BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, Pages 113–118
S 0273-0979(04)01038-9
Article electronically published on October 5, 2004

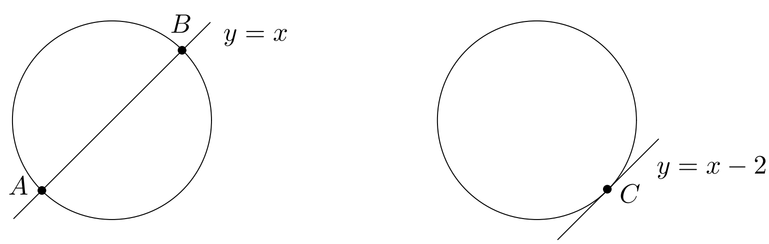
Computational algebraic geometry, by H. Schenck, London Mathematical Society Student Texts, vol. 58, Cambridge University Press, Cambridge, 2003, xiv+193 pp., $70.00 (hardback), ISBN 0-521-82964-X; $28.00 (paperback), ISBN 0-521-53650-2

Algebraic geometry is a powerful combination of algebra and geometry, with a rich history and many applications, both theoretical and practical. It also has an intimidating reputation. Hence it is important to have a variety of introductory texts at different levels of sophistication. We will see that Schenck's book offers an interesting path into this wonderful subject.

1. What is algebraic geometry?

Algebraic geometry goes back to the coordinate geometry of Descartes, which enables one to describe curves and surfaces by means of equations. When the equations involve only polynomials, then the algebra of the polynomials is deeply linked to the geometry of the corresponding curves and surfaces.

But there is more to algebraic geometry than just equations and solutions, for the solutions often exhibit "extra structure". Consider what happens when a line meets the circle $x^2 + y^2 = 2$:



The picture on the left has two points of intersection A and B, while on the right we have the single point of intersection C. However, since the line is tangent to the circle at this point, we say that C has *multiplicity two.*

Multiplicity can be defined rigorously in various ways. At the point C, we can proceed as follows. Let

$$\mathbb{R}[x,y]_C = \left\{ \frac{f}{g} \;\middle|\; f,g \in \mathbb{R}[x,y], g(C) \neq 0 \right\}$$

be the ring of rational functions in x, y that are defined at C. Inside this ring we have the ideal $\langle x^2 + y^2 - 2, y - x + 2 \rangle$ generated by $x^2 + y^2 - 2$ and $y - x + 2$. Then one can show that the quotient ring

$$\mathcal{O}_C = \mathbb{R}[x,y]_C / \langle x^2 + y^2 - 2, y - x + 2 \rangle$$

2000 *Mathematics Subject Classification.* Primary 14-01; Secondary 13Pxx, 14Qxx.

is a vector space over $\mathbb{R}$ of dimension two. This dimension is the multiplicity of C. In contrast, the rings $\mathcal{O}_A$ and $\mathcal{O}_B$ corresponding to A and B are one-dimensional vector spaces over $\mathbb{R}$, so that A and B have multiplicity one.

A classic result in algebraic geometry is *Bézout's Theorem*, which says that the number of points of intersection of two plane curves (the geometry) is the product of the degrees of the curves (the algebra). As shown by the above example, we need to count points with multiplicity in order for this to work.

However, there is more to an intersection than just its multiplicity. In algebraic topology, the first invariants were the Betti numbers. As the subject developed, people realized that Betti numbers come from more basic objects, the homology groups. Similarly, multiplicities of intersections come from more basic objects, which are rings such as the above. In recognition of this, Grothendieck and others developed the theory of schemes during the 1950s and 1960s. Given any commutative ring R, one gets the *affine scheme* $\mathrm{Spec}(R)$, which consists of a topological space and a sheaf of rings. These are then patched together to construct schemes. Scheme theory has proved to be exceptionally powerful. For example, the proof of Fermat's Last Theorem uses schemes in many places in the argument.

In this language, the intersection of $x^2+y^2=2$ and $y=x-2$ is the affine scheme $\mathrm{Spec}(\mathcal{O}_C)$. More generally, if k is a field, then a system of polynomial equations

$$f_1(x_1,\ldots,x_n)=\cdots=f_s(x_1,\ldots,x_n)=0$$

gives the affine scheme $\mathrm{Spec}(R)$, where

$$R=k[x_1,\ldots,x_n]/\langle f_1,\ldots,f_s\rangle.$$

The scheme $\mathrm{Spec}(R)$ subsumes not only the solutions of the above equations but also their multiplicities and (when k is not algebraically closed) their Galois theory.

One can also think about the ring $\mathcal{O}_C$ from a more algebraic point of view. It is a quotient of $\mathbb{R}[x,y]_C$ by the ideal $I=\langle x^2+y^2-2, y-x+2\rangle$. Thus elements of I are of the form $h_1\cdot(x^2+y^2-2)+h_2\cdot(y-x+2)$, where $h_1,h_2,\in\mathbb{R}[x,y]_C$. A natural question is whether this representation is unique. The answer is obviously no, since $0=0\cdot(x^2+y^2-2)+0\cdot(y-x+2)$ is also represented by

$$0=-(y-x+2)(x^2+y^2-2)+(x^2+y^2-2)(y-x+2). \tag{1}$$

In general, one uses *syzygies* to keep track of this lack of uniqueness. In our situation, $\mathbb{R}[x,y]_C$ is a unique factorization domain and $x^2+y^2-2, y-x+2$ are relatively prime. This makes it easy to see that the syzygy (1) generates the syzygy module. In the language of commutative algebra, we have the exact sequence

$$0\longrightarrow R\xrightarrow{\left(\begin{smallmatrix}-f_2\\ f_1\end{smallmatrix}\right)} R^2\xrightarrow{(\,f_1\ f_2\,)} R\longrightarrow\mathcal{O}_C\longrightarrow 0, \tag{2}$$

where $R=\mathbb{R}[x,y]_C$, $f_1=x^2+y^2-2$, and $f_2=y-x+2$. This is an example of a *free resolution.* (In fact, (2) is an example of a *Koszul complex*, which is a free resolution since f_1,f_2 form a *regular sequence.*)

This discussion touches on some of the many threads that make up the fabric of algebraic geometry. It is a subject that begins with equations and their solutions but also includes more sophisticated objects such as schemes, free resolutions, and a *lot* more. There are also applications to an amazing range of subjects, including number theory, coding theory, robotics, statistics, geometric modeling, chemical kinetics, and mathematical physics.

2. Learning algebraic geometry I

Every generation of algebraic geometers has had to face the challenge of learning this immense subject. When I was in graduate school in the early 1970s, a standard text was the encyclopedic *Éléments de Géométrie Algébrique* by Grothendieck and Dieudonné [10], supplemented by Mumford's *Introduction to Algebraic Geometry*, also called *The Red Book* [19]. In these books, schemes played the central role. In the late 1970s, books by Hartshorne [12], Griffiths and Harris [9], and Shafarevich [21] were published. In comparison to *EGA*, these books are user-friendly, but they still require a substantial background. Hartshorne and Shafarevich devote a lot of space to scheme theory, while Griffiths and Harris instead work over the complex numbers and take a more analytic approach. These classic texts are still in use, supplemented by more recent books in the same tradition such as [14] and [24].

Starting with Mumford's 1976 book *Algebraic Geometry I: Complex Projective Varieties* [18], a new approach developed, where the focus was purely on *varieties*, which are the solutions of polynomial equations in affine or projective space. Other books on varieties include [15], [11], and most recently [23]. At a more elementary level, we also have Reid's *Undergraduate Algebraic Geometry* [20] and the recent book [13]. These books don't say much about schemes and sheaf cohomology, nor do they go into detail about the commutative algebra which lies at the heart of much of algebraic geometry. Hence students will need to learn these important topics elsewhere. For schemes, the book *The Geometry of Schemes* [6] by Eisenbud and Harris was written explicitly to fill this gap, and *Commutative Algebra with a View toward Algebraic Geometry* [5] by Eisenbud plays a similar role with respect to commutative algebra.

These books (and other fine books not mentioned here) make it possible for readers with a variety of backgrounds to get started in algebraic geometry. However, due to the advent of computers, these books are no longer the whole story.

3. The influence of computational methods

In the mid 1960s, algorithms for manipulating polynomials and ideals were introduced into algebraic geometry, culminating in the 1980s with the appearance of programs like CoCoA [3], Macaulay (now Macaulay 2) [17], and Singular [22]. These programs use *Gröbner bases*, which are generators of ideals and modules that are well-suited for computational purposes. In particular, Gröbner basis methods enable one to compute syzygy modules and by extension free resolutions. From here, one can do an amazing variety of computations in algebraic geometry.

These programs enable one to compute sophisticated examples and test conjectures. This has been a boon to researchers, who now routinely reference these programs in their papers. Furthermore, the ideas underlying Gröbner bases have led to purely theoretical results in commutative algebra and algebraic geometry.

Because of the high complexity of the algorithms, there are still many interesting computations that are currently (and possibly permanently) out of our reach. Nevertheless, computational methods have had a profound impact on how people approach research in algebraic geometry and commutative algebra.

4. Learning algebraic geometry II

In graduate school, I studied algebraic geometry from a purely abstract point of view. The idea of computing a nontrivial free resolution never entered my head. In the late 1980s, I was thrilled to learn that one can actually compute the objects I loved. But for the younger generation, the problem is more complicated: somehow they have to learn not only the classical material but also the newer computational methods. How does one do this in a finite amount of time?

The first solution was a series of books such as [1], [2], and [4], where the focus is primarily on Gröbner basis methods. These books are very useful, though they are not sufficient by themselves—their coverage of algebraic geometry and commutative algebra is far from complete. Another solution can be found in the book [7], which assumes a substantial background and explains how to compute sophisticated objects in algebraic geometry using Macaulay 2.

More recently, we find the books *Computational Commutative Algebra 1* [16] and *A* ***Singular*** *Introduction to Commutative Algebra* [8]. These books discuss commutative algebra within the context of CoCoA (for the former) and SINGULAR (for the latter). The advantage of these books is that they combine substantial computational experience with a fairly complete introduction to commutative algebra. Their main disadvantage is that they contain relatively little algebraic geometry. But as we've seen, there are a wide variety of algebraic geometry texts one can use once the reader has the background provided by these books.

5. SCHENCK'S BOOK

Schenck's book *Computational Algebraic Geometry* represents a different solution to the dilemma of how to begin the study of algebraic geometry. As indicated by the title, it deals with computational matters, but in contrast to the books just mentioned, it includes substantial amounts of algebraic geometry. The book uses Macaulay 2, though an instructor should have no difficulty in adapting the text to CoCoA or SINGULAR.

Another difference between this book and the others is that Schenck's book is not self-contained. Many results, such as the Hilbert Nullstellensatz, Nakayama's Lemma, Schreyer's Syzygy Theorem, or Local Duality, are given without proof. For other results, the proofs are often terse. This is deliberate, with the twin goals of keeping the book short and encouraging the reader to check the details in other books. Ample and enthusiastic references are provided.

The book is intended to introduce many of the key players in algebraic geometry. In the preface, Schenck refers to his book as "Snapshots of Commutative Algebra via Macaulay 2". In the first three chapters, we encounter primary decompositions and their geometric meaning, Hilbert functions, Hilbert polynomials, degrees, dimensions, free resolutions, mapping cones, and regular sequences. There are numerous Macaulay 2 examples, though the algorithms that underlie these examples aren't explained until Chapter 4, which treats Gröbner bases. Another example of the "state early, prove later" philosophy is the Hilbert Syzygy Theorem, which appears in Chapter 2 but is proved in Chapter 8 by an elegant application of Tor and the Koszul complex.

The beginning ideas of homological algebra come in Chapter 5 with a discussion of the homology of a simplicial complex. This is followed by the corresponding algebraic object, the Stanley-Reisner ideal. The primary decomposition of this ideal has a lovely combinatorial interpretation. In this way, Schenck gives the reader a taste of the emerging field of algebraic combinatorics. The ideas of localization, Hom, and tensor are introduced in Chapter 6, while Ext, Tor, and derived functors are postponed until Chapter 8.

To me, the heart of the book is Chapter 7, which treats the geometry of points in projective space. Given three points in the plane, the crucial distinction is whether they are collinear or not. Generalizing this to higher dimensions and more points leads to sophisticated ideas such as *regularity*. The relation between the geometry and the algebra is quite remarkable and is reinforced by many examples and exercises. Anyone who reads this chapter and does the exercises will come away with a good understanding of how Hilbert functions behave and what they mean geometrically.

Chapter 9 discusses sheaves, cohomology, and curves, and Chapter 10 goes more deeply into some of the commutative algebra, including Cohen-Macaulay rings, depth, arithmetically Cohen-Macaulay varieties, and local duality. The high point of the chapter is a sketch of Stanley's proof of Motzkin's upper bound conjecture for triangulations of the sphere. The argument is algebraic, using Reisner's theorem on when the Stanley-Reisner ring is arithmetically Cohen-Macaulay.

The book is written in a terse but energetic style—Schenck is clearly in love with the material. To get the full benefit of the book, you need to read it with a computer at hand. The experience of working out the examples is intrinsic to understanding the text. The only place the terseness causes problems is in the discussion of depth on pages 147–148. The brevity also means that students need to be either strong enough to fill in the details or engaged enough to ask lots of questions. Any student who completes this book will be excited about algebraic geometry and well-equipped for further reading.

References

[1] W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, Graduate Studies in Math. **3**, AMS, Providence, 1994. MR1287608 (95g:13025)

[2] T. Becker and V. Weispfenning, *Gröbner Bases: A Computational Approach to Commutative Algebra*, Graduate Texts in Math. **141**, Springer-Verlag, New York, Heidelberg, Berlin, 1993. MR1213453 (95e:13018)

[3] CoCoA, available at `http://cocoa.dima.unige.it/`.

[4] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Second Edition, Undergraduate Texts in Math., Springer-Verlag, New York, Heidelberg, Berlin, 1997. MR1417938 (97h:13024)

[5] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Math. **150**, Springer-Verlag, New York, Heidelberg, Berlin, 1995. MR1322960 (97a:13001)

[6] D. Eisenbud and J. Harris, *The Geometry of Schemes*, Graduate Texts in Math. **197**, Springer-Verlag, New York, Heidelberg, Berlin, 2002. MR1730819 (2001d:14002)

[7] D. Eisenbud, D. Grayson, M. Stillman, and B. Sturmfels (editors), *Computations in Algebraic Geometry with Macaulay 2*, Algorithms and Computations in Math. **8**, Springer-Verlag, New York, Heidelberg, Berlin, 2002. MR1949544 (2004b:14002)

[8] G.-M. Greuel and G. Pfister, *A **Singular** Introduction to Commutative Algebra*, Springer-Verlag, New York, Heidelberg, Berlin, 2002. MR1930604 (2003k:13001)

[9] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Series in Pure & Applied Mathematics, Wiley, New York, 1978. MR0507725 (80b:14001)

[10] A. Grothendieck and J. Dieudonné, *Éléments de Géométrie Algébrique*, Publications mathématiques d'IHES **4, 8, 11, 17, 20, 24, 28, 32**, Bures-sur-Yvette, 1960–1967. Scanned versions available at `http://grothendieck-circle.org/`. The first volume (in a second edition) published as *Éléments de Géométrie Algébrique I*, Grundlehren der mathematischen

Wissenschaften **166**, Springer-Verlag, New York, Heidelberg, Berlin, 1971. MR0217083 (36:177a); MR0163909 (29:1208); MR0163910 (29:1209); MR0163911 (29:1210); MR0173675 (30:3885); MR0199181 (33:7330); MR0217086 (36:178); MR0238860 (39:220)

[11] J. Harris, *Algebraic Geometry*, Graduate Texts in Math. **133**, Springer-Verlag, New York, Heidelberg, Berlin, 1992. MR1182558 (93j:14001)

[12] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. **52**, Springer-Verlag, New York, Heidelberg, Berlin, 1977. MR0463157 (57:3116)

[13] K. Hulek, *Elementary Algebraic Geometry*, Student Mathematical Library **20**, AMS, Providence, 2003. Translation of *Elementare Algebraische Geometrie*, Vieweg, Braunschweig Wiesbaden, 2000. MR1955795 (2003m:14002)

[14] S. Iitaka, *Algebraic Geometry: An Introduction to Birational Geometry of Algebraic Varieties*, Graduate Texts in Math. **76**, Springer-Verlag, New York, Heidelberg, Berlin, 1982. MR0637060 (84j:14001)

[15] K. Kendig, *Elementary Algebraic Geometry*, Graduate Texts in Math. **44**, Springer-Verlag, New York, Heidelberg, Berlin, 1977. MR0447222 (56:5537)

[16] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 1*, Springer-Verlag, New York, Heidelberg, Berlin, 2000. MR1790326 (2001j:13027)

[17] Macaulay 2, available at `http://www.math.uiuc.edu/Macaulay2/`.

[18] D. Mumford, *Algebraic Geometry 1: Complex Projective Varieties*, Grundlehren der mathematischen Wissenschaften **221**, Springer-Verlag, New York, Heidelberg, Berlin, 1976. MR1344216 (96d:14001)

[19] D. Mumford, *The Red Book of Varieties and Schemes*, Lectures Notes in Math. **1358**, Springer-Verlag, New York, Heidelberg, Berlin, 1999. MR1748380 (2001b:14001)

[20] M. Reid, *Undergraduate Algebraic Geometry*, London Mathematical Society Student Texts **12**, Cambridge Univ. Press, Cambridge, 1988. MR0982494 (90a:14001)

[21] I. R. Shafarevich, *Basic Algebraic Geometry*, Grundlehren der mathematischen Wissenschaften **213**, Springer-Verlag, New York, Heidelberg, Berlin, 1974. MR0366917 (51:3163)

[22] Singular, available at `http://www.singular.uni-kl.de/`.

[23] K. Smith, L. Kahanpää, P. Kekäläinen, and W. Traves, *An Invitation to Algebraic Geometry*, Universitext, Springer-Verlag, New York, Heidelberg, Berlin, 2000. MR1788561 (2001k:14002)

[24] K. Ueno, *Algebraic Geometry 1, 2, and 3*, Trans. Math. Mono. **185**, **197**, and **218**, AMS, Providence, 1999, 2001, and 2003. Translation of *Diasū Kika*, Iwanami Shoten, Tokyo, 1977. MR1714917 (2000g:14001); MR1810311 (2001j:14001); MR1989998 (2004g:14001)

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