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Jürgen Moser, selected chapters in the calculus of variations, by Oliver Knill, Birkhäuser Verlag, Basel-Boston-Berlin, 2003, iv+132 pp., EUR 25.68, ISBN 3-7643-2185-7

The interplay between Hamiltonian dynamics and the calculus of variations has been long and fruitful. In the last two decades an exciting new chapter has been added to this story by the development of Aubry-Mather theory. Seen from the variational standpoint, Aubry-Mather theory consists of the study of a very natural class of "minimizing solutions" to one-dimensional variational problems. Seen from Hamiltonian systems, the variational condition "minimizing" singles out an (often small) class of orbits whose global behaviour is much more tractable than the behaviour of general orbits. The book under review is based on lectures on this topic that were given by Jürgen Moser at the ETH Zürich during the summer semester 1988. It is an excellent introduction to the fundamentals of Aubry-Mather theory and prepares the reader for the numerous and far-reaching new developments in this area (see below).

Bi-infinite minimizers. The most basic problem in the calculus of variations is to minimize a variational integral. So, in the one-dimensional case, what could be more natural than to look for bi-infinite curves γ such that every compact segment of γ minimizes the variational integral in the class of curves with the same endpoints. To be more explicit, let

$$F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

be a smooth integrand and consider the variational integral

$$I_F(\gamma) = I(\gamma) = \int_a^b F(t, \gamma(t), \dot{\gamma}(t))\, dt$$

for piecewise C^1-curves $\gamma : [a, b] \to \mathbb{R}^n$. A C^1-curve $\gamma^* : \mathbb{R} \to \mathbb{R}^n$ is called *F-minimal* if, for *every* compact interval $[a, b] \subseteq \mathbb{R}$ and every piecewise C^1-curve $\gamma : [a, b] \to \mathbb{R}^n$ with $\gamma(a) = \gamma^*(a)$ and $\gamma(b) = \gamma^*(b)$, we have

$$I(\gamma^* \mid [a, b]) \leq I(\gamma).$$

In the special case $F(t, x, y) = \frac{1}{2} \sum_{i=1}^n y_i^2$, the integral $I_F(\gamma)$ is the (euclidean) energy of γ, and the nonconstant minimizers $\gamma^* : \mathbb{R} \to \mathbb{R}^n$ are precisely the affine lines in $\mathbb{R}^n$ parametrized proportionally to arclength. More generally, if

$$F(t, x, y) = \frac{1}{2} \sum_{i,j=1}^n g_{ij}(x) y_i y_j$$

is given by a (time-independent) Riemannian metric $g = (g_{ij})$ on $\mathbb{R}^n$, then a minimizer $\gamma^* : \mathbb{R} \to \mathbb{R}^n$ is a geodesic such that every finite segment $\gamma^* \mid [a, b]$ is the shortest curve joining $\gamma^*(a)$ to $\gamma^*(b)$. Such bi-infinite minimizers $\gamma^* : \mathbb{R} \to \mathbb{R}^n$ need not exist in general, even if the problem to find curves $\gamma : [a, b] \to \mathbb{R}^n$ that minimize I with fixed endpoints is solvable. If g is a complete Riemannian metric on the

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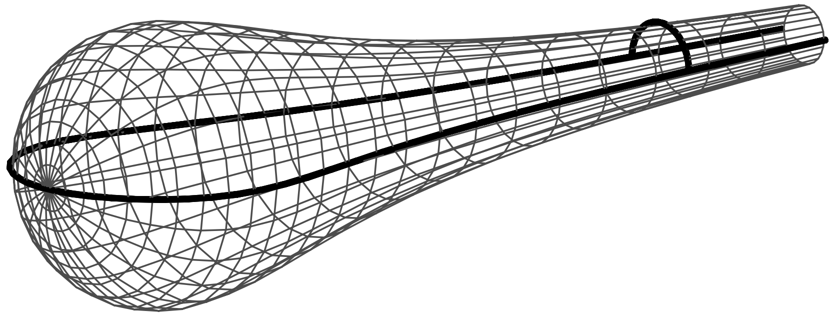

FIGURE 1. Both ends of a minimizer would have to go to infinity, but this obviously contradicts minimality.

plane of finite total area, then it is pretty obvious that such bi-infinite minimizers do not exist; see Fig. 1.

Usually, one adds some compactness and assumes that F is periodic in the first $n+1$ variables or, more generally, that $\overline{M}$ is an infinite covering space of a compact manifold M and that F is the lift to $\mathbb{R}/\mathbb{Z} \times T\overline{M}$ of an integrand on $\mathbb{R}/\mathbb{Z} \times TM$. In this case, existence of bi-infinite minimizers can be proved under natural convexity and growth conditions on F; cf. [Mat3].

One might say that Aubry-Mather theory is the study of the existence and properties of such F-minimal curves $\gamma : \mathbb{R} \to \overline{M}$. Here is a brief outline of the subject following its historic development. About 60 years before S. Aubry and J. Mather started independently to work on this topic, M. Morse published his beautiful article [Mor] in which he considered an arbitrary Riemannian metric g on a closed surface M of genus greater than one. He studies "geodesics of class A" on M, i.e. geodesics $c : \mathbb{R} \to M$ whose lifts to the universal cover $\overline{M}$ are minimal with respect to the lift $\overline{g}$ of g to $\overline{M}$. He compares these geodesics to the geodesics of a background hyperbolic metric g_0 on M, and proves that - in the universal cover $\overline{M}$ - every $\overline{g}$-minimal geodesic lies at finite distance from a hyperbolic geodesic. Conversely, for every hyperbolic geodesic $\overline{c}_0$ in $\overline{M}$ there exists a minimal $\overline{g}$-geodesic at finite distance from $\overline{c}_0$. M. Morse's fundamental estimate depends on the hyperbolicity of $\overline{g}_0$ and carries over to higher dimensions [Kli]. It plays an important role in M. Gromov's theory of hyperbolic groups and spaces [Gro]. G.A. Hedlund [Hed] extended M. Morse's results to the case of Riemannian 2-tori. Here the background metric is chosen locally Euclidean, and the minimal geodesics on the universal cover $\mathbb{R}^2$ are at finite distance from straight lines. At the end of his paper Hedlund gives an example of a Riemannian metric on the 3-torus. The point of this example is that the following fact that is crucial for his results on the 2-torus is not true on this 3-torus.

Fact: *If on a complete orientable Riemannian surface M a closed curve c has minimal length in its free homotopy class, then the same is true for all iterates c^n of c. In particular, the lifts $\overline{c} : \mathbb{R} \to \overline{M}$ of c to the universal cover are minimal.*

Hedlund's example is studied more closely in [Ban1], and it is shown that, indeed, Hedlund's results on minimal geodesics on 2-tori do not carry over to higher dimensions. Bi-infinite minimizers can become very rare in dimensions $n \geq 3$. Still, there are interesting results in this direction [Mat4], [BIK].

Monotone twist maps. In the different framework of monotone twist maps similar questions and results appeared in the work by S. Aubry (see e.g. [AL]) and J. Mather [Mat1] in the early 1980s. Monotone twist maps are area and orientation preserving maps

$$\varphi = (f, g) : \mathbb{R}/\mathbb{Z} \times [0, 1] \to \mathbb{R}/\mathbb{Z} \times [0, 1]$$

that preserve the boundary components and bend the curves $\{x\} \times [0, 1]$ in a given fixed direction, e.g. $\frac{\partial f}{\partial y}(x, y) > 0$ for all $(x, y) \in \mathbb{R}/\mathbb{Z} \times [0, 1]$. Such maps appear in 2-dimensional Hamiltonian dynamics at various occasions, e.g. as Poincaré maps of generic elliptic periodic orbits or to describe the motion of a billiard ball in a smooth strictly convex domain. The simplest examples of monotone twist maps are

$$\varphi_c(x, y) = (x + cy, y)$$

with $c > 0$. Here the phase space $\mathbb{R}/\mathbb{Z} \times [0, 1]$ is foliated by the φ_c-invariant circles $\mathbb{R}/\mathbb{Z} \times \{y\}$ on which φ_c acts as a rotation by $\alpha = cy$. Now the celebrated invariant curve theorem of KAM theory says roughly that for monotone twist maps φ that differ only very little from some φ_c and for rotation numbers $\alpha = cy$ satisfying a diophantine condition, there exists a φ-invariant closed curve close to $\mathbb{R}/\mathbb{Z} \times \{y\}$ on which φ is smoothly conjugate to a rotation by α. Such an invariant curve imposes some stability on the system since any orbit of φ that starts on one side of the curve will stay on this side of the curve for all times. On the other hand one easily finds examples of monotone twist maps φ without any homotopically nontrivial φ-invariant simple closed curve in the interior of the annulus, so-called Birkhoff regions of instability. The starting point of Aubry-Mather theory is the discovery (see [Mat1]) of compact invariant sets M_α - called Mather sets - of an arbitrary monotone twist map φ such that all orbits on M_α have rotation number α. Here α is an irrational number in the "twist interval" (α_0, α_1), where α_i denotes the rotation number of the φ-invariant circle $\mathbb{R}/\mathbb{Z} \times \{i\}$. An important point is that the Mather set M_α coincides with the KAM invariant curve of rotation number α provided such a curve exists. In many cases, however, M_α will be a Cantor set, more specifically the graph of a Lipschitz function $\varphi_\alpha : C_\alpha \to [0, 1]$, where $C_\alpha \subseteq \mathbb{R}/\mathbb{Z}$ is a Cantor set. Obviously, a Cantor set M_α will not impose the same type of stability on φ as an invariant curve, since the orbits of φ can "leak" through the gaps of M_α.

The relation between minimizers and Mather sets. The Mather sets M_α can be defined using the generating function h of φ and calculus of variations as follows. If an orbit $(x_i, y_i) = \varphi^i(x_0, y_0)$ of φ is in M_α, then the sequence $(x_i)_{i\in\mathbb{Z}}$ is minimal in the sense that every finite segment $(x_{i_0}, \dots, x_{i_1})$ of $(x_i)_{i\in\mathbb{Z}}$ minimizes $\sum_{i=i_0}^{i_1-1} h(x_i, x_{i+1})$ when compared to arbitrary sequences $(y_{i_0}, \dots, y_{i_1})$ with $y_{i_0} = x_{i_0}$ and $y_{i_1} = x_{i_1}$. So this is a discrete analogue of the condition defining F-minimality for a variational integrand F. Conversely, in generic cases M_α consists precisely of those orbits $(x_i, y_i) = \varphi^i(x_0, y_0)$ such that $(x_i)_{i\in\mathbb{Z}}$ is minimal and $\lim_{i\to\infty} \frac{x_i}{i} = \alpha$. Actually, this implies $|x_i - (x_0 + i\alpha)| < 1$ for all $i \in \mathbb{Z}$, and this corresponds to Hedlund's result that minimal geodesics of a $\mathbb{Z}^2$-periodic metric on $\mathbb{R}^2$ lie at finite distance from straight lines.

Back to the book. Jürgen Moser recognized the close connection between Aubry-Mather theory and the work by Morse and Hedlund and the problem of analyzing the minimal solutions $\gamma : \mathbb{R} \to \mathbb{R}$ for a variational integrand $F(t, x, \dot{x})$ that is periodic in the first two variables. The material described above is approximately the content of J. Moser's lecture notes. They retain much of the unique way of teaching of this great mathematician. They include the variational foundations of the theory from the 19^{th} century and lead the reader to the newest results available in the 1980s. J. Moser's insight into connections between seemingly different theories is well reflected in this book. It also contains a short survey on more recent results with an updated list of references.

To close this review I will describe the new developments which I personally find most interesting.

Local minimizers. An important new idea due to J. Mather is to use the concept of "constrained minimizers" to construct orbits that connect sets of minimizing orbits with different rotation vectors. This turned out to be very successful in two degrees of freedom [Mat2], and there are promising results also in higher dimensions; cf. [Xia], [Mat5]. Here the ultimate goal is to prove the existence of Arnold diffusion for a large class of perturbations of an integrable Hamiltonian system with more than two degrees of freedom.

Hamilton-Jacobi equations. The close relation between Aubry-Mather theory and the existence of viscosity solutions to Hamilton-Jacobi equations was exhibited and investigated by A. Fathi [Fat]. In the Riemannian setting the natural Hamilton-Jacobi equation is $|\text{grad}\, u| = 1$ where the unknown u is a real function on the manifold. If u is a C^1-solution of $|\text{grad}\, u| = 1$, then every flow line of grad u minimizes Riemannian arclength. So a global C^1-solution implies the existence of a foliation by minimal geodesics, and, conversely, the nonexistence of such foliations is related to the appearance of discontinuities in the first derivatives of solutions of $|\text{grad}\, u| = 1$.

Minimizing hypersurfaces. J. Moser realized that many of the results of Aubry-Mather theory generalize to higher dimensions if one looks for area-minimizing hypersurfaces instead of length-minimizing curves. Moser's groundbreaking paper [Mos1] gave rise to continuing activity by the reviewer and F. Auer [AB], by R. de la Llave and L. Caffarelli [CL] and by P. Rabinowitz and E. Stredulinsky [RS].

Pseudoholomorphic curves. In [Mos2] J. Moser proved a stability result of KAM type for foliations of almost complex tori by pseudoholomorphic lines. This is a perturbation result where the almost complex structure is close to a standard complex structure. The question arises if - in analogy to Aubry-Mather theory - there is a non-perturbative theory valid for pseudoholomorphic curves in an arbitrary tame almost complex 4-torus. Here the restriction to dimension 4 will be essential for the finer aspects of the theory since it provides one with the topological tool "positivity of intersection numbers between pseudoholomorphic curves". This should replace the fact that curves in surfaces locally separate the space that is crucial for Aubry-Mather theory in two degrees of freedom. A first step in this direction is [Ban2].

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Victor Bangert

Mathematisches Institut der Universität Freiburg

E-mail address: `bangert@email.mathematik.uni-freiburg.de`