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Spectral theory of linear operators—and spectral systems in Banach algebras, by Vladimir Müller, Birkhäuser-Verlag, Basel, 2003, x + 381 pp., CHF 248, ISBN 3-7643-6912-4

Let A be a bounded operator on a Banach space X. A scalar λ is in the spectrum of A if the operator $A - \lambda$ is not invertible. Case closed. What more is there to say? As anyone with the slightest exposure to operator theory will testify, there is so much out there that no book could come close to being comprehensive. What authors do in such situations is choose a small area or topic of interest to the author and concentrate on that. The present book is no exception.

The spectral theory of operators has its roots in the theory of matrices and in the theory of integral equations. In the early years of matrix theory, the terms "proper value", "characteristic value", "secular value", and "latent root" were all used for what we now call an eigenvalue. Laguerre constructed the exponential function of a matrix, and Frobenius obtained expansions for the resolvent operator in the neighborhood of a pole. Sylvester constructed arbitrary functions of a matrix with distinct eigenvalues. This was generalized by Buchheim to the case of multiple eigenvalues.

It was F. Reisz who extended these concepts to the space l^2. Dealing with compact operators on this space, he showed that the resolvent set is open, that the resolvent operator is analytic, and that the Cauchy integral theorem can be used in the case of a pole to obtain a projection operator commuting with the given operator.

Wiener showed that Cauchy's integral theorem and Taylor's theorem remain valid for analytic functions with values in a complex Banach space. Nagumo extended some of the results of F. Riesz to Banach algebras. Hille applied similar ideas in the study of semi-groups. Gelfand developed the ideal theory of Banach algebras. He used the contour integral to obtain idempotents. The spectral mapping theory is due to Dunford. He introduced the concepts of continuous and residual spectrum. The concepts of ascent and decent of a bounded operator are also due to him.

Fredholm studied integral equations. He gave a detailed representation of the resolvent as the quotient of two entire functions in terms of expansions in determinants. Schmidt used the method of approximating a compact operator by operators of finite rank in Hilbert space. Considerable work has been done by many authors concerning the computation and distribution of eigenvalues.

The present monograph is an attempt to organize recent progress in certain areas of spectral theory. The aim is to present a survey of results concerning various types of spectra in a unified, axiomatic way. The setting is a Banach algebra $\mathcal{A}$, and the generalized spectrum is defined to be $\sigma_R(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin R\}$, where R is a set of "nice" elements. He defines three sets of "nice" elements.

Definition 1. Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called a "regularity" if it satisfies the following conditions:

(i) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in R$ iff $a^n \in R$;

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(ii) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1$, then

$$ab \in R \text{ iff } a \in R \text{ and } b \in R.$$

This definition sets the tone for the entire book. (Unfortunately, the definition is not stated correctly there.) It is chosen in such a way that there are many natural, interesting classes satisfying its requirements. At the same time, the requirements are strong enough to produce non-trivial consequences such as the spectral mapping theorem. Spectra of n-tuples of commuting elements of a Banach algebra are described similarly by means of a notion of joint regularity.

There are two other definitions of "nice" elements.

Definition 2. A non-empty subset R of $\mathcal{A}$ is called a "lower semi-regularity" if it satisfies the following conditions:

(i) if $a \in \mathcal{A}$, $n \in \mathbb{N}$ and $a^n \in R$, then $a \in R$;

(ii) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac + bd = 1$ and $ab \in R$, then $a \in R$ and $b \in R$.

Definition 3. A subset R of $\mathcal{A}$ containing a neighborhood of the unit element is called an "upper semi-regularity" if it satisfies the following conditions:

(i) if $a \in R$ and $n \in \mathbb{N}$, then $a^n \in R$;

(ii) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac + bd = 1$ and $a, b \in R$, then $ab \in R$.

The first chapter discusses Banach algebras. In addition to the basic material, the author includes some areas and results concerning commutative Banach algebras, approximate point spectrum in commutative Banach algebras, permanently singular elements, and non-removable ideals.

In the second chapter, the author considers bounded operators on a Banach space. He pays particular attention to operator-valued functions whose ranges behave continuously. He includes results concerning operators with closed ranges, factorization of vector-valued functions, generalized inverses, local spectrum and specialized operators.

The third chapter deals with essential spectra with emphasis on Fredholm and semi-Fredholm operators. (Φ_+ is the set of those operators A with closed range $R(A)$ and null space $N(A)$ having finite dimension $\alpha(A)$. Φ_- is the set of operators A having closed range with finite codimension $\beta(A)$. $\Phi = \Phi_+ \cap \Phi_-$.) He describes some perturbation properties of these operators. He studies ascent, descent, some classes of operators defined by means of kernels and ranges, measures of non-compactness, and related operator quantities. There are several definitions of essential spectrum. The essential spectrum

$$\sigma_e(A) = \bigcap_{K \in K(X)} \sigma(A + K)$$

is near and dear to the reviewer's heart, mainly because he invented it [1], [2] (some authors erroneously refer to this set as the Weyl spectrum; Weyl had nothing to do with it). This set consists of those points of the spectrum $\sigma(A)$ of A which cannot be removed from the spectrum by the addition to A of a compact operator. It is characterized by the fact that $\lambda \notin \sigma_e(A)$ if and only if $\lambda \in \Phi_A$ and $i(A - \lambda) = 0$, where

$$\Phi_A = \{\lambda \in \mathbb{C} : A - \lambda \in \Phi\}$$

and $i(A) = \alpha(A) - \beta(A)$. Thus, it is defined by an upper semi-regularity, not by a lower semi-regularity as stated in the book. Since it is not defined by a regularity, the author gives it short shrift.

It might also be mentioned that Theorem 16.8 is due to the reviewer [3]. There he considered the quantity

$$\nu(A) = \sup_{\text{codim } M<\infty} \inf_{\substack{x\in M \\ \|x\|=1}} \|Ax\|.$$

Among other things, he proved

Theorem 4. $A \in \Phi_+(X, Y)$ *if and only if* $\nu(A) \neq 0$.

This implies Theorem 16.8 in the book.

The next chapter is devoted to the Taylor spectrum and the functional calculus for functions analytic on a neighborhood of the spectrum.

The last chapter studies orbits of operators, i.e., sequences of the form $\{T^n x : n = 0, 1, ...\}$, where x is a fixed vector. Included are results concerning the joint spectral radius, capacity, local spectral radii, orbits and weak orbits, and local capacity.

All results are presented in an elementary way. Only a basic knowledge of functional analysis, topology and complex analysis is assumed. The book is well written and contains a wealth of material. (However, there are results that have been left out.) The author made a concerted effort to simplify proofs taken from many sources. He is to be commended for doing such yeoman's service in assembling all this material in one place in a clear, concise manner.

References

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