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State spaces of operator algebras: Basic theory, orientations, and C-products*, by Erik M. Alfsen and Frederic W. Shultz, Birkhäuser Boston, Boston, MA, 2001, xii + 350 pp., $69.95, ISBN 0-8176-3890-3

Geometry of state spaces of operator algebras, by Erik M. Alfsen and Frederic W. Shultz, Birkhäuser Boston, Boston, MA, 2003, xiv + 467 pp., $74.95, ISBN 0-8176-4319-2

One of the most important auxiliary objects associated with an operator algebra is its state space. The two books under review describe the authors' solutions, obtained together with H. Hanche-Olsen and B. Iochum [1], [2], [9], to the problems: What data must be added to a state space so that the operator algebra can be recovered? and Which convex sets can arise as state spaces? As that work is now around twenty years old, they are able to present it here in a very finished form.

Operator algebras come in two varieties, C*-algebras and von Neumann algebras. (My friends who work in the non self-adjoint theory will forgive me for using the term in this way for the purposes of this review.) Concretely, a C*-algebra is a linear subspace of $B(H)$ (the space of bounded operators on a complex Hilbert space H) which is algebraically closed under operator products and adjoints and is topologically closed in norm. Concrete von Neumann algebras are defined similarly, now requiring closure in the weak* topology. There are abstract characterizations as well: C*-algebras are complex Banach algebras equipped with an involution satisfying $\|x^*x\| = \|x\|^2$, and von Neumann algebras are C*-algebras that have a Banach space predual. We write $A, B, \ldots$ for elements of concrete operator algebras and $x, y, \ldots$ for elements of abstract operator algebras.

What are they good for? The central motivation in the subject has always been physics — more about this later — but in recent decades attention has been shifting toward connections with other areas of mathematics. Indeed, operator algebras arise naturally in a wide range of settings. If Ω is a compact Hausdorff topological space, then $C(\Omega)$, the set of continuous functions from Ω into $\mathbf{C}$, is a C*-algebra. If X is a σ-finite measure space, then $L^\infty(X)$ is a von Neumann algebra. If G is a locally compact group, then its left representation on the Hilbert space $L^2(G)$ generates both a C*-algebra $C^*(G)$ and a von Neumann algebra $W^*(G)$. There are operator algebras naturally associated to foliated manifolds [5], directed graphs [6], Euclidean Bruhat-Tits buildings [13], and Poisson manifolds [15]. It sometimes seems that almost every mathematical object has a naturally associated operator algebra! Moreover, operator algebra techniques have paid off handsomely with, for example, major applications to group representations [7], the Novikov conjecture [12], Connes' index theorem for foliations [5], and Jones' work in knot theory [10].

In order to appreciate Alfsen and Shultz's contribution, one needs to know a little about states and order. Every concrete operator algebra admits a partial order defined by setting $A \le B$ if $B - A$ is positive semidefinite, i.e., $\langle (B-A)v, v\rangle \ge 0$ for all $v \in H$. This is equivalent to the abstract definition $x \le y$ if $y - x = z^*z$ for some z, so it does not depend on the representation.

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Now if $\mathcal{A}$ is an operator algebra, then a *state* on $\mathcal{A}$ is a bounded linear functional $\rho : \mathcal{A} \to \mathbf{C}$ such that $\|\rho\| = 1$ and $x \geq 0 \Rightarrow \rho(x) \geq 0$. In the von Neumann algebra case, one is mainly interested in *normal* states, meaning that ρ should be weak* continuous. The term "state" has to do with the fact that in quantum mechanics the states of a physical system are modelled by unit vectors in a Hilbert space. Given such a vector v, the map $A \mapsto \langle Av, v\rangle$ is an operator algebraic state on $B(H)$, and hence also on any concrete operator algebra nondegenerately contained in $B(H)$. Moreover, there is a sort of converse to this observation called the *GNS construction*: if ρ is any state on an operator algebra $\mathcal{A}$, one can define an inner product on $\mathcal{A}$ by setting $\langle x, y\rangle = \rho(y^*x)$; factoring out null vectors and completing then yields a Hilbert space on which $\mathcal{A}$ acts by left multiplication. The end result will be a representation in which the original state ρ arises from a unit vector in the preceding manner. Thus, states "are" unit vectors.

Following Alfsen and Shultz, from here on we assume that our C*-algebras are unital. (Readers beware: this is sufficiently emphasized in the first volume, but not in the second.) In the last paragraph we saw that states are closely related to representations. The *state space* $\mathcal{S}(\mathcal{A})$ of all states on a C*-algebra $\mathcal{A}$ can also be a useful tool via the following result. Note first that $\mathcal{S}(\mathcal{A})$ is a weak* compact convex subset of the dual Banach space $\mathcal{A}'$, and for each self-adjoint element $x \in \mathcal{A}$ we have a continuous affine function $\hat{x} : \mathcal{S}(\mathcal{A}) \to \mathbf{R}$ defined by $\hat{x}(\rho) = \rho(x)$. Now an old theorem of Kadison [11] states that the map $x \mapsto \hat{x}$ is an isometric isomorphism from $\mathcal{A}_{sa}$, the self-adjoint part of $\mathcal{A}$, onto the space of all continuous affine functions from $\mathcal{S}(\mathcal{A})$ into $\mathbf{R}$. For von Neumann algebras one considers the *normal state space* $\mathcal{S}_*(\mathcal{A})$ of weak* continuous states and gets an isometric isomorphism of $\mathcal{A}_{sa}$ with the space of bounded affine functions on $\mathcal{S}_*(\mathcal{A})$. Alas, normal state spaces are generally not compact in any useful topology, which complicates matters in the von Neumann algebra case.

By Kadison's theorem, given $\mathcal{S}(\mathcal{A})$ one can immediately recover $\mathcal{A}_{sa}$ as a real Banach space; less obviously, this actually determines $\mathcal{A}$ as a complex Banach space. Moreover, it is elementary that $x \geq 0$ if and only if $\hat{x}(\rho) \geq 0$ for all ρ and that $\hat{1}_{\mathcal{A}}(\rho) = 1$ for all ρ where $1_{\mathcal{A}}$ is the unit of $\mathcal{A}$. Thus, one also recovers the order and the unit of $\mathcal{A}$ from $\mathcal{S}(\mathcal{A})$. However, the product in $\mathcal{A}$ is not uniquely determined in general. Indeed, granting that there exist C*-algebras which are not isomorphic to their "opposite" algebra obtained by reversing the order of the product (a surprisingly difficult fact; see e.g. [16]), it follows that one cannot hope to entirely recover $\mathcal{A}$ from $\mathcal{S}(\mathcal{A})$, since $\mathcal{S}(\mathcal{A}) \cong \mathcal{S}(\mathcal{A}^{op})$ always holds.

The extra structure that needs to be added to $\mathcal{S}(\mathcal{A})$ in order to fully determine the C*-algebra $\mathcal{A}$ is simple but unexpected. It hinges on a basic fact about $\mathcal{S}(\mathcal{A})$ that every operator algebraist should know and that I did not know before reviewing these books: the smallest face of $\mathcal{S}(\mathcal{A})$ containing a given pair of distinct pure states is a line segment if the states are inequivalent, and it is a 3-ball if the states are equivalent. (A state is *pure* if it is an extreme point of $\mathcal{S}(\mathcal{A})$; two pure states are *equivalent* if they give rise via the GNS construction to unitarily equivalent representations; a *3-ball* is a convex set that is affinely isomorphic to a closed Euclidean ball in $\mathbf{R}^3$.) Now $\mathcal{A}$ induces an orientation of every 3-ball face of $\mathcal{S}(\mathcal{A})$, and these orientations vary continuously in an appropriate sense. The central result of the first volume under review states that the product in $\mathcal{A}$ can be recovered from this extra data. Even more remarkably, continuously varying orientations of the facial 3-balls of $\mathcal{S}(\mathcal{A})$ are in one-to-one correspondence with products which make $\mathcal{A}$ as an ordered Banach space into a C*-algebra. There is also a von Neumann algebra version of this result which is simpler in some respects but less quotable.

The second volume is directed toward establishing a characterization of those convex sets which arise as state spaces or normal state spaces. The solution is slightly complicated — not terribly so, but enough that I prefer not to try to describe it here. Jordan algebras, which were already a background presence in the first volume, now become crucial; their general theory occupies almost half of the second volume, so let me say a little about them.

The *Jordan product* in an operator algebra is the symmetrized product $x \circ y = \frac{1}{2}(xy + yx)$. Recall that the C*-algebra product in general cannot be recovered from $\mathcal{S}(\mathcal{A})$. However, the Jordan product can. In fact, given the underlying vector space of $\mathcal{A}$, the order plus the unit determines the Jordan product, and vice versa (and both determine the norm). This shows why Jordan algebras are relevant to the problem of characterizing state spaces of operator algebras: the natural first step is to characterize state spaces of Jordan algebras, where orientation issues are not present.

Many readers will already know that a (real) *Jordan algebra* is a vector space over $\mathbf{R}$ together with a commutative, but generally not associative, bilinear product $\circ$ that satisfies the identity $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$. Here we are interested in *JB-algebras*, which are Banach spaces equipped with a Jordan product that is compatible with the norm in a natural way, and *JBW-algebras*, which are JB-algebras that are dual Banach spaces. With the Jordan product described above, the self-adjoint part of any C*-algebra is a JB-algebra, and the self-adjoint part of any von Neumann algebra is a JBW-algebra. The work of Alfsen and Shultz et al. is the best example of an application of JB-algebras to the theory of operator algebras, and they are also useful in the study of infinite-dimensional complex domains [14] and in other areas [17].

I mentioned earlier that operator algebras have important connections to physics; let me come back to that now. In quantum mechanics a physical system is modelled by a Hilbert space, with states of the system represented by unit vectors and "observable" or physically meaningful variables represented by self-adjoint operators. Alternatively, one can take an operator algebra as primary and obtain Hilbert space representations via the GNS construction. Physicists probably find this approach most compelling in situations where intuitively "the same" physical system can be inequivalently realized on different Hilbert spaces. Formally, this means that one considers multiple representations of the same operator algebra. Specifically, this happens (1) in quantum statistical mechanics [4] where different values of temperature typically give rise to inequivalent representations of a single operator algebra and (2) in quantum field theory on curved spacetimes [18] where the vacuum representations according to different observers are generally inequivalent.

It seems to be fairly widely believed that Jordan algebras, more specifically JB-algebras, are really the proper tool in this arena, although very few people actually use them. (A rare partial exception is [8].) Alfsen and Shultz raise this point and emphasize the connection with physics repeatedly throughout their two books. Indeed, Jordan algebras were originally conceived as a model for the bounded observables of a quantum-mechanical system. The idea was to find axioms satisfied by the set of self-adjoint operators in $B(H)$ which could be justified on physical grounds, with the dual aims of conceptually clarifying the quantum theory and of broadening that theory in hopes of accommodating problematic physical systems such as those involving quantum fields. However, neither of these goals was ever really achieved. The physical justification for even the Jordan algebra axioms (in particular, that the sum of two observables must be observable) was never all that convincing, and Jordan algebras turned out to be no help at all in dealing with field theoretic issues. Still, hopes of finding a profitable use of Jordan algebras in quantum mechanics persist (see e.g. [3]).

The exposition in these two volumes is excellent, and the work they describe is certainly a tour de force, but the ultimate results characterizing state spaces seem difficult to apply because one is unlikely to be able to actually verify the stated conditions in any given case. Probably the recovery of $\mathcal{A}$ from $\mathcal{S}(\mathcal{A})$ with orientation described in the first volume will be more useful for the working operator algebraist. On the other hand, many people now feel that "matrix order", the sequence of order structures on the matrix algebras $M_n(\mathcal{A})$ for $n \in \mathbf{N}$, is in some sense more fundamental than order at the level $n = 1$. For instance, the sequence of state spaces $\mathcal{S}(M_n(\mathcal{A}))$ does completely determine the C*-algebra $\mathcal{A}$. This raises the question: Which sequences of convex sets can arise as the sequence of state spaces of matrix algebras over a C*-algebra?

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