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Stochastic analysis, by Ichiro Shigekawa, Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 2004, xii+182 pp., $39.00, ISBN 0-8218-2626-3

Analysts and probabilists look at the world through different eyes. Analytically a diffusion in $\mathbf{R}^n$ is described by a parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{1\le i,j\le n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{1\le i\le n} b_i(x) \frac{\partial u}{\partial x_i} \tag{1}$$

though most analysts would not put a $1/2$ in front of the first term. For a probabilist it is natural to start with an $n \times m$ matrix σ so that $\sigma\sigma^t = a$ and consider a process X_t with

$$X_t - X_0 = \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds. \tag{2}$$

Here B_t is an m-dimensional Brownian motion, first observed in the 1800's in the dance of pollen grains under a microscope. Later at the beginning of the twentieth century Einstein gave an explanation of the observed motion and Bachelier [1] put it to work as a model of the stock market. Equation (2), which is often written informally as

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt,$$

is a stochastic differential equation. If $\sigma \equiv 0$ it reduces to the ordinary differential equation

$$dX/dt = b(X_t). \tag{3}$$

One can think of (3) as describing the motion of a particle in a fluid with velocity vector $b(x)$ at the point x. From this viewpoint (2) adds fluctuations to the path given by the matrix $\sigma(X_t)$ multiplied by the increments of B_t. This is natural if X_t represents a vector of stock prices. $b(X_t)$ represents the general trend of the process while $\sigma(X_t)\, dB_t$ introduces correlated random fluctuations.

Ignoring, as we will throughout most of this review, that theorems need assumptions to make them true, the connection between the two viewpoints is that if X_t is a solution of (2), then $u(t,x) = E(f(X_t)|X_0 = x)$ solves (1). In words the last equation says that the average value of $f(X_t)$ for the process starting with $X_0 = x$ is $u(t,x)$. In symbols more familiar to analysts

$$u(t,x) = \int p_t(x,y) f(y)\, dy$$

where $p_t(x,y)$ is the fundamental solution to (1) with pole at x.

Probabilistically $p_t(x,y)$ is the transition probability of X_t, i.e., $P(X_t = y|X_0 = x)$. One can, as Dynkin [2] did in his classic book on Markov processes, start with $p_t(x,y)$ and use two theorems of Kolmogorov to construct a process with continuous paths. However, it is more satisfying for probabilists to build the process without

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borrowing results from analysis. Itô [4] was the first to do this by using a Picard iteration scheme that works when σ and b are both Lipschitz continuous. Since that time other researchers have produced solutions to (1) under weaker assumptions, but until Malliavin [5] did his work, probabilists were not able to give their own proofs of the existence of transition densities.

This achievement is the main focus of Shigekawa's book. Theorem 6.7 gives the result for nondegenerate diffusions, i.e., where the differential operator L that appears on the right-hand side of (1) can be written as

$$L = \frac{1}{2}\sum_{m=1}^{n} V_m^2 + V_0$$

where the V_m are vector fields that span the tangent space of $\mathbf{R}^n$ at each point. As the reader might guess from the way we have just written the operator, this result is just a warm-up for Theorem 6.16, which gives the result under Hormander's hypoellipticity conditions.

No analyst will think that the probabilistic derivation is a better proof that Hormander's original, but analytically inclined probabilists will enjoy the developments. Shigekawa's book is short, but it begins from the definition of the Weiner process and introduces all of the necessary machinery: stochastic integration, abstract Weiner space, Hermite polynomials, and multiple Weiner integrals, though some of the elementary results are stated and not proved. Along the way the reader meets the Ornstein-Uhlenbeck process and its semigroup, hypercontractivity, logarithmic Sobolev inequalities, Burkholder-Davis-Gundy inequalities, the L^p maximal ergodic theorem, Littlewood-Paley G-functions, and Sobolev spaces on abstract Weiner space. All of this builds up to Chapter 5, which establishes that nondegenerate Weiner functionals have a smooth density and introduces a useful integration by parts formula on Weiner space. Chapter 6 studies the behavior of stochastic integrals as a function of their starting point, introduces Malliavin's covariance matrix, and shows that it is nondegenerate under Hormander's condition.

The book is wonderfully organized with succinct chapter summaries that have allowed the reviewer to fake an understanding of its contents. With four chapters of preliminaries to wade through before the plot really starts to develop, this is not light bedtime reading. However, the parts I have read are clearly written, so I am confident that the serious reader will find it a valuable account of a result that is a striking achievement of modern stochastic analysis. It should be a useful text for an informal seminar or course on these developments. However, depending on the participant's level of sophistication it may need to be supplemented from other sources. Shigekawa is to be congratulated on writing a nice book and the AMS for bringing it out at a reasonable price.

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