BOOK REVIEWS

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Momentum maps and Hamiltonian reduction, by J.-P. Ortega and T. S. Ratiu, Progress in Math., vol. 222, Birkhäuser Boston, Inc., Boston, MA, 2004, xxxiv + 497 pp., $69.95, ISBN 0-8176-4307-9

The term *reduction of a Hamiltonian system* is used in the literature in several different ways. It may describe the induced dynamics on an invariant manifold or the dynamics of the system projected to the space of orbits of the symmetry group. It may include a study of the structure of the orbit space or a search for a structure of the original system which is reproduced on the orbit space. Because of these ambiguities, there are different approaches to reduction which give a rich variety of descriptions of Hamiltonian dynamics.

1. Hamiltonian systems and symmetries

The phase space of a Hamiltonian system is a symplectic manifold (P, ω), that is a smooth manifold endowed with a closed, non-degenerate form ω. Non-degeneracy of ω means that, for each smooth function f on P, there exists a unique vector field X_f on P such that

$$X_f \lrcorner \omega = df. \tag{1}$$

Here $\lrcorner$ denotes the left interior product (contraction) of vector fields and forms. The vector field X_f is called the *Hamiltonian vector field* of f. Thus, we have a linear map $f \mapsto X_f$ of the space $C^\infty(P)$ of smooth functions on P into the space $\mathcal{X}(P)$ of smooth vector fields on P. If P is connected, the kernel of this map consists of constant functions on P. The symplectic form ω on P induces a bracket on $C^\infty(P)$, called the *Poisson bracket*, which is defined by

$$\{f, h\} = -X_f h, \tag{2}$$

where the right hand side denotes the derivative of h in the direction X_f. The Poisson bracket defines the structure of a Lie algebra on $C^\infty(P)$ such that the map $h \mapsto X_h$ is a Lie algebra anti-homomorphism. In other words,

$$X_{\{f,h\}} = -[X_f, X_h] \tag{3}$$

for all $f, h \in C^\infty(P)$. Equation (2) implies that, for each $f \in C^\infty(P)$ the map $h \mapsto \{f, h\}$ is a derivation of $C^\infty(P)$. Thus, the space $C^\infty(P)$ of smooth functions on a symplectic manifold (P, ω) has the structure of a *Poisson algebra.* In other

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words, $C^\infty(P)$ is an associative algebra and a Lie algebra such that the bracket operation defines a derivation of the associative algebra.

Let G be a Lie group and $\Phi : G \times P \to P : (g,p) \mapsto \Phi_g(p) = gp$ an action of G on P preserving the symplectic form ω. The action Φ is *proper* if, for every convergent sequence $\{p_n\}$ in P and every sequence $\{g_n\}$ in G such that the sequence $\{g_n p_n\}$ converges, the sequence $\{g_n\}$ converges and

$$\lim_{n\to\infty}(g_n p_n) = (\lim_{n\to\infty} g_n)(\lim_{n\to\infty} p_n). \tag{4}$$

The action Φ is *Hamiltonian* if there exists an Ad^*-equivariant *momentum map* J from P to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G such that, for every $\xi \in \mathfrak{g}$, the action on P of the one-parameter subgroup $\exp t\xi$ of G is given by translation by t along integral curves of the Hamiltonian vector field X_{J_ξ} of J_ξ, where $J_\xi(p) = \langle J(p) \mid \xi \rangle$ is the evaluation of $J(p)$ on ξ. In other words, for every $f \in C^\infty(P)$,

$$\frac{d}{dt}(\exp t\xi)^* f_{|t=0} = \{J_\xi, f\}. \tag{5}$$

From the point of view of the theory of reduction, properness and the existence of an Ad^*-equivariant momentum map are key characteristics of the G-action.

For every G-invariant function $h \in C^\infty(P)$, the group G is a *symmetry group* of the Hamiltonian system (P,ω,h). The first Noether theorem ensures that the momentum map J is constant on integral curves of the Hamiltonian vector field X_h.

2. Dirac's reduction

The first systematic study of reduction is due to P.A.M. Dirac from the point of view of quantization of field theories with constraints, for example electrodynamics and gravity, [8]. Dirac considered a family of smooth functions on a symplectic manifold (P,ω), which can be considered as components of a smooth map J from P to some vector space V. They are *primary constraint functions* defining a constraint set $C = J^{-1}(0)$. In Dirac's terminology, *constraints* are smooth functions on P that vanish on C. They form an associative ideal

$$\mathcal{I} = \{h \in C^\infty(P) \mid h_{|C} = 0\} \tag{6}$$

in $C^\infty(P)$. A function $f \in C^\infty(P)$ is *first class* if the Poisson bracket $\{f,h\} \in \mathcal{I}$ for every $h \in \mathcal{I}$. In other words, first class functions form a Lie algebra normalizer $\mathcal{N}$ of $\mathcal{I}$ in the Poisson algebra $C^\infty(P)$. If all constraints are first class, that is $\mathcal{I} \subseteq \mathcal{N}$, then the Jacobi identity for the Poisson bracket implies that $\mathcal{N}$ is a Poisson subalgebra of $C^\infty(P)$ and the quotient $\mathcal{N}/\mathcal{I}$ inherits from $\mathcal{N}$ the structure of a Poisson algebra, called the *reduced* Poisson algebra.

Second class constraints are constraints that are not first class. In the presence of second class constraints, the associative ideal $\mathcal{I}$ fails to be a Poisson ideal. Hence, taking Poisson brackets of functions in $C^\infty(P) \bmod \mathcal{I}$ does not lead to a well defined Poisson bracket. For example, if C is a symplectic submanifold of (P,ω), then all constraints are second class. In this case the pull-back ω_C of ω to C is symplectic, and $C^\infty(C)$ is the Poisson algebra of (C,ω_C). However, we cannot compute the Poisson bracket of $f_C, h_C \in C^\infty(C)$ by extending f_C and h_C to smooth functions f and h on P, evaluating the Poisson bracket $\{f,h\}$ in $C^\infty(P)$ and restricting the result to C. Dirac's recipe for quantizing of systems with second class constraints was to impose the second class constraints on the classical level and to quantize the obtained system with only first class constraints. He also gave a prescription of how to modify the Poisson bracket on $C^\infty(P)$ to obtain a new Poisson bracket $\{\cdot,\cdot\}_D$ for which all constraints are first class. The modified Poisson bracket $\{\cdot,\cdot\}_D$ is called the *Dirac bracket*. We denote by $(C^\infty(P),\{\cdot,\cdot\}_D)$ the Poisson algebra consisting of smooth functions on P with the bracket operation given by the Dirac bracket. Let $\mathcal{N}_D$ denote the normalizer of the ideal $\mathcal{I}$ in the Poisson algebra $(C^\infty(P),\{\cdot,\cdot\}_D)$. Since all constraints are first class, $\mathcal{I} \subseteq \mathcal{N}_D$, and the Jacobi identity for the Dirac bracket implies that $\mathcal{N}_D$ is a Poisson subalgebra of $(C^\infty(P),\{\cdot,\cdot\}_D)$. Hence, the quotient $\mathcal{N}_D/\mathcal{I}$ inherits from $\mathcal{N}_D$ the structure of a Poisson algebra, called the *Dirac reduced* Poisson algebra. This construction is supposed to be independent of the choice of Dirac bracket.

3. Geometric reduction

Geometric reduction was developed as an attempt to understand Dirac's reduction in geometric terms, [11], [12], [13] and [24].

If the constraint set C is a submanifold of P, then the symplectic form ω on P pulls back to a form ω_C on C. If

$$\ker \omega_C = \{u \in TC \mid u \lrcorner \omega_C = 0\}$$

has constant rank, then it is an involutive distribution on C. By Frobenius' theorem, $\ker \omega_C$ is integrable. We denote the space of maximal integral manifolds of $\ker \omega_C$ by $\bar{P} = C/\ker \omega_C$. Let $\rho : C \to \bar{P}$ be the projection associating to each $p \in C$ the maximal integral manifold of $\ker \omega_C$ through p. If $\bar{P}$ has a manifold structure such that $\rho : C \to \bar{P}$ is a submersion, then there exists a unique symplectic form $\bar{\omega}$ on $\bar{P}$ such that $\rho^*\bar{\omega} = \omega_C$. The symplectic manifold $(\bar{P},\bar{\omega})$ is a *geometrically reduced* phase space corresponding to the constraint submanifold C of (P,ω).

Dirac's principle that one should implement all second class constraints on the classical level can be interpreted as a requirement that functions on P should be pulled back to a symplectic submanifold D of P that contains the constraint manifold C as a co-isotropic submanifold. With this interpretation, the corresponding Dirac bracket coincides with the usual Poisson bracket on (D,ω_D), where ω_D is the pull-back of ω to D, [20]. Expressing D locally as the level set of second class constraints explains the Bergmann-Goldberg formula for the Dirac bracket, [4]. In this case, the Dirac reduced Poisson algebra is isomorphic to the Poisson algebra of geometrically reduced phase space $(\bar{P},\bar{\omega})$. This result is independent of the choice of symplectic submanifold D of P that contains C as a co-isotropic manifold.

The inherent weakness of geometric reduction is the necessity of making a series of assumptions: smoothness of C, constant rank of $\ker \omega_C$ and quotient manifold structure of $\bar{P}$.

4. Regular reduction

The term *regular reduction* is used to describe reduction when the constraint set is a level set of an Ad^*-equivariant momentum map $J : P \to \mathfrak{g}^*$ for a free and proper action of a Lie group G on (P,ω). In other words, $C = J^{-1}(\mu)$ for some $\mu \in \mathfrak{g}^*$.

Since the action of G on P is free and proper, the space $\bar{P} = P/G$ of orbits of G has the structure of a smooth manifold such that the orbit map $\rho : P \to \bar{P} : p \mapsto Gp$

is a submersion. Moreover, for each $\mu \in \mathfrak{g}^*$, the level set $J^{-1}(\mu)$ is a submanifold of P. Let

$$G_\mu = \{g \in G \mid Ad_g^*\mu = \mu\}$$

be the isotropy group of μ. It is a closed subgroup of G which acts on $J^{-1}(\mu)$. Since the action of G on P is free and proper, it follows that the action of G_μ on $J^{-1}(\mu)$ is free and proper. Hence, the space $\bar{P}_\mu = J^{-1}(\mu)/G_\mu$ of G_μ-orbits in $J^{-1}(\mu)$ is a manifold and the orbit map $\rho_\mu : J^{-1}(\mu) \to \bar{P}_\mu : p \mapsto G_\mu p$ is a submersion. Moreover, the kernel of the pull-back $\omega_{J^{-1}(\mu)}$ of ω to $J^{-1}(\mu)$ has constant rank, and it is spanned by Hamiltonian vector fields X_{J_ξ}, for ξ in the Lie algebra $\mathfrak{g}_\mu$ of G_μ. Thus, $\bar{P}_\mu = J^{-1}(\mu)/G_\mu$ has an induced symplectic form $\bar{\omega}_\mu$ such that $\rho_\mu^*\bar{\omega}_\mu = \omega_{J^{-1}(\mu)}$. The symplectic manifold $(\bar{P}_\mu, \bar{\omega}_\mu)$ is often called the *Marsden-Weinstein reduced space*, [14].

The power of this approach is that all the necessary assumptions of geometric reduction are consequences of the assumption that the action of G on (P, ω) is free and proper. Since $(\bar{P}_\mu, \bar{\omega}_\mu)$ can also be obtained from the constraint set $J^{-1}(\mu)$ by geometric reduction, it follows that its Poisson algebra can be obtained by Dirac reduction as described in the preceding section.

Unlike the case of geometric reduction, where we have a single constraint manifold, here we have a partition of P by a family $\{J^{-1}(\mu) \mid \mu \in \mathfrak{g}^*\}$ of constraint manifolds. One can ask the question: how do the reduced symplectic manifolds $(\bar{P}_\mu, \bar{\omega}_\mu)$ fit together?

Since the action of G on P is free and proper, the space $\bar{P} = P/G$ of orbits of G has the structure of a smooth manifold such that the orbit map $\rho : P \to \bar{P} : p \mapsto Gp$ is a submersion. The orbit space $\bar{P}_\mu = J^{-1}(\mu)/G_\mu$ is naturally diffeomorphic to the projection $\rho(J^{-1}(\mu))$ of $J^{-1}(\mu)$ to the orbit space $\bar{P} = P/G$. Hence, each $\rho(J^{-1}(\mu))$ carries a natural symplectic structure. For the sake of simplicity of notation, we may identify $\bar{P}_\mu$ with $\rho(J^{-1}(\mu))$. Since ρ and J are continuous, each $P_\mu = \rho(J^{-1}(\mu))$ is an embedded submanifold of $\bar{P}$. The family $\{\bar{P}_\mu = \rho(J^{-1}(\mu)) \mid \mu \in \mathfrak{g}^*\}$ gives a singular foliation of $\bar{P}$ in the sense of Stefan, [23].

The G-invariance of the symplectic form ω on P implies that the Poisson bracket on $C^\infty(P)$ is G-invariant. Hence, $C^\infty(P)^G$ is a Poisson subalgebra of $C^\infty(P)$, and we can use the isomorphism $\rho^* : C^\infty(\bar{P}) \to C^\infty(P)^G$ to pull back the Poisson algebra structure on $C^\infty(P)^G$ to $C^\infty(\bar{P})$. Thus, passing to the orbit space preserves the Poisson algebra structure. Let $\{\bar{f}, \bar{h}\}$ denote the Poisson bracket of $\bar{f}$, and $\bar{h}$ in $C^\infty(\bar{P})$; that is, for every $\bar{f}$, $\bar{h} \in C^\infty(\bar{P})$,

$$\rho^*\{\bar{f}, \bar{h}\} = \{\rho^*\bar{f}, \rho^*\bar{h}\}. \tag{7}$$

It should be noted that the Poisson algebra structure on $C^\infty(P)$, induced by the symplectic form ω, may be somewhat more special than that on $C^\infty(\bar{P})$ because $\bar{P}$ need not be symplectic. Manifolds, for which the ring of smooth functions have a Poisson algebra structure are called *Poisson manifolds*.

We can describe the singular foliation of $\bar{P}$ by symplectic manifolds $(\bar{P}_\mu, \bar{\omega}_\mu)$ directly in terms of the Poisson bracket on $C^\infty(\bar{P})$. To each $\bar{h} \in C^\infty(\bar{P})$, we associate a vector field $\bar{X}_{\bar{h}}$ on $\bar{P}$, called the *Poisson vector field* of $\bar{h}$, such that, for every $\bar{f} \in C^\infty(\bar{P})$,

$$\bar{X}_{\bar{h}}\bar{f} = -\{\bar{h}, \bar{f}\}. \tag{8}$$

The left hand side of (8) denotes the derivative of $\bar{f}$ in the direction $\bar{X}_{\bar{h}}$. Let $\exp t\bar{X}_{\bar{h}}$ denote the local one-parameter group of local diffeomorphisms of $\bar{P}$ generated by

$\bar{X}_{\bar{h}}$. For each $\bar{p} \in \bar{P}$, the orbit $O_{\bar{p}}$ through $\bar{p}$ of the family $\mathcal{H}(\bar{P}) = \{\bar{X}_{\bar{h}} \mid \bar{h} \in C^\infty(\bar{P})\}$ of Poisson vector fields on $\bar{P}$ is the set of points in $\bar{P}$ that can be reached from $\bar{p}$ by piecewise smooth curves in $\bar{P}$ with tangent vectors in $\mathcal{H}(\bar{P})$. In other words,

$$O_{\bar{p}} = \left\{ (\exp t_1 \bar{X}_{\bar{h}_1} \circ ... \circ \exp t_n \bar{X}_{\bar{h}_n})(\bar{p}) \mid \begin{array}{l} n \in \mathbb{N}, t_1, ..., t_n \in \mathbb{R} \ , \\ \bar{h}_1, ..., \bar{h}_n \in C^\infty(\bar{P}) \end{array} \right\}. \tag{9}$$

Each orbit $O_{\bar{p}}$ has the natural structure of a symplectic manifold. It is the symplectic leaf through $\bar{p}$ of the Poisson manifold $\bar{P}$. In general, the inclusion of $O_{\bar{p}}$ in $\bar{P}$ may be an immersion. However, in the case of a free and proper Hamiltonian action discussed here, symplectic leaves coincide with the symplectic submanifolds $(\bar{P}_\mu, \bar{\omega}_\mu)$ of $\bar{P}$.

If the action of G on (P, ω) is free and proper but does not admit a momentum map, then the constraint set C is not defined and we cannot use either Dirac's reduction or the geometric form of regular reduction described above. Nevertheless, we can use Poisson reduction to get a singular foliation of the orbit space $\bar{P}$ by symplectic orbits of the family $\mathcal{H}(\bar{P})$ of Poisson vector fields on $\bar{P}$. In this case, symplectic leaves are immersed in $\bar{P}$.

5. Singular reduction

The term *singular reduction* is used in the case of a proper action of a Lie group G on (P, ω) which is not free. For each $p \in G$, the isotropy group of p is $G_p = \{g \in G \mid gp = p\}$. The assumed properness of the action implies that all isotropy groups are compact. For each compact subgroup H of G, the set

$$P_{(H)} = \{p \in P \mid G_p \text{ is conjugate to } H \text{ in } G\} \tag{10}$$

of points of *orbit type* (H) is a local submanifold of P. This means that connected components of $P_{(H)}$ are submanifolds of P. Similarly, connected components of the projection $\bar{P}_{(H)} = \rho(P_{(H)})$ of $P_{(H)}$ under the G-orbit map ρ are manifolds. For each $\mu \in \mathfrak{g}^*$, the connected components of the intersection $J^{-1}(\mu) \cap P_{(H)}$ are submanifolds of P. Moreover, each connected component of the projection $\bar{P}_{\mu,(H)} = \rho(J^{-1}(\mu) \cap P_{(H)})$ of $J^{-1}(\mu) \cap P_{(H)}$ to $\bar{P}$ is a manifold endowed with a symplectic form $\omega_{\mu,(H)}$. Thus, singular reduction leads from a G-invariant Hamiltonian system (P, ω, h) to a collection of Hamiltonian systems $(\bar{P}_{\mu,(H)}, \bar{\omega}_{\mu,(H)}, \bar{h}_{\mu,(H)})$, where H is a compact subgroup of G, $\bar{h}_{\mu,(H)}$ is the restriction of $\bar{h} = \rho_* h$ to $\bar{P}_{\mu,(H)}$ and $\mu \in \mathfrak{g}^*$, [16].

To complete the description of the structure of the orbit space $\bar{P}$ we need to discuss how the manifolds $\bar{P}_{(H)}$ fit together. Consider a slice neighbourhood $S_p \times Gp$ of a point $p \in P$ with isotropy group $G_p = H$. It follows from the slice theorem, [17], that $\rho(S_p)$ is a neighbourhood of $\bar{p} = \rho(p)$ in $\bar{P}$, which is homeomorphic to an open subset of the space of orbits of a linear action of H on $T_p S_p$. Since H is compact, we can use the invariant theory to describe the H-orbits in this neighbourhood.

Let $\mathbb{R}[T_p S_p]^H$ denote the algebra of H-invariant polynomials on P. Hilbert's theorem ensures that $\mathbb{R}[T_p S_p]^H$ is finitely generated, ([25], p. 274). Let $\sigma_1, ..., \sigma_k$ be a Hilbert basis for $\mathbb{R}[T_p S_p]^H$. The corresponding Hilbert map

$$\sigma = (\sigma_1, ..., \sigma_k) : T_p S_p \to \mathbb{R}^k : v \mapsto (\sigma_1(v), ..., \sigma_k(v)) \tag{11}$$

induces a map $\tilde{\sigma} : T_p S_p / H \to \mathbb{R}^k : Hv \mapsto \sigma(v)$. By the Tarski-Seidenberg theorem, the range of $\tilde{\sigma}$ is a semi-algebraic set in $\mathbb{R}^k$, ([9], p. 218). Thus, the orbit space

T_pS_p/H is homeomorphic to a semi-algebraic set $\tilde{\sigma}(T_pS_p/H)$ in $\mathbb{R}^k$. Hence, the orbit space $\bar{P} = P/G$ is locally homeomorphic to an open subset of a semi-algebraic set. In the context of reduction of symmetries of Hamiltonian systems, this construction was first used in [6].

Every semi-algebraic set admits a canonical Whitney stratification into a finite number of semi-algebraic subsets. By a theorem of Bierstone, [5], this stratification coincides with the stratification of T_pS_p/H by orbit type and gives rise to a local stratification of the orbit space $\bar{P} = P/G$. Gluing these local strata with non-empty intersections together gives rise to a global stratification of $\bar{P}$ with strata given by connected components of $\bar{P}_{(H)}$ as H runs over compact subgroups of G.

Since the action of G on P is proper, the space $C^\infty(P)^G$ of G-invariant smooth functions on P separates G-orbits. Hence,

$$C^\infty(\bar{P}) = \{\bar{f} : \bar{P} \to \mathbb{R} \mid \rho^* f \in C^\infty(P)\} \tag{12}$$

defines on $\bar{P}$ the structure of a differential space in the sense of Sikorski, [19]. A theorem of Schwarz, [18], and results of [7] imply that, in the category of differential spaces, $\bar{P}$ is locally diffeomorphic to an open subset of a semi-algebraic set in $\mathbb{R}^k$. Hence, $\bar{P}$ with the differential structure $C^\infty(\bar{P})$ is a subcartesian space in the sense of Aronszajn, [1]. The category of subcartesian spaces contains the category of smoothly stratified spaces, [21], and it is easier to work with.

The space $C^\infty(\bar{P})$ of smooth functions on the orbit space inherits from $C^\infty(P)^G$ the structure of a Poisson algebra as in the case of regular reduction. The stratification of the reduced space $\bar{P}$ and its singular foliation, as described above, can be determined from the structure of $C^\infty(\bar{P})$ as follows. A *derivation* of $C^\infty(\bar{P})$ is a linear map $\bar{X} : C^\infty(\bar{P}) \to C^\infty(\bar{P})$ that satisfies Leibniz' rule:

$$\bar{X}(\bar{h}_1\bar{h}_2) = (\bar{X}\bar{h}_1)\bar{h}_2 + \bar{h}_1(\bar{X}\bar{h}_2)$$

for every $\bar{h}_1, \bar{h}_2 \in C^\infty(\bar{P})$. Since $\bar{P}$ is a subcartesian space, for every derivation $\bar{X}$ of $C^\infty(\bar{P})$ and every point $\bar{p} \in \bar{P}$ there exists a unique maximal smooth integral curve $t \mapsto c(t)$ in $\bar{P}$ through $\bar{p}$. A derivation $\bar{X}$ is a *vector field* on $\bar{P}$ if translations along maximal integral curves of $\bar{X}$ give rise to a local one-parameter group $\exp t\bar{X}$ of local diffeomorphisms of $\bar{P}$. Let $\mathcal{X}(\bar{P})$ be the family of all vector fields on $\bar{P}$. Orbits of this family, defined as in equation (9) are smooth manifolds which coincide with strata of the stratification of $\bar{P}$ discussed above. Similarly, orbits of the family of Poisson vector fields on $\bar{P}$, defined by equation (8), are symplectic leaves of the singular foliation of $\bar{P}$ by connected components of manifolds $\bar{P}_{\mu,(H)}$, [21].

As in the case of regular reduction, the stratification of the orbit space $\bar{P}$ by orbits of the family of all vector fields is independent of the existence of a momentum map. If the action of G on (P, ω) does not admit a momentum map, then we still get a singular foliation of strata by symplectic orbits of the family of Poisson vector fields on $\bar{P}$. In this case, symplectic leaves are immersed submanifolds of strata of $\bar{P}$.

6. Algebraic reduction

In the case of improper actions, invariant functions need not separate orbits. Hence, the space of invariant functions does not carry full information about orbit space. For improper actions, one has to find the right questions to ask from reduction.

The problem of commutativity of quantization and reduction was formulated in the setting of representation theory by V. Guillemin and S. Sternberg, [10]. They studied a free action of a compact group G on a compact symplectic manifold (P, ω), with an Ad^*-equivariant momentum map J, such that geometric quantization of (P, ω) gives rise to a unitary representation of G on the space of holomorphic sections of a complex line bundle over P. Since P is compact, the representation space is finite dimensional and it can be decomposed into the direct sum of irreducible unitary representations of G. Under technical assumptions, they could express multiplicities of irreducible representations appearing in the decomposition in terms of geometric data provided by reduction. This problem has been extensively studied by several authors under the assumption of compactness, which allows the techniques of algebraic geometry to be utilized; see [15] and the references quoted therein.

Algebraic reduction was introduced as an attempt to address the problem of commutativity of quantization and reduction when the group and the symplectic manifold are not compact. It has been successful in a few examples when the action of the group is not proper. Algebraic reduction at $\mu \in \mathfrak{g}^*$ gives rise to a Poisson algebra $(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu}$ of G_μ-invariant elements of the quotient of $C^\infty(P)$ by the ideal $\mathcal{J}_\mu$ generated by components of $(J - \mu)$, and G_μ is the isotropy group of μ, [2], [22]. If the action of G on P is free and proper, then the Poisson algebra $(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu}$ is canonically isomorphic to the Poisson algebra $C^\infty(\bar{P}_\mu)$ of the symplectic manifold $(\bar{P}_\mu, \bar{\omega}_\mu)$. If the action of G on P is proper but not free, the Poisson algebra $(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu}$ need not be isomorphic to the Poisson algebra obtained by singular reduction, [3].

7. The book

The book *Momentum Maps and Hamiltonian Reduction* by Juan-Pablo Ortega and Tudor S. Ratiu concentrates on regular and singular reduction.

The presentation of the subject matter starts in a way suitable for a graduate text in mathematics, with introductory chapters on manifolds and smooth structures, Lie group actions and pseudogroups and groupoids. The next two chapters deal with various momentum maps: from the equivariant momentum map, described above, to the optimal momentum map which assigns to each point $p \in P$ the orbit $O_{\bar{p}}$ through $\bar{p} = \rho(p)$ of the family $\mathcal{H}(\bar{P}) = \{\bar{X}_{\bar{h}} \mid \bar{h} \in C^\infty(\bar{P})\}$ of Poisson vector fields on $\bar{P}$.

The second half of the book deals with various forms of reduction. Here the pace accelerates and reading becomes more difficult. Statements of theorems reach over a page in length. At this stage the book becomes too difficult for beginners.

The authors missed several opportunities they had created by providing a good review of the background material. For example:

- A proof of the stratification theorem in chapter 2 would give the beginner a chance to see how the tools introduced in chapter 1 are used.
- Chapter 2 would benefit from a more detailed description of how the theorems of Weyl, Tarski-Seidenberg and Schwarz are used to show that the space of orbits of a linear action of a compact Lie group is a semi-algebraic set and why semi-algebraic sets are stratified. The authors' comment that this is "a well known result" is somewhat unsatisfactory.
- It would have been valuable from a pedagogical point of view to give a complete proof in section 2.5.4 of the smooth structure of the orbit space.

- Singular Poisson reduction leads naturally from manifolds to differential spaces. A stratified space is a special case of a differential space, but the stratification structure is much more complicated than the structure of a subcartesian differential space. The authors decided not to discuss reduced Poisson algebras of globally defined functions on the orbit space and restricted their attention to presheafs of Poisson algebras. Therefore, they were unable to describe the geometric structure of singularly reduced Poisson space directly in terms of its Poisson algebra.

In spite of the shortcomings listed above, *Momentum Maps and Hamiltonian Reduction* is a very valuable reference book. It covers a wealth of material, has a good index and a quite comprehensive bibliography.

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