations of a basis B of $\mathbb{R}^n$. Here B is called a *basis of the lattice* L and the matrix of its inner products $G(B) = ((b_i, b_j))_{i,j} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ a *Gram matrix* of L.

The most important measure for the quality of a lattice is the density of the associated sphere packing. This is a set of spheres of common diameter d whose centers form the lattice L such that d is maximal with the condition that no two of the spheres overlap. It is easily seen that d equals the minimal distance of two distinct lattice points $d =: \sqrt{\min(L)}$, the square root of the minimum of L. Then the *density* of the sphere packing, which is frequently referred to as the density of the lattice, is proportional to its *Hermite invariant*

$$\gamma(L) := \frac{\min(L)}{\det(G(B))^{1/n}},$$

where $\det(G(B)) = \mathrm{vol}(E/L)^2$ is the square of the covolume of L in E. Note that the density of the lattice is not only invariant under Euclidean motions but also under scalings. Therefore most of the lattices are considered up to similarity. The main goal in lattice theory is to find dense lattices.

Martinet's book concentrates on the most classical approach to find the densest lattices in a given dimension and hence to determine the *Hermite constant*

$$\gamma_n := \max\{\gamma(L) \mid L \text{ is an } n\text{-dimensional lattice}\}.$$

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TABLE 1. Number of perfect lattices

dimension	1	2	3	4	5	6	7	8
perfect	1	1	1	2	3	7	33	$\geq$ 10170
extreme	1	1	1	2	3	6	30	
densest	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{D}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$

The idea, which goes back to Korkine and Zolotareff [10], is to construct all finitely many local maxima of the density function on the space of similarity classes of lattices. Lattices representing these local maxima are called *extreme* lattices.

Extreme lattices L can be characterized by the geometry of their minimal vectors

$$\mathrm{Min}(L) = \{\lambda \in L \mid (\lambda, \lambda) = \min(L)\}.$$

A lattice L is called *perfect* if the projections along its minimal vectors span the space of all symmetric endomorphisms of E, that is if

$$\langle \lambda^{tr}\lambda \mid \lambda \in \mathrm{Min}(L)\rangle = \mathbb{R}^{n\times n}_{sym}.$$

Since the trace induces a nondegenerate bilinear form on the space of symmetric matrices, this means that a lattice is perfect if and only if its Gram matrix $G = G(B)$ is determined (up to a scalar multiple) by the coordinates of the minimal vectors with respect to the chosen lattice basis B: In these coordinates the equality $(\lambda, \lambda) = m := \min(L)$ reads as

$$\mathrm{trace}(\lambda^{tr}\lambda G) = \mathrm{trace}(\lambda G \lambda^{tr}) = \lambda G \lambda^{tr} = m \text{ for all } \lambda \in \mathrm{Min}(L).$$

If L is perfect, then for any given m this linear system has a unique symmetric solution G. Choosing $m \in \mathbb{Q}$, all coefficients of the system are rationals, and hence also the solution G is rational. This shows that perfect lattices are proportional to integral lattices, where a lattice L is called *integral* if $(\lambda, \mu) \in \mathbb{Z}$ for all $\lambda, \mu \in L$ or equivalently if L is contained in its *dual lattice*

$$L^{\#} := \{v \in E \mid (v, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in L\}.$$

Up to similarity, there are only finitely many perfect lattices in each dimension. They can be calculated effectively with Voronoi's algorithm [15] and are known up to dimension 7 (see Table 1).

The notion of perfection is also useful in other situations. Based on Voronoi's algorithm, J. Opgenorth [12] gave the best known algorithm to calculate the normalizer of a finite unimodular group in the full unimodular group $\mathrm{GL}_n(\mathbb{Z})$, where the size of the computations mainly depends on the dimension of the space of invariant quadratic forms and not so much on the degree of the matrices.

Being perfect is a necessary but not a sufficient condition for a lattice L to be extreme. It has to fulfill an additional convexity condition. A lattice L is called *eutactic* (resp. *weakly eutactic*) if the quadratic form is a positive (non-negative) linear combination of the projections along the minimal vectors. A famous theorem due to Voronoi states that a lattice is extreme if and only if it is perfect and eutactic.

The densest lattices are known up to dimension 8 and in dimension 24 (see Table 1). In dimension ≤ 8 the densest lattices are root lattices familiar from the classification of semi-simple Lie algebras. All perfect lattices in dimensions 3, 4, and 5 have been classified by Korkine and Zolotareff [10]. Hofreiter [8] determined the extreme forms in dimension 6, and Blichfeldt's work [3] calculates the Hermite

TABLE 2. Number of eutactic lattices

dimension	1	2	3	4	5
minimal classes	1	2	5	18	136
weakly eutactic	1	2	5	17	127
eutactic	1	2	5	16	118

constants γ_6, γ_7, and γ_8. Barnes [1], Stacey [13], and Jaquet [9] later verified Blichfeldt's results by determining all perfect lattices in dimensions 6 and 7. In dimension 8 one already knows more than 10,000 perfect lattices, indicating the limit of Korkine and Zolotareff's approach. However, Mordell's inequality relating γ_n and γ_{n-1}, which is an equality for $n = 8$, permits a short proof of the fact that the root lattice $\mathbb{E}_8$ is the unique densest lattice in dimension 8. This lattice is *unimodular*, which means that $\mathbb{E}_8^{\#} = \mathbb{E}_8$ and is *even*; that is the quadratic form $\lambda \mapsto (\lambda, \lambda)$ takes only even values on $\mathbb{E}_8$. Very recent work [4] shows that the remarkable Leech lattice is the densest lattice in dimension 24. After $\mathbb{E}_8$ the Leech lattice is a second example of an even unimodular lattice that realizes the maximal density in its dimension, showing that the arithmetic and geometric properties of lattices are sometimes miraculously related.

Whereas eutaxy seems to play a minor role in dimensions ≤ 7, it becomes more important in dimension 8 (and presumably in higher dimensions). Among one family of 1,175 perfect 8-dimensional lattices, O. Jaquet verified in 1993 that only 383 are extreme. Approaches to classify all eutactic lattices in a given dimension are based on the notion of *minimal classes*. Two lattices L and L' are called *minimally equivalent* if there is a map $g \in \mathrm{GL}(E)$ with $Lg = L'$ and $\mathrm{Min}(L)g = \mathrm{Min}(L')$. Replacing the equality above by the inclusion $\mathrm{Min}(L)g \subseteq \mathrm{Min}(L')$, one defines an ordering relation on the set of all minimal classes of lattices. The maximal elements are precisely the classes of perfect lattices which are the only minimal classes containing only one similarity class of lattices. There are only finitely many minimal classes of lattices in a given dimension.

From the point of view of classifying eutactic lattices, the only minimal classes that are of interest are those that consist of well-rounded lattices, where a lattice is called *well-rounded* if its minimal vectors span the space E. Weakly eutactic lattices as well as perfect lattices are well-rounded. The well-rounded minimal classes are classified up to dimension 5 (see Table 2).

Each minimal class contains at most one weakly eutactic lattice, which is then the unique lattice of minimal density in the minimal class. This also implies that there are only finitely many weakly eutactic lattices in a given dimension. Up to dimension 3 all minimal classes contain a eutactic lattice, and these lattices are integral. In dimension 4 there are two eutactic lattices on which the quadratic form takes values in a real quadratic field; in dimension 5 totally real number fields of degrees up to 9 occur. It is still an open question whether every number field occurs as a field of definition of a eutactic lattice. There are examples where these fields are not totally real.

The mostly classical theory described above has been adapted to more specific situations. There is a notion of relative extremeness, perfection and eutaxy, where the space of all lattices is replaced by a suitable subspace, for instance cyclotomic lattices or, more generally, the ones that are invariant under a finite unimodular group. Since the relative Voronoi algorithm mainly depends on the dimension of the space of lattices and not on the dimension of E, this notion allows one to find dense lattices as relative local maxima of the density function also in higher dimensions.

A very natural point of view is to consider the lattice and its dual simultaneously. A lattice L is called *dual-extreme* if L is a local maximum for the *Bergé-Martinet function* γ' defined by

$$\gamma'(L)^2 := \gamma(L)\gamma(L^{\#}) = \min(L)\min(L^{\#}).$$

There are only finitely many similarity classes of dual-extreme lattices, and they are classified up to dimension 4, where one finds $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_3^{\#}$, $\mathbb{D}_4$, $\mathbb{A}_4$ and $\mathbb{A}_4^{\#}$ ([2, Théorème 5.1]). Since the densest lattices in dimension 8 and 24 are similar to their dual lattices, they also realize the global maximum of γ'. Though dual-extreme lattices are characterized as those being dual-perfect and dual-eutactic, there is no algorithm à la Voronoi. The only known classification method is based on the partition of the minimal classes into dual-minimal classes.

Strongly related to the theory of extreme lattices is the notion of strongly perfect lattices introduced by B.B. Venkov [14] using spherical designs: A lattice L is called *strongly perfect* if $\mathrm{Min}(L)$ is a spherical 4-design, which means that the average value over $\mathrm{Min}(L)$ equals the O_n-invariant integral over the sphere for all polynomials of degree ≤ 4. Strongly perfect lattices are extreme. The design property yields combinatorial means to classify them and also implies the lower bound $\gamma'(L)^2 \geq \frac{n+2}{3}$ for an n-dimensional strongly perfect lattice L. Up to dimension 11 the strongly perfect lattices are the root lattices and their dual lattices, $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{D}_4$, $\mathbb{E}_6$, $\mathbb{E}_6^{\#}$, $\mathbb{E}_7$, $\mathbb{E}_7^{\#}$, $\mathbb{E}_8$, and the two 10-dimensional lattices K'_{10} and ${K'_{10}}^{\#}$. Also the Leech lattice, the Coxeter-Todd lattice in dimension 12 and the Barnes-Wall lattice in dimension 16, which are the densest known lattices in their dimension, are strongly perfect.

This connection to spherical designs opens up the possibility of using new methods, such as representation theory or modular forms, to find extreme lattices and also to prove extremeness. For example the representation theory of the automorphism group of the even unimodular 248-dimensional Thompson Smith lattice Λ_{248} makes it possible to show that this lattice is strongly perfect and hence extreme. It is still impossible to calculate $\min(\Lambda_{248})$ and hence also to check extremeness without the use of the automorphism group. The bound above and explicit construction of short vectors show that $\min(\Lambda_{248})$ is either 10 or 12.

Compared to other books on lattices, such as Ebeling's nice introductory book [7], which concentrates on the arithmetic theory of lattices, in particular their connection to modular forms, and, of course, the "bible" [5], which collects a lot of important information on lattices, Martinet's book is the only modern treatise on the geometric theory of lattices. Jacques Martinet is one of the leading experts in this theory. The book reflects both the author's teaching experience as well as his deep geometrical insights, which often allow him to simplify the known proofs. Though it is the first systematic account that treats the modern results on perfect lattices, the style equally inspires its use as a textbook for a course on lattices. Starting from the classical inequalities in the geometry of numbers, Martinet leads the reader to recent developments and open questions. Comments on related research encourage further reading. Concrete examples and elementary proofs (for instance for the classification results above whenever this is possible) make the book accessible also to non-experts and interested graduate students.

References

[1] E.S. Barnes, *The perfect and extreme senary forms.* Canad. J. Math. **9** (1957), 235–242. MR **19:**251e

[2] A.-M. Bergé, J. Martinet, *Sur un problème de dualité lié aux sphères en géométrie des nombres.* J. Number Theory **32** (1989), no. 1, 14–42. MR **90g:**11088

[3] H.F. Blichfeldt, *The minimum value of positive quadratic forms in six, seven, and eight variables.* Math. Z. **39** (1935), 1–15.

[4] H. Cohn, A. Kumar, *The optimality of the Leech lattice.* (preprint, 2003) available via: http://research.microsoft.com/∼cohn/Leech/

[5] J.H. Conway, N.J.A. Sloane, *Sphere packings, lattices and groups.* Grundlehren **290**, Springer (1988). MR **89a:**11067

[6] L.E. Dickson, *History of the theory of numbers III, quadratic and higher forms.* Chelsea, New York (1966). MR **39:**6807c

[7] W. Ebeling, *Lattices and codes.* Vieweg (1994). MR **95c:**11084

[8] N. Hofreiter, *Über Extremformen.* Monatshefte f. Math. **40** (1933), 129–152.

[9] D.-O. Jaquet-Chiffelle, *Énumération complète des classes de formes parfaites en dimension* 7. Ann. Inst. Fourier (Grenoble) **43** (1993), no. 1, 21–55. MR **94d:**11048

[10] A. Korkine, G. Zolotareff, *Sur les formes quadratiques.* Math. Ann. **5** (1872) 581–583, ibid. **6** (1873) 366–389, ibid. **11** (1877) 242–292.

[11] H. Minkowski, *Geometrie der Zahlen.* Teubner, Leipzig (1910).

[12] J. Opgenorth, *Dual cones and the Voronoi algorithm.* Experiment. Math. **10** (2001), no. 4, 599–608. MR **2003c:**11077

[13] K.C. Stacey, *The enumeration of perfect septenary forms.* J. London Math. Soc. (2) **10** (1975), 97–104. MR **51:**5507

[14] B. B. Venkov, *Réseaux et designs sphériques.* Réseaux euclidiens, designs sphériques et formes modulaires, 10–86, Monogr. Enseign. Math. **37**, Enseignement Math., Geneva (2001). MR **2002m:**11061

[15] G.F. Voronoi, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques.* Journal f. d. Reine und Angewandte Mathematik **133** (1908), ibid. **134** (1908), ibid. **136** (1909).

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