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PREVALENCE

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ABSTRACT. Many problems in mathematics and science require the use of infinite-dimensional spaces. Consequently, there is need for an analogue of the finite-dimensional notions of 'Lebesgue almost every' and 'Lebesgue measure zero' in the infinite-dimensional setting. The theory of prevalence addresses this need and provides a powerful framework for describing generic behavior in a probabilistic way. We survey the theory and applications of prevalence.

CONTENTS

1. INTRODUCTION

A central problem in mathematics is the description of generic behavior. Given a set of objects, what is the nature of a generic element of the set? This question applies to diffeomorphisms, Riemannian metrics, algebraic varieties, function spaces, and linear operators, just to name several examples. The perturbative strategy provides a powerful method of inquiry. One learns a great deal about a mathematical object by studying how it behaves under small perturbations. Indeed, the success

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of analysis can be attributed in part to its ability to handle the linear problems that arise in the study of perturbations. Often the study of perturbations leads to nonlinear problems, a phenomenon which accounts for many fascinating aspects of the theory of dynamical systems.

In order to describe generic behavior, the notion of genericity must be made precise. Lebesgue measure leads to a natural notion in the finite-dimensional case. A subset $G \subset \mathbb{R}^n$ is said to be measure-theoretically generic if $\mathbb{R}^n \setminus G$ has zero Lebesgue measure. The ubiquity of problems involving infinite-dimensional spaces strongly suggests the need for an analogue of the finite-dimensional notions of 'Lebesgue almost every' and 'Lebesgue measure zero' in the infinite-dimensional setting. Mathematicians have long desired such an analogue. We would like to preserve the following properties of Lebesgue measure zero sets:

(1) A measure zero set has no interior ('almost every' implies dense).
(2) Every subset of a measure zero set also has measure zero.
(3) A countable union of measure zero sets also has measure zero.
(4) Every translate of a measure zero set also has measure zero.

Hunt, Sauer, and Yorke [24, 25] formulate the theory of prevalence, a measure-theoretic notion of genericity for infinite-dimensional vector spaces. Prevalence provides the needed analogue of 'Lebesgue almost every'. Lebesgue measure itself cannot be generalized to infinite-dimensional vector spaces. Consequently, one defines prevalence in terms of the class of compactly supported probability measures. Let X be a completely metrizable topological vector space. A Borel subset $S \subset X$ is said to be ***shy*** if there exists a compactly supported Borel probability measure μ such that $\mu(S + x) = 0$ for all $x \in X$. The complement of a shy set is called a ***prevalent*** set. One may think of μ as describing a family of perturbations in X. In this sense, $E \subset X$ is prevalent if for all $x \in X$, the process of choosing a perturbation at random with respect to μ and adding it to x yields a point in E with probability one. Shy sets satisfy the aforementioned desirable properties of Lebesgue measure zero sets, and shyness is equivalent to Lebesgue measure zero in $\mathbb{R}^n$.

Lacking a measure-theoretic notion of genericity, mathematicians traditionally invoke the topological notion of genericity based on the category theorem of Baire. A countable intersection of open, dense sets is said to be a ***residual***, or topologically generic, set. The Baire category theorem asserts that residual subsets of complete metric spaces (or, more generally, Baire spaces) are dense. Prevalence and topological genericity do not coincide. This is evident even in $\mathbb{R}^n$, where residual sets may have zero Lebesgue measure. To see this, let $\{x_k\}$ be a countable dense subset of $\mathbb{R}^n$ and fix $\epsilon > 0$. For each k, put an open ball around x_k of Lebesgue measure $\epsilon/2^k$. The union of these balls is open and dense in $\mathbb{R}^n$ but has total measure at most ϵ. Intersecting such sets over a sequence of values of ϵ tending to zero, we obtain a residual set with zero Lebesgue measure. We now consider several natural examples that indicate the importance of a measure-theoretic perspective.

Example 1.1 (Normal Numbers)**.** For each $\omega \in [0, 1]$, let $(d_k(\omega))$ denote the sequence of binary digits in the dyadic expansion of ω. In terms of the digit sequence, $\omega = \sum 2^{-k} d_k(\omega)$. Define $f_n : [0, 1] \to [0, 1]$ by

$$f_n(\omega) = \frac{1}{n} \#\{1 \leqslant k \leqslant n : d_k(\omega) = 1\}.$$

The value $f_n(\omega)$ is the fraction of 1s that appear in the first n digits of the dyadic expansion of ω. Viewed probabilistically, the digit functions d_k are independent, identically distributed random variables. Therefore, $f_n(\omega) \to 1/2$ as $n \to \infty$ for Lebesgue almost every $\omega \in [0,1]$ by the strong law of large numbers. This is the Borel normal number theorem.

While statistically regular behavior is measure-theoretically generic, irregular behavior is residual. We now construct a residual set V such that for $\omega \in V$,

$$\liminf_{n\to\infty} f_n(\omega) = 0, \quad \limsup_{n\to\infty} f_n(\omega) = 1.$$

Let $\alpha > 1/2$. For each $n \in \mathbb{Z}^+$, define the sets

$$W_{\alpha,n} = \{\omega \in [0,1] : f_i(\omega) \geqslant \alpha \text{ and } f_j(\omega) \leqslant 1-\alpha \text{ for some } i, j \geqslant n\},$$
$$V_{\alpha,n} = \mathrm{Interior}(W_{\alpha,n}).$$

The set $V_{\alpha,n}$ is open and dense in $[0,1]$ for each n. Intersecting over $n \in \mathbb{Z}^+$, we obtain a residual set V_α such that for $\omega \in V_\alpha$,

$$\liminf_{n\to\infty} f_n(\omega) \leqslant 1-\alpha, \quad \limsup_{n\to\infty} f_n(\omega) \geqslant \alpha.$$

Moreover, intersecting the sets V_α over a sequence of values of α converging to 1, we obtain the required residual set V.

The following example uses the same construction as the above example.

Example 1.2 (A residual set on which the Lyapunov exponent does not exist)**.** Lyapunov exponents play a crucial role in smooth ergodic theory. Let M be a compact finite-dimensional Riemannian manifold and let $f : M \to M$ be a smooth map. For $(x,v) \in TM$, $\|v\| \neq 0$, the number

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\|,$$

should the limit exist, is called the Lyapunov exponent of f at (x,v). We say that $x \in M$ is a ***regular point*** for f if there are Lyapunov exponents $\lambda_l(x) > \cdots > \lambda_1(x)$ and a collection of subspaces

$$\{0\} = E_0(x) \subset E_1(x) \subset \cdots \subset E_{l-1}(x) \subset E_l(x) = T_xM$$

such that for $1 \leqslant j \leqslant l$ and $v \in E_j(x) \setminus E_{j-1}(x)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\| = \lambda_j(x).$$

From a measure-theoretic point of view, regularity is generic. The multiplicative ergodic theorem of Oseledec asserts that the set of regular points for f has full measure with respect to any f-invariant Borel probability measure on M. However, the set of regular points is frequently quite small in the topological sense.

The following simple example illustrates that Lyapunov exponents may not exist for a residual set of points. Let $p > 1$ and $q > 1$ satisfy $(1/p) + (1/q) = 1$ and $p > q$. Consider the map $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} px, & \text{if } 0 \leqslant x < \frac{1}{p}; \\ qx - \frac{q}{p}, & \text{if } \frac{1}{p} \leqslant x \leqslant 1. \end{cases}$$

Lebesgue measure is invariant under f and the transformation is ergodic. Therefore, the Birkhoff pointwise ergodic theorem implies that for Lebesgue almost every $x \in [0,1]$,

$$\lim_{n\to\infty} \frac{1}{n} \log(f^n)'(x) = \frac{\log(p)}{p} + \frac{\log(q)}{q}.$$

Nevertheless, there exists a residual set R on which the Lyapunov exponent does not exist. More precisely, for $x \in R$ we have

$$\liminf_{n\to\infty} \frac{1}{n} \log(f^n)'(x) = \log(q), \quad \limsup_{n\to\infty} \frac{1}{n} \log(f^n)'(x) = \log(p).$$

Example 1.3 (Schrödinger type operators)**.** The presence of a singular continuous spectrum for a Schrödinger type operator is often considered accidental and even undesirable. However, Simon [47] shows that this phenomenon is Baire generic. Let $C_0(\mathbb{R}^n)$ denote the space of continuous real-valued functions on $\mathbb{R}^n$ vanishing at infinity in the uniform norm. For $V \in C_0(\mathbb{R}^n)$, let $\mathcal{S}(V)$ be the Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^n)$. Simon proves that for a residual set of functions V, the operator $\mathcal{S}(V)$ has purely singular continuous spectrum on all of $(0,\infty)$.

Prevalence has been constructed to reflect probabilistic intuition. Rather than topological notions of genericity, prevalence should be used when one desires a probabilistic result on the likelihood of a given property in a function space.

In this article, we survey the theory and applications of prevalence. Prevalence has been formulated for topological vector spaces [24, 25], abelian topological groups [11], convex subsets of topological vector spaces [2], and nonlinear spaces [26, 27]. We discuss these theoretical developments and examine an interesting collection of applications. The applications examined here emphasize the role of prevalence in dynamical systems and analysis.

2. Notation

- Let λ denote Lebesgue measure on $\mathbb{R}$.
- For $n \geqslant 1$, let λ_n denote Lebesgue measure on $\mathbb{R}^n$.
- For a finite-dimensional subspace V of a topological vector space, let λ_V denote Lebesgue measure on V.
- Let $B(x,\epsilon)$ denote the open ball centered at x of radius ϵ.

3. Prevalence in linear spaces

Let X be a topological vector space. A sound theory of genericity for topological vector spaces should satisfy the following ***genericity axioms***.

Axiom 1. *A generic subset of X is dense in X.*

Axiom 2. *If $L \supset G$ and G is generic, then L is generic.*

Axiom 3. *A countable intersection of generic sets is generic.*

Axiom 4. *Every translate of a generic set is generic.*

Axiom 5. *A subset G of $\mathbb{R}^n$ is generic if and only if G is a set of full Lebesgue measure in $\mathbb{R}^n$.*

The topological theory of genericity fails to satisfy Axiom 5. One might try to define a measure-theoretic notion of genericity on a given topological vector space in terms of a specific translation-invariant measure. This approach leads to serious difficulties. In an infinite-dimensional, separable Banach space, every translation-invariant Borel measure which is not identically zero assigns infinite measure to all open sets. To see this, let B denote an infinite-dimensional, separable Banach space. Suppose that for some $x \in B$ and some $\epsilon > 0$, the open ball $B(x, \epsilon)$ centered at x of radius ϵ has finite measure. Since B is infinite-dimensional, $B(x, \epsilon)$ contains infinitely many disjoint open balls of radius $\epsilon/4$. Each of these balls has the same measure and the sum of their measures is finite, so the $(\epsilon/4)$-balls must all have zero measure. The separable space B may be covered by countably many $(\epsilon/4)$-balls, and therefore B has measure zero.

In the absence of a reasonable translation-invariant measure, one might nevertheless hope to define a measure such that the property of having full measure is preserved by translation; such a measure is called quasi-invariant. In $\mathbb{R}^n$, any measure that is absolutely continuous with respect to Lebesgue measure and has a positive density is quasi-invariant. However, for an infinite-dimensional, locally convex topological vector space, a σ-finite, quasi-invariant measure defined on the Borel sets must be identically zero.

These difficulties suggest that looking for an analogue of Lebesgue measure on function spaces will not bear fruit. Hunt, Sauer, and Yorke [24, 25] develop the theory of prevalence for linear spaces by invoking a characterization of Lebesgue zero measure that extends to function spaces. This characterization depends upon the Tonelli theorem. A Borel subset $S \subset \mathbb{R}^n$ has zero Lebesgue measure if and only if there exists ***some*** compactly supported Borel probability measure μ on $\mathbb{R}^n$ such that $\mu(S + x) = 0$ for every $x \in \mathbb{R}^n$.

Constructed to reflect probabilistic intuition, prevalence satisfies Axioms 1-5. Although we present the theory for linear spaces, it extends naturally to topological groups. Christensen [11] defines the notion of 'Haar zero set' for abelian Polish groups, topological abelian groups with a complete separable metric. Borwein and Moors [10] generalize the work of Christensen by treating the nonseparable case.

We motivate the theory by considering how the notion of 'Lebesgue almost every' on $\mathbb{R}^n$ can be formulated in terms of the same notion on lower-dimensional spaces. Foliate $\mathbb{R}^n$ by k-dimensional planes. These planes may be thought of as translates of $\mathbb{R}^k \subset \mathbb{R}^n$ by elements of $\mathbb{R}^{n-k}$. If 'Lebesgue almost every' translate of $\mathbb{R}^k$ intersects a Borel set $E \subset \mathbb{R}^n$ in full k-dimensional Lebesgue measure, then E has full n-dimensional Lebesgue measure by the Fubini/Tonelli theorem.

If $\mathbb{R}^n$ is replaced by an infinite-dimensional space X, we cannot formulate the same condition because the space of translations of a k-dimensional subspace is infinite-dimensional. However, we can impose the stronger condition that *every* translate of the subspace intersects E in a set of full Lebesgue measure. A preliminary notion of prevalence is obtained by declaring that a Borel set $E \subset X$ is prevalent if there exist some finite k and some k-dimensional subspace V such that every translate of V intersects E in a set of full k-dimensional Lebesgue measure. In order to ensure that a countable intersection of prevalent sets is prevalent (Axiom 3), we must enlarge the space of measures under consideration beyond Lebesgue measure supported on finite-dimensional subspaces.

Definition 3.1. Let X be a completely metrizable topological vector space. A Borel set $E \subset X$ is said to be ***prevalent*** if there exists a Borel measure μ on X such that

(1) $0 < \mu(C) < \infty$ for some compact subset C of X, and

(2) the set $E + x$ has full μ-measure (that is, the complement of $E + x$ has measure zero) for all $x \in X$.

More generally, a subset F of X is prevalent if F contains a prevalent Borel set. The complement of a prevalent set is called a ***shy*** set. If $F \subset X$ is prevalent, we say that ***almost every*** element of X lies in F.

Condition 1 ensures that it is sufficient to consider compactly supported Borel probability measures. If X is separable, then all Borel measures which take on a value other than 0 and ∞ satisfy Condition 1 [41]. The measure μ may be Lebesgue measure on a finite-dimensional subspace of X. Such measures may be used effectively in many applications. More generally, one may think of μ as describing a family of perturbations in X. In this sense, E is prevalent if for all $x \in X$, the process of choosing a perturbation at random with respect to μ and adding it to x yields a point in E with probability one.

Definition 3.2. Prevalence may be viewed as a form of measure-theoretic transversality. A Borel measure μ is said to be ***transverse*** to a Borel set $S \subset X$ if

(1) $0 < \mu(C) < \infty$ for some compact subset C of X, and
(2) $\mu(S + x) = 0$ for all $x \in X$.

The Borel set S is therefore shy if there exists a measure transverse to S. Philosophically, the less concentrated a measure is, the more sets it is transverse to. At one extreme, a point mass is transverse to only the empty set. We now verify that prevalence satisfies the genericity axioms.

Proposition 3.3 ([24])**.** *The theory of prevalence satisfies Axioms* 1-5*.*

Before proving this proposition, we present an important fact about shy sets.

Proposition 3.4. *Every shy Borel set $S \subset X$ has a transverse probability measure with compact support. Furthermore, the support of this measure can be taken to have arbitrarily small diameter.*

Proof. Let μ be a measure transverse to S. Let $\epsilon > 0$. By Condition 1 of Definition 3.2, μ can be restricted to a compact subset C of finite, positive measure, and this restriction is also transverse to S. Since C is compact, it can be covered by finitely many balls of radius ϵ. At least one of these balls must intersect C in a set of positive measure. Let $B(x, \epsilon)$ denote one such ball. The restriction of μ to the compact set $C \cap \overline{B}(x, \epsilon)$, normalized so that $\mu(C \cap \overline{B}(x, \epsilon)) = 1$, is transverse to S. $\square$

Proof of Axiom 1*.* We prove that all prevalent sets are dense. Let $S \subset X$ be a shy set. We will show that S has no interior in X. It suffices to assume that S is Borel. Suppose by way of contradiction that S contains an open ball B. By Proposition 3.4, S has a transverse probability measure μ supported on a compact set of diameter much smaller than the diameter of B. For $x \in X$ such that $\mathrm{supp}(\mu) \subset B + x$, $\mu(S + x) = 1$. This contradiction completes the proof. $\square$

Proof of Axiom 2*.* This property follows immediately from the definition of prevalence. $\square$

Proof of Axiom 3*.* We prove that a countable intersection of prevalent sets is prevalent. We will show that a countable union of shy sets is shy. First, suppose S and

T are shy Borel subsets of X. We show that $S \cup T$ is shy by constructing a measure transverse to $S \cup T$. By Proposition 3.4, there exist compactly supported Borel probability measures μ and ν transverse to S and T, respectively. Define the convolution $\mu * \nu$ by

$$\mu * \nu(A) = \mu \times \nu(A^{\Sigma}),$$

where $A^{\Sigma} = \{(x, y) \in X \times X : x + y \in A\}$. The convolution $\mu * \nu$ is a probability measure with compact support because its support is contained in the continuous image of $\operatorname{supp}(\mu) \times \operatorname{supp}(\nu)$ under the mapping $(x, y) \mapsto x + y$. By the Fubini theorem, for every $x \in X$ we have

$$\mu * \nu(S + x) = \int_X \mu(S + x - y)\, d\nu(y) = 0,$$
$$\mu * \nu(T + x) = \int_X \nu(T + x - y)\, d\mu(y) = 0.$$

Therefore, $\mu * \nu$ is transverse to both S and T and hence to their union.

Thus far we have established that the union of two shy sets is shy. It follows inductively that the union of a finite collection of shy sets is shy. The countable case requires the theory of infinite product measures. See [13] for a presentation of this theory. Let $\{S_i : i \in \mathbb{Z}^+\}$ be a collection of shy Borel subsets of X. By Proposition 3.4, there exist Borel probability measures μ_i transverse to S_i with compact support C_i such that $\operatorname{diam}(C_i) \leqslant 2^{-i}$. By translating these measures, we may assume that each C_i contains the origin. We construct the infinite convolution of the μ_i and we show that this convolution is transverse to each S_i and hence to their union.

The infinite product $C^{\Pi} = C_1 \times C_2 \times \cdots$ is compact by the Tychonoff theorem and has a probability measure $\mu^{\Pi} = \mu_1 \times \mu_2 \times \cdots$ defined on its Borel subsets. Since X is complete and each element of C_i lies at most 2^{-i} away from zero, the summation mapping from $C^{\Pi} \to X$ defined by

$$(c_i) \mapsto \sum_{i=1}^{\infty} c_i$$

is well-defined and continuous. Here we have made use of the fact that the topology of X is generated by a complete metric for the first time. The image C of C^{Π} under the summation mapping is compact, and μ^{Π} induces a measure μ supported on C. The measure μ is given by $\mu(A) = \mu^{\Pi}(A^{\Sigma})$, where

$$A^{\Sigma} = \left\{(c_i) \in C^{\Pi} : \sum_{i=1}^{\infty} c_i \in A\right\}.$$

We show that μ is transverse to each S_i. Fix $i \in \mathbb{Z}^+$. Write $\mu^{\Pi} = \mu_i \times \nu_i^{\Pi}$ where

$$\nu_i^{\Pi} = \mu_1 \times \cdots \times \mu_{i-1} \times \mu_{i+1} \times \cdots .$$

Let ν_i be the compactly supported probability measure induced by ν_i^{Π} under the summation mapping. We have that $\mu = \mu_i * \nu_i$. Since μ_i is transverse to S_i, $\mu_i * \nu_i$ is transverse to S_i by the Fubini theorem. □

Proof of Axiom 4. The fact that every translate of a prevalent set is prevalent follows immediately from the definition of prevalence. □

Proof of Axiom 5. We verify that $S \subset \mathbb{R}^n$ is shy if and only if it has Lebesgue measure zero. It suffices to assume that S is Borel, because every Lebesgue measure zero set is contained in a Borel set of measure zero. If S has Lebesgue measure zero, then $\lambda_n(S+x) = \lambda_n(S) = 0$ for each $x \in \mathbb{R}^n$, so S is shy because λ_n is transverse to S. If S is shy, there exists a compactly supported Borel probability measure μ such that $\mu(S+x) = 0$ for all $x \in \mathbb{R}^n$. Applying the Tonelli theorem, we have

$$0 = \int_{\mathbb{R}^n} \mu(S-y)\, d\lambda_n(y) = \int_{\mathbb{R}^n} \lambda_n(S-x)\, d\mu(x) = \lambda_n(S)\mu(\mathbb{R}^n) = \lambda_n(S).$$

□

In order to prove that a set is prevalent, a measure μ satisfying Conditions 1 and 2 of Definition 3.1 must be found. A good candidate for μ is Lebesgue measure supported on some finite-dimensional subspace of X.

Definition 3.5. A finite-dimensional subspace $P \subset X$ is said to be a ***probe*** for a set $F \subset X$ if there exists a Borel set $E \subset F$ such that $E + x$ has full λ_P-measure for all $x \in X$. The existence of a probe is a sufficient (but not necessary) condition for a set F to be prevalent. We say that $F \subset X$ is ***k-prevalent*** if there exists a k-dimensional probe for F. The set $S \subset X$ is said to be ***k-shy*** if $X \setminus S$ is k-prevalent.

We now examine several examples. In each case, prevalence is established using a probe.

Example 3.6. Almost every function $f \in L^1[0,1]$ satisfies

$$\int_0^1 f(x)\, dx \neq 0.$$

To see this, let P denote the one-dimensional subspace of $L^1[0,1]$ consisting of the constant functions. Recall that λ_P denotes Lebesgue measure on P. For any $f \in L^1[0,1]$,

$$\int_0^1 (f(x) + \alpha)\, dx = 0$$

for one and only one value of α, namely $\alpha = -\int f(x)\, dx$. Therefore, P is a probe because

$$\lambda_P\left(\left\{\alpha \in \mathbb{R} : \int_0^1 (f(x)+\alpha)\, dx = 0\right\}\right) = 0$$

for every $f \in L^1[0,1]$. The next example also admits a one-dimensional probe.

Example 3.7. For $1 < p \leqslant \infty$, almost every sequence $(a_i) \in \ell^p$ has the property that $\sum_{i=1}^{\infty} a_i$ diverges. To see this, let v be the harmonic sequence $(1/i)$ and let P be the subspace of ℓ^p spanned by v. For any sequence $(a_i) \in \ell^p$, the series

$$\sum_{i=1}^{\infty} a_i + \ell \sum_{i=1}^{\infty} \frac{1}{i}$$

converges for at most one value of $\ell \in \mathbb{R}$. Therefore, P is a probe for each $1 < p \leqslant \infty$.

Example 3.8 ([20])**.** Almost every function in $C[0,1]$ is nowhere differentiable.

One cannot construct a one-dimensional probe in this case. Suppose there exists $g \in C[0,1]$ with the property that for each $f \in C[0,1]$, the function $f + \gamma g$ is nowhere differentiable for almost every $\gamma \in \mathbb{R}$. Setting $f(x) = -xg(x)$, $f + \gamma g$ is differentiable at $x = \gamma$ for every $\gamma \in [0,1]$. Hunt proves the result by constructing a two-dimensional probe. The set of nowhere differentiable functions is thus 2-prevalent in $C[0,1]$ but not 1-prevalent.

Example 3.9 ([40])**.** Let n and m be positive integers and let $A \subset \mathbb{R}^n$ be an unbounded set. The image $\varphi(A)$ is unbounded for almost every $\varphi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$.

Proof. It suffices to assume $m = 1$. We show that the set

$$V = \{\varphi \in C^1(\mathbb{R}^n, \mathbb{R}) : \varphi(A) \text{ is bounded}\}$$

is shy. Let $P = (\mathbb{R}^n)^*$, the dual of $\mathbb{R}^n$. We show that P is a probe. For $v \in \mathbb{R}^n$, let v^* be the element of $(\mathbb{R}^n)^*$ defined by $v^*(w) = (v, w)$. Let $\{e_i : i = 1, \dots, n\}$ be the standard basis of $\mathbb{R}^n$. For $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$ and $(\alpha_i) \in \mathbb{R}^n$, write

$$\varphi_\alpha = \varphi + \sum_{i=1}^{n} \alpha_i e_i^*.$$

If V is not shy, there exists some $g \in C^1(\mathbb{R}^n, \mathbb{R})$ such that

$$\lambda_n\big(\{\alpha \in \mathbb{R}^n : g_\alpha(A) \text{ is bounded}\}\big) > 0.$$

Without loss of generality, we may assume that $g \in V$; that is, $g(A) \subset [-d, d]$ for some $d > 0$. Since $\{\alpha \in \mathbb{R}^n : g_\alpha(A) \text{ is bounded}\}$ has positive n-dimensional measure, there must exist n linearly independent vectors $\{v_i\}$ such that $g + v_i^* \in V$; that is, there exist scalars $c_i > 0$ such that the functions $g + v_i^*$ map A into $[-c_i, c_i]$. Thus A is contained in the set

$$\bigcap_{i=1}^{n} (v_i^*)^{-1}\big([-c_i - d, c_i + d]\big),$$

a bounded solid polygon. This contradiction establishes that V is shy. $\square$

Example 3.10 ([24])**.** This result establishes the prevalence of hyperbolicity for periodic orbits of maps. We say that a periodic point x of period p for a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is ***hyperbolic*** if $Df^p(x)$ has no eigenvalues of norm 1. For $1 \leqslant k \leqslant \infty$, almost every C^k map on $\mathbb{R}^n$ has the property that all of its periodic points are hyperbolic. One establishes this result by first fixing the period p and proving that almost every C^k map on $\mathbb{R}^n$ has the property that all of its periodic points of period p are hyperbolic. The space of polynomial functions of degree at most $2p-1$ serves as a probe. Intersecting over $p \in \mathbb{N}$ finishes the argument because the countable intersection of prevalent sets is prevalent.

4. Relative prevalence

Suppose that a parameter is constrained to vary over a shy subset of the ambient vector space. This situation arises frequently in dynamical systems and economics. In order to remain applicable, the notion of prevalence must be relativized. Let C be a convex subset of a completely metrizable topological vector space X. We wish to define shyness and prevalence relative to C. One could relativize the notion of shyness simply by adding the requirement that the measure μ be supported on C. However, this relativization does not lead to a theory that satisfies relative versions of Axioms 1-5. To gain some understanding of the difficulty, consider the proof that the union of two shy sets is shy. Let S_1 and S_2 be shy Borel subsets of X. There exist compactly supported measures μ_1 and μ_2 such that

$$\mu_1(S_1 + x) = 0 = \mu_2(S_2 + x)$$

for every $x \in X$. The convolution $\mu_1 * \mu_2$ has compact support and satisfies

$$\mu_1 * \mu_2\big((S_1 \cup S_2) + x\big) = 0$$

for every $x \in X$. Therefore, $S_1 \cup S_2$ is shy. Notice that the convolution is supported on $S_1 + S_2$. Consequently, if μ_1 and μ_2 are supported on C, then $\mu_1 * \mu_2$ is supported on $C + C$ and not on C.

Anderson and Zame [2] have formulated relative notions of shyness and prevalence that satisfy relative versions of Axioms 1-5. The definition is quite subtle due to the delicate issue of supports.

Definition 4.1 ([2])**.** Let X be a topological vector space and let $C \subset X$ be a convex subset which is completely metrizable in the relative topology induced from X. Let $c \in C$. We say that a universally measurable set $E \subset C$ is ***shy in C at c*** if for each $1 \geqslant \delta > 0$ and each neighborhood W of 0 in X, there is a regular Borel probability measure μ on X with compact support such that

$$\operatorname{supp} \mu \subset \big(\delta(C - c) + c\big) \cap (W + c)$$

and $\mu(E + x) = 0$ for every $x \in X$. Notice that by convexity,

$$\delta(C - c) + c = \delta C + (1 - \delta)c \subset C.$$

We say that E is ***shy in C*** if it is shy at each point $c \in C$. An arbitrary subset $F \subset C$ is shy in C if it is contained in a shy universally measurable set. A subset $Y \subset C$ is ***prevalent in C*** if its complement $C \setminus Y$ is shy in C.

Remark 4.2. Recall that a subset $C \subset X$ is universally measurable if it is measurable with respect to the completion of every regular Borel probability measure on X.

The definition of relative prevalence may seem inadequate because of the apparent special role played by the point $c \in C$. In fact, relative prevalence is a homogeneous property.

Proposition 4.3 ([2])**.** *If E is shy at some point $c \in C$, then E is shy at every point of C and therefore E is shy in C.*

Relative prevalence satisfies relative versions of the genericity axioms.

Proposition 4.4 ([2])**.** *Let C be a completely metrizable convex subset of a topological vector space X. The following hold.*

(1) *No relatively open subset of C is shy in C.*
(2) *Every subset of a set that is shy in C is shy in C.*
(3) *The countable union of sets that are shy in C is shy in C.*
(4) *If E is shy in C, then $E + x$ is shy in $C + x$ for every $x \in X$.*
(5) *If $X = \mathbb{R}^n$ and C has nonempty interior in X, then $E \subset C$ is shy in C if and only if the Lebesgue measure of E is zero.*

Relative prevalence generalizes the global notion of Christensen, Hunt, Sauer, and Yorke. If $C = X$, then a Borel set $E \subset C$ is shy in C if and only if E is shy in the sense of Christensen, Hunt, Sauer, and Yorke. A useful sufficient condition for relative shyness may be expressed in terms of finite-dimensional subspaces.

Definition 4.5. A universally measurable subset $E \subset C$ is ***k-shy in C*** if there is a k-dimensional subspace $P \subset X$ such that $\lambda_P(C + a) > 0$ for some $a \in X$ and $\lambda_P(E + x) = 0$ for every $x \in X$. An arbitrary subset $F \subset X$ is k-shy in C if it is contained in a k-shy universally measurable set. The complement of a k-shy set in C is said to be ***k-prevalent in C***.

Proposition 4.6. *Every set that is k-shy in C is shy in C.*

As mentioned previously, the subtlety in the definition of relative prevalence arises from the delicate issue of supports. We show that if either of the two requirements on the supports of the measures were eliminated from the definition, then relative shyness would no longer be closed under countable unions. We construct the examples in the cone $L^1_+[0,1] = \{f \in L^1[0,1] : f \geqslant 0\}$, a shy subset of $L^1[0,1]$. Let λ be Lebesgue measure on $[0,1]$. For $f \in L^1[0,1]$ and $s \in [0,1]$, let f_s be the translation defined by

$$f_s(t) = f(t - s),$$

where the subtraction is computed modulo 1. We begin with a preliminary lemma.

Lemma 4.7. *Let $U \subset [0,1]$ be a dense open set with $\lambda(U) < 1$, and let $f = 1 - \chi_U$ be the characteristic function of the complement of U. If $A \subset [0,1]$ is a set of positive Lebesgue measure, then*

$$\left\| \inf_{\alpha \in A} f_\alpha \right\|_1 = 0.$$

We now weaken the definition of relative shyness in two different ways. Each alternative defines a notion that is not closed under countable unions.

Definition 4.8. Let X be a topological vector space and let $C \subset X$ be a convex subset that is completely metrizable in the relative topology induced from X. Let $c \in C$. We say that a universally measurable set $E \subset C$ is ***shy-1 in C at c*** if for each neighborhood W of 0 in X, there is a regular Borel probability measure μ on X with compact support such that

$$\operatorname{supp} \mu \subset C \cap (W + c),$$

and $\mu(E + x) = 0$ for every $x \in X$. If E is shy-1 at some point $c \in C$, then E is shy-1 at every point of C and we say that E is shy-1 in C.

Definition 4.9. Let X be a topological vector space and let $C \subset X$ be a convex subset that is completely metrizable in the relative topology induced from X. Let $c \in C$. We say that a universally measurable set $E \subset C$ is ***shy-2 in C at c*** if for each $1 \geqslant \delta > 0$ there is a regular Borel probability measure μ on X with compact support such that

$$\operatorname{supp} \mu \subset \big(\delta(C - c) + c\big)$$

and $\mu(E + x) = 0$ for every $x \in X$. If E is shy-2 at some point $c \in C$, then E is shy-2 at every point of C and we say that E is shy-2 in C.

Example 4.10. We define a convex set $C \subset L^1_+$ such that $C = K \cup Y$ where K and Y are shy-1 in C. Let $U \subset [0,1]$ be an open dense set with $\lambda(U) < 1$. Let $f = 1 - \chi_U$ and set

$$K = \{f_s : s \in [0,1]\} \subset L^1_+.$$

Let C be the closed convex hull of K, a compact convex subset of L^1. Set $Y = C \setminus K$. We construct probability measures λ_α and λ_β supported on C for which $\lambda_\alpha(g+Y) = 0$ and $\lambda_\beta(g+K) = 0$ for every $g \in L^1$. Define mappings $\alpha, \beta : [0,1] \to C$ by

$$\alpha(s) = f_s,$$
$$\beta(s) = \frac{2}{3} f_s + \frac{1}{3} f_{1-s}.$$

Let λ_α and λ_β be the image measures $\lambda_\alpha = \alpha^*(\lambda)$ and $\lambda_\beta = \beta^*(\lambda)$.

We show that $\lambda_\alpha(g+Y) = 0$ for every $g \in L^1$. Let $g \in L^1$. If $g = 0$, then there is nothing to prove because Y is disjoint from K, the support of λ_α. Suppose by way of contradiction that $g \neq 0$ and that $\lambda_\alpha(g+Y) > 0$. Write

$$A = \{s \in [0,1] : f_s - g \in Y\}.$$

By hypothesis, $\lambda(A) = \lambda_\alpha(g+Y) > 0$. For each $s \in A$, we have

$$f_s - g = f_s - g^+ + g^- \geqslant 0$$

because $Y \subset L^1_+$. Since $f_s \geqslant 0$ and $\min\{g^+, g^-\} = 0$, it follows that $f_s \geqslant g^+$ for each $s \in A$. By Lemma 4.7,

$$\left\| \inf_{s \in A} f_s \right\|_1 = 0$$

and therefore $g^+ = 0$. Consequently, for each $s \in A$ we have $f_s - g = f_s + g^-$. Fixing $s \in A$ and using the fact that $\|h\|_1 = 1 - \lambda(U)$ for every $h \in C$, we have

$$\begin{aligned} 1 - \lambda(U) &= \|f_s - g\|_1 \\ &= \|f_s\|_1 + \|g^-\|_1 \\ &= 1 - \lambda(U) + \|g^-\|_1, \end{aligned}$$

so $g^- = 0$. Therefore $g = 0$, a contradiction. Similarly, one may show that $\lambda_\beta(g+K) = 0$ for every $g \in L^1$. The above arguments may be carried out with λ^ϵ_α and λ^ϵ_β, the normalized images of the restriction of Lebesgue measure to $[0,\epsilon]$, for any $\epsilon > 0$. Consequently, by choosing ϵ sufficiently small, the supports of λ^ϵ_α and λ^ϵ_β can be chosen to reside in translates of arbitrarily small neighborhoods of $0 \in L^1$.

Example 4.11. We define a collection $\{C_n \subset L^1_+ : n \in \mathbb{N}\}$ such that $\bigcup_{n=1}^\infty C_n = L^1_+$ and each C_n is shy-2 in L^1_+. For each $n \in \mathbb{N}$, let $p_n = \chi_{[0,1/n]}$ and let

$$C_n = \{y \in L^1_+ : \|p_n y\|_1 \leqslant 1\}.$$

It is clear that each C_n is closed in L^1_+, that $C_n \subset C_{n+1}$, and that $\bigcup_{n=1}^\infty C_n = L^1_+$. We construct compactly supported probability measures μ_n for which $\mu_n(C_n + g) = 0$ for every $g \in L^1$ and for which the supports $\operatorname{supp} \mu_n$ all sit in a bounded subset of L^1_+. Setting $c = 0$, we conclude that each C_n is shy-2 in L^1_+ at 0 and therefore each C_n is shy-2 in L^1_+. For each n, let $U_n \subset [0,1]$ be a dense open set such that

$$\rho_n = \lambda\big(U_n \cap [0, 1/n]\big) < \frac{1}{n},$$
$$\sigma_n = \lambda\big(U_n \cap [1/n, 1]\big) < \frac{n-1}{n}.$$

Define $f_n \in L^1_+$ by

$$f_n(t) = \begin{cases} \frac{3n}{1-n\rho_n}(1 - \chi_{U_n}(t)), & \text{if } t \leqslant \frac{1}{n}; \\ \frac{3n}{n-1-n\sigma_n}(1 - \chi_{U_n}(t)), & \text{if } t > \frac{1}{n}. \end{cases}$$

Observe that $\|f_n\|_1 = 6$ and that $\|p_n f_n\|_1 = 3$. By the continuity of translation, there exists $0 < \gamma_n < 1/n$ such that $\|(f_n)_s\|_1 = 6$ and $\|p_n(f_n)_s\|_1 > 2$ for each $s \in [0, \gamma_n]$. Define $\alpha_n : [0, \gamma_n] \to L^1_+$ by $\alpha_n(s) = (f_n)_s$ and let $\lambda_n = \alpha_n^*(\lambda/\gamma_n)$. Notice that λ_n is a probability measure and that $\operatorname{supp} \lambda_n = \alpha_n([0, \gamma_n])$ is a compact subset of L^1_+. Arguing as in Example 4.10, one may show that $\lambda_n(C_n + g) = 0$ for every $g \in L^1$.

5. Equilibrium in financial models

We consider the equilibrium foundations of continuous-time finance. The continuous in time capital asset pricing model (CAPM) of Breeden is the most fundamental model in the field. See [2] and the references contained therein for details. For this discussion, we shall need only the following properties of the model. The economy consists of n agents and each agent i is associated with an endowment $e_i \in L^2_+$. Araujo and Monteiro [3] prove that the set of endowments $(e_1, \ldots, e_n)$ for which an equilibrium exists is of the first category in $(L^2_+)^n$. Therefore, from the topological point of view, most endowments lead to an economy for which equilibrium does not exist. The notion of prevalence leads to a dramatically different conclusion about generic equilibrium existence.

Duffie and Zame [12] prove that if the aggregate endowment $\sum_{i=1}^n e_i$ is uniformly bounded away from zero, then there exists an equilibrium. This scenario is prevalent, as the following result indicates.

Theorem 5.1 ([2]). *The set of endowments*

$$\{(e_1, \ldots, e_n) \in (L^2_+)^n : \sum_{i=1}^n e_i \textit{ is uniformly bounded away from zero}\}$$

is 1*-prevalent in* $(L^2_+)^n$*.*

This theorem follows from a result for general Banach lattices.

Theorem 5.2. *For every Banach lattice* X *and every* $e \in X_+$*, the set*

$$P = \{x \in X_+ : x \geqslant \gamma e \textit{ for some } \gamma > 0\}$$

is 1*-prevalent in* X_+*.*

Proof. The set P is a Borel set. Let $E = X_+ \backslash P$. Let $V \subset X$ be the one-dimensional subspace spanned by e, and let $x \in X$. Recall that λ_V denotes Lebesgue measure on V. We show that $V \cap (E + x)$ is either empty or a singleton. If this is not the case, there exist real numbers $t_1 > t_2$ and elements w_1, $w_2 \in E$ such that $t_1 e = w_1 + x$ and $t_2 e = w_2 + x$. Then

$$w_1 = w_2 + (t_1 - t_2)e \geqslant (t_1 - t_2)e,$$

so that $w_1 \in P$. This contradiction establishes that $V \cap (E + x)$ is either empty or a singleton. Therefore, $\lambda_V(E + x) = 0$. Since $\lambda_V(X_+) > 0$, we conclude that P is 1-prevalent in X_+. □

6. Prevalence in nonlinear spaces

Infinite-dimensional spaces may possess no natural linear structure. Kaloshin [26, 27] has extended the notion of prevalence to the space of smooth mappings between two manifolds, a fundamental object in dynamical systems and singularity theory. Kaloshin bases his extension of prevalence on Kolmogorov's heuristic suggestion that the infinite-dimensional problem may be reduced to a finite-dimensional one [1]:

> *In order to obtain negative results concerning insignificant or exceptional character of some phenomenon, we shall apply the following, somewhat haphazard, technique: If in a class K of functions f one can introduce a finite number of functionals $\{F_1, F_2, \ldots, F_r\}$ which in some sense can naturally be considered as taking generally arbitrary values $(F_1(f) = C_1, F_2(f) = C_2, \ldots, F_r(f) = C_r)$ from some region in the r-dimensional space of points $C = (C_1, \ldots, C_r)$, then any phenomenon that can take place only if C belongs to a set of zero r-dimensional measure will be regarded as exceptional and subject to neglect.*

Let $C^k(M, N)$ denote the space of mappings of a smooth manifold M into a smooth manifold N of class C^k. Endow $C^k(M, N)$ with the Whitney (strong) C^k topology. Let $D^n \subset \mathbb{R}^n$ be the closed unit ball. The space $C^k(M \times D^n, N)$ may be thought of as the space of n-parameter families $\{f_\epsilon : \epsilon \in D^n\} \subset C^k(M, N)$, C^k-smoothly depending on the parameter.

Definition 6.1. Let n be a positive integer. A set $P \subset C^k(M, N)$ is said to be ***strictly n-prevalent*** if there exists an open, dense set of n-parameter families $F(P) \subset C^k(M \times D^n, N)$ such that the following two conditions hold:

(1) For each n-parameter family $\{f_\epsilon : \epsilon \in D^n\} \in F(P)$, the set $\{\epsilon \in D^n : f_\epsilon \notin P\}$ has Lebesgue measure zero.
(2) For each $f \in C^k(M, N)$, there exists a family $\{f_\epsilon : \epsilon \in D^n\} \in F(P)$ such that $f_0 = f$.

Definition 6.2. A set $P \subset C^k(M, N)$ is said to be ***n-prevalent*** if P can be represented as the intersection of a countable number of strictly n-prevalent sets.

The notion of n-prevalence satisfies analogues of the genericity axioms. In particular, if $P \subset \mathbb{R}^N$ is an n-prevalent set for some $n < N$, then P is a set of full measure in $\mathbb{R}^N$. In order to relate n-prevalence to the notion of prevalence in linear spaces, we introduce a local form of prevalence for linear spaces.

Definition 6.3. Let X be a completely metrizable topological vector space. A set $S \subset X$ is ***locally shy*** if each point in X has a neighborhood whose intersection with S is shy. The complement of a locally shy set is said to be ***locally prevalent***.

All shy sets are locally shy. It is not known if the converse holds in general. Nevertheless, if X is separable then local shyness implies shyness. The space $C^k(M,N)$ is a separable Banach manifold. Therefore, following Kaloshin, it is natural to define a nonlinear Christensen-Hunt-Sauer-Yorke (CHSY)-shy set in terms of Banach charts.

Definition 6.4. The set $S \subset C^k(M,N)$ is said to be ***nonlinear CHSY-shy*** if for every mapping $f \in C^k(M,N)$, there exists a Banach chart (ψ, U) such that $f \in U$ and $\psi(S \cap U)$ is shy in the corresponding Banach space. The set $P \subset C^k(M,N)$ is said to be ***nonlinear CHSY-prevalent*** if the complement of P is nonlinear CHSY-shy.

Observe that if $C^k(M,N)$ is a Banach space, then local shyness is equivalent to nonlinear CHSY-shyness.

Proposition 6.5. *If $P \subset C^k(M,N)$ is an n-prevalent set, then P is nonlinear CHSY-prevalent.*

Kaloshin [26, 27] reformulates several classical theorems of singularity theory and the geometric theory of dynamical systems from the prevalent point of view. In particular, he establishes prevalent versions of the Whitney embedding theorem, Mather's stability theorem, and the Kupka-Smale theorem.

7. Topological entropy and periodic orbit growth rate

Let M be a finite-dimensional smooth compact manifold with $\dim(M) \geqslant 2$. Let $\mathrm{Diff}^r(M)$ be the space of C^r diffeomorphisms of M, an open subset of $C^r(M,M)$. Topological entropy, denoted $h(f)$, is the most important numerical invariant related to orbit growth. This quantity describes the total exponential complexity of the orbit structure. Roughly speaking, it represents the exponential growth rate of the number of orbit segments distinguishable with arbitrarily fine but finite precision.

We consider the problem of how fast the number of periodic points with period n grows as a function of n for a generic diffeomorphism of M. Bowen has conjectured that generically the growth rate is exponential with exponent given by the topological entropy. It turns out that the prevalent behavior of the growth of the number of periodic points contrasts sharply with the topologically generic behavior. While for prevalent diffeomorphisms the growth rate is not much faster than exponential, arbitrarily fast growth is Baire generic.

Let $f \in \mathrm{Diff}^r(M)$. Define

$$P_n(f) = \#\{x \in M : x = f^n(x)\},$$

$$P_n^{iso}(f) = \#\{x \in M : x = f^n(x) \text{ and } y \neq f^n(y) \text{ for } y \neq x \text{ in some nbd of } x\}.$$

The number of isolated points of period n is considered for technical reasons.

Definition 7.1. A diffeomorphism $f \in \mathrm{Diff}^r(M)$ is said to be an ***Artin-Mazur diffeomorphism*** (AM diffeomorphism) if the number of isolated periodic orbits of f grows at most exponentially fast. That is, there exists $C > 0$ such that

$$P_n^{iso}(f) \leqslant \exp(Cn)$$

for all $n \in \mathbb{Z}^+$.

Artin and Mazur [4] prove that for $0 \leqslant r \leqslant \infty$, AM diffeomorphisms are dense in $\mathrm{Diff}^r(M)$. The theorem of Artin and Mazur may be extended by considering the hyperbolicity of the periodic orbits. A point $x \in M$ of period n for f is hyperbolic if $df^n(x)$ has no eigenvalues of modulus one.

Definition 7.2. A diffeomorphism $f \in \mathrm{Diff}^r(M)$ is said to be a ***strongly Artin-Mazur diffeomorphism*** if all periodic points of f are hyperbolic and if there exists $C > 0$ such that

$$P_n(f) \leqslant \exp(Cn)$$

for all $n \in \mathbb{Z}^+$.

Kaloshin [28] demonstrates that for $0 \leqslant r < \infty$, strongly AM diffeomorphisms are dense in $\mathrm{Diff}^r(M)$. The set of AM diffeomorphisms spectacularly fails to be topologically generic in $\mathrm{Diff}^r(M)$.

Definition 7.3. Let (a_n) be a sequence. We say that $f \in \mathrm{Diff}^r(M)$ ***has*** $\boldsymbol{(a_n)}$***-growth*** if there exists a subsequence (a_{n_k}) such that

$$P^{iso}_{n_k}(f) > a_{n_k}$$

for all $k \in \mathbb{Z}^+$. We say that $G \subset \mathrm{Diff}^r(M)$ has (a_n)-growth if every element of G has (a_n)-growth.

Theorem 7.4 ([29])**.** *For any $2 \leqslant r < \infty$, there exists an open set $\mathcal{N} \subset \mathrm{Diff}^r(M)$ with the following property. For each sequence (a_n), $\mathcal{N}$ has a residual subset $\mathcal{R}_{(a_n)}$ with (a_n)-growth.*

In particular, arbitrarily fast growth is topologically generic in $\mathcal{N}$. Since the topological entropy of any C^r $(r \geqslant 1)$ diffeomorphism of M is finite, it follows that the equation

$$h(f) = \limsup_{n\to\infty} \frac{\log P_n(f)}{n}$$

is not topologically generic. Therefore, Bowen's conjecture is false.

Despite the preceding results, it seems natural that for a randomly chosen diffeomorphism, the growth rate of the number of periodic points should not be super-exponential. This is indeed the case. Hunt and Kaloshin have proven the following remarkable result.

Theorem 7.5 ([30])**.** *Let $1 < r \leqslant \infty$. There exists a prevalent set $\mathcal{F} \subset \mathrm{Diff}^r(M)$ such that for every $f \in \mathcal{F}$, we have the following stretched exponential bound. For all $\delta > 0$, there exists $C = C(\delta) > 0$ such that*

$$P_n(f) \leqslant \exp(Cn^{1+\delta}).$$

Moreover, it is possible to obtain a bound on the decay of the hyperbolicity of the periodic points as a function of the period.

Definition 7.6. The ***hyperbolicity*** of a linear operator $L : \mathbb{R}^N \to \mathbb{R}^N$, denoted $\gamma(L)$, is given by

$$\gamma(L) = \inf_{\phi\in[0,1)} \inf_{|v|=1} |Lv - \exp(2\pi i\phi)v|.$$

For a periodic point $x = f^n(x)$, its hyperbolicity $\gamma_n(x, f)$ is defined as the hyperbolicity of the derivative $df^n(x)$. That is, $\gamma_n(x, f) = \gamma(df^n(x))$.

Minimizing the hyperbolicity over a given period, define

$$\gamma_n(f) = \min_{\{x:x=f^n(x)\}} \gamma_n(x, f).$$

For a prevalent diffeomorphism f, $\gamma_n(f)$ decays at a stretched exponential rate.

Theorem 7.7 ([30])**.** *Let $1 < r \leqslant \infty$. There exists a prevalent set $\mathcal{F} \subset \mathrm{Diff}^r(M)$ such that for every $f \in \mathcal{F}$, we have the following stretched exponential bound. For all $\delta > 0$, there exists $C = C(\delta) > 0$ such that*

$$\gamma_n(f) \geqslant \exp(-Cn^{1+\delta}).$$

Theorem 7.5 provides a partial solution to the following open problem posed by Arnold.

Problem 7.8. Prove that a prevalent diffeomorphism $f \in \mathrm{Diff}^r(M)$ is an Artin-Mazur diffeomorphism.

8. Infinite-dimensional dynamical systems

The theory of dynamical systems has illuminated the nature of the asymptotic behavior of finite-dimensional systems. Inspired by this success, the methods of dynamical systems have been brought to bear on the infinite-dimensional systems generated by dissipative partial differential equations. Surprisingly, these *a priori* infinite-dimensional systems often have finite-dimensional global attractors. If a system possesses a global attractor, the study of the asymptotic behavior of the system essentially reduces to the analysis of the dynamics on the attractor.

We consider a dissipative partial differential equation written as an evolution equation

$$\frac{du}{dt} = F(u) \tag{8.1}$$

on a Hilbert space H. The evolution equation generates a semigroup $\{S(t) : t \geqslant 0\}$ such that for any initial condition $u_0 \in H$, there exists a unique solution to (8.1) given by $u(t, u_0) = S(t)u_0$. Using the dissipative nature of the equation, one can prove the existence of a compact set $\mathcal{A} \subset H$ such that $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}$ and $\mathrm{dist}(S(t)\mathcal{D}, \mathcal{A}) \to 0$ as $t \to \infty$ for all bounded sets $\mathcal{D} \subset H$.

A priori, $\mathcal{A}$ need not be a finite-dimensional set. Surprisingly, many evolution equations generate finite-dimensional attractors. The remarkable Sobolev-Lieb-Thirring inequalities [19, 32] have been used to establish physically relevant upper bounds on the box-counting and Hausdorff dimensions of the global attractors of many evolution equations. Examples include nonlinear wave equations, reaction-diffusion equations, and the two-dimensional incompressible Navier-Stokes system.

The existence of a finite-dimensional global attractor $\mathcal{A}$ leads to a fundamental question. In what sense are the ***dynamics*** on $\mathcal{A}$ finite-dimensional? This question is of great theoretical and computational importance, the latter because any numerical simulation of a partial differential equation is necessarily finite-dimensional. Ideally, we would like to construct a finite-dimensional dynamical system that completely describes the dynamics on $\mathcal{A}$. Approaches to this problem fall into two general classes.

The dynamics on $\mathcal{A}$ may be studied intrinsically by viewing $\mathcal{A}$ as a subset of the ambient phase space H. The theory of inertial manifolds seeks to find a finite-dimensional Lipschitz manifold containing $\mathcal{A}$ that is invariant under the dynamics and that attracts all trajectories at an exponential rate. The inertial manifold may be used to reduce the asymptotic dynamics of (8.1) to a finite-dimensional set of ordinary differential equations, the 'inertial form' of (8.1). Unfortunately, the known conditions that imply the existence of an inertial manifold are very restrictive. In particular, the existence of an inertial manifold for the two-dimensional Navier-Stokes equations remains an open problem.

Although the global attractor is finite-dimensional, estimates of its dimension may be prohibitively large. Birnir and Grauer [9] suggest that the complexity of the situation may be reduced by considering only the essential core of $\mathcal{A}$, the basic attractor. The basic attractor describes the asymptotic behavior of a typical trajectory. We now make this idea precise.

Definition 8.1. Let $\mathcal{M}$ be invariant under the action of the semigroup $\{S(t)\}$. The ***basin of attraction*** of $\mathcal{M}$, denoted $\mathrm{basin}(\mathcal{M})$, is the set of initial conditions $u_0 \in H$ such that $S(t)u_0 \to \mathcal{M}$.

By definition, $\mathrm{basin}(\mathcal{A}) = H$. We isolate the smallest part of $\mathcal{A}$ that attracts a prevalent set.

Definition 8.2. An invariant set $\mathcal{B} \subset H$ is a ***basic attractor*** if the basin of attraction of $\mathcal{B}$ is prevalent and if $\mathcal{B}$ is minimal with respect to this property. Minimality here means that there exists no strictly smaller $\mathcal{B}' \subset \mathcal{B}$ with $\mathrm{basin}(\mathcal{B}) \subset \mathrm{basin}(\mathcal{B}')$ up to shy sets.

Generalizing a theorem of Milnor [37], Birnir [7, 8] proves that the global attractor $\mathcal{A}$ may be decomposed into a basic attractor and a negligible remainder.

Theorem 8.3. *The global attractor $\mathcal{A}$ may be written as the union of a basic attractor $\mathcal{B}$ and a remainder $\mathcal{C}$ such that* $\mathrm{basin}(\mathcal{B})$ *is prevalent and* $\mathrm{basin}(\mathcal{C}) \setminus \mathrm{basin}(\mathcal{B})$ *is shy.*

In physical experiments or numerical simulations, one expects to see only the basic attractor after a sufficiently long transient period. For some systems, the basic attractor is low-dimensional and amenable to analysis. The viscous Moore-Greitzer equation is an example of such a system. This equation describes the flow of air through turbomachines such as jet engines. Birnir and Hauksson [6] conduct a detailed study of the basic attractor associated with the Moore-Greitzer model.

The approaches discussed thus far are intrinsic in the sense that they view $\mathcal{A}$ in its natural setting. Viewing the problem extrinsically, one tries to embed $\mathcal{A}$ into an appropriate Euclidean space $\mathbb{R}^m$ and then construct a dynamical system on $\mathbb{R}^m$ that replicates the dynamics on $\mathcal{A}$. The following would be ideal.

Conjecture 8.4 ([43])**.** For some m, comparable with the dimension of $\mathcal{A}$, there exists a map $\varphi : H \to \mathbb{R}^m$ that is injective on $\mathcal{A}$ and a smooth ordinary differential equation on $\mathbb{R}^m$ that replicates the dynamics on $\mathcal{A}$. More precisely, the ordinary differential equation generates a flow $\{T(t)\}$ and a global attractor X such that

$$T(t)\big|_X = \varphi \circ S(t) \circ \varphi^{-1}.$$

Currently this conjecture lies beyond reach, although some progress has been made.

The extrinsic approach requires that the attractor $\mathcal{A}$ be parametrized by finitely many coordinates. If $\varphi : H \to \mathbb{R}^m$ is injective on $\mathcal{A}$, then φ^{-1} provides such a parametrization. The search for a nice class of parametrizations motivates the following general questions.

(1) How is the dimension of a set or of a measure affected by a generic projection into a finite-dimensional Euclidean space?
(2) Given a set, under what conditions is a generic projection one-to-one on the set? How regular is the inverse?

By ***projection***, we mean simply a (possibly nonlinear) mapping into a finite-dimensional Euclidean space. These general questions address various aspects of the accuracy of projections. We now discuss recent progress on these problems.

8.1. **The Dimension-Theoretic Perspective.** One may define the dimension of an attractor in many different ways. Setting aside dynamics, the attractor may be viewed as a compact set of points in a metric space. Viewing the attractor in this light, the dimension of the attractor may be defined as the box-counting dimension or the Hausdorff dimension of the attracting set. Measure-dependent notions of attractor dimension take into account the distribution of points induced by the dynamics and are thought to be more accurately measured from numerical or experimental data. One often analyzes the 'natural measure', the probability measure induced by the statistics of a typical trajectory that approaches the attractor. Natural measure is not known to exist for arbitrary systems, but it does exist for Axiom A attractors and for certain classes of systems satisfying conditions weaker than uniform hyperbolicity. See [23, 50] for expository discussions of systems that are known to have natural measures.

The dimension spectrum (D_q spectrum) characterizes the multifractal structure of an attractor. Given a Borel measure μ with compact support X in some metric space, for $q \geqslant 0$ and $q \neq 1$ let

$$D_q(\mu) = \lim_{\epsilon \to 0} \frac{\log \int_X \left[\mu(B(x,\epsilon))\right]^{q-1} d\mu(x)}{(q-1)\log \epsilon}$$

provided the limit exists, where $B(x,\epsilon)$ is the ball of radius ϵ centered at x. If the limit does not exist, define $D_q^+(\mu)$ and $D_q^-(\mu)$ to be the lim sup and lim inf, respectively. Let

$$D_1(\mu) = \lim_{q \to 1} D_q(\mu),$$

again provided the limit exists. This spectrum includes the box-counting dimension (D_0), the information dimension (D_1), and the correlation dimension (D_2). The Hausdorff dimension of a set X may be recovered from the dimension spectrum via a variational principle. Let $\mathcal{M}(X)$ denote the set of Borel probability measures on X. The Hausdorff dimension of X, denoted $\dim_H(X)$, may be expressed in terms of the lower correlation dimension of measures supported on X [14]. We have

$$\dim_H(X) = \sup_{\mu \in \mathcal{M}(X)} D_2^-(\mu).$$

We first consider the case of a compactly supported measure in $\mathbb{R}^n$. The following general principles express the character of the finite-dimensional results.

(1) If an $(m-\epsilon)$-dimensional measure is projected into m-dimensional space, then generically its dimension is preserved.
(2) If an $(m+\epsilon)$-dimensional measure is projected to m-dimensional space, then typically the projected measure is absolutely continuous with a density in L^2.

These principles were first discovered by Marstrand, Kaufman, and Mattila in the study of orthogonal projections. The following results make these principles precise.

Theorem 8.5 (Preservation of Hausdorff Dimension [45]). *Let $X \subset \mathbb{R}^n$ be a compact set. For almost every function $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, one has*

$$\dim_H(f(X)) = \min\{m, \dim_H(X)\}.$$

Theorem 8.6 (Preservation of the Dimension Spectrum [21]). *Let μ be a Borel probability measure on $\mathbb{R}^n$ with compact support and let q satisfy $1 < q \leqslant 2$. Assume that $D_q(\mu)$ exists. Then for almost every function $f \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $D_q(f(\mu))$ exists and is given by*

$$D_q(f(\mu)) = \min\{m, D_q(\mu)\}.$$

Theorem 8.7 (Smoothness of Projections [42]). *Let μ be a Borel probability measure on $\mathbb{R}^n$ with correlation dimension greater than $m + 2\gamma$. Then for a prevalent set of C^1 maps $f : \mathbb{R}^n \to \mathbb{R}^m$, the image of μ under f has a density with at least γ fractional derivatives in $L^2(\mathbb{R}^m)$.*

For the preservation results, the space $C^1(\mathbb{R}^n, \mathbb{R}^m)$ can be replaced by any space that contains the linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ and is contained in the locally Lipschitz functions. Theorem 8.5 extends to smooth functions a result of Mattila [35] (generalizing earlier results of Marstrand [34] and Kaufman [31]) that makes the same conclusion for almost every linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$, in the sense of Lebesgue measure on the space of m-by-n matrices. Theorems 8.5 and 8.6 and their predecessors follow from a potential-theoretic characterization of the dimensions involved. Roughly speaking, the dimension is the largest exponent for which a certain singular integral converges. Precisely speaking, we have the following:

Proposition 8.8 ([21]). *If $q > 1$ and μ is a Borel probability measure, then*

$$D_q^-(\mu) = \sup\left\{ s \geqslant 0 : \int_X \left(\int_X \frac{d\mu(y)}{|x-y|^s} \right)^{q-1} d\mu(x) < \infty \right\}. \tag{8.2}$$

Sauer and Yorke [45] establish (8.2) for $q = 2$. Proposition 8.8 generalizes the result of Sauer and Yorke to a significant part of the dimension spectrum.

Suppose now that the ambient space is not finite-dimensional. Many infinite-dimensional dynamical systems have been shown to have compact finite-dimensional attractors. Such attractors exist for a variety of the evolution equations of mathematical physics, including the Navier-Stokes system, various classes of reaction-diffusion systems, nonlinear dissipative wave equations, and complex Ginzburg-Landau equations. The remarkable Sobolev-Lieb-Thirring inequalities [32, 19] have been invoked to establish physically significant upper bounds on attractor dimension in a number of cases. Nevertheless, a fundamental question remains. In what sense are the ***dynamics*** on the attractor finite-dimensional? While some progress has been made in answering this question, the problem remains fundamentally open.

When the ambient space is not finite-dimensional, one does not expect a dimension preservation result analogous to Theorem 8.5 or Theorem 8.6 to hold. We express the extent to which the dimension spectrum is affected by a generic projection from a Banach space to $\mathbb{R}^m$ in terms of the thickness exponent. This exponent measures how well a compact subset X of a Banach space B can be approximated by finite-dimensional subspaces of B.

Definition 8.9 (Thickness Exponent)**.** The ***thickness exponent*** $\tau(X)$ of a compact set $X \subset B$ is defined as follows. Let $d(X, \epsilon)$ be the minimum dimension of all finite-dimensional subspaces $V \subset B$ such that every point of X lies within ϵ of V; if no such V exists, then $d(X, \epsilon) = \infty$. Let

$$\tau(X) = \limsup_{\epsilon \to 0} \frac{\log d(X, \epsilon)}{\log(1/\epsilon)}.$$

There is no general relationship between the thickness exponent and the Hausdorff dimension. Finite-dimensional disks have thickness exponent zero but can have arbitrarily high Hausdorff dimension. A countable set, which necessarily has Hausdorff dimension zero, can have positive thickness. For example, one can show that the compact subset $\{0, e_2/\log 2, e_3/\log 3, \ldots\}$ of the real Hilbert space with basis $\{e_1, e_2, \ldots\}$ has an infinite thickness exponent. On the other hand, the thickness exponent is bounded above by the upper box-counting dimension.

The following theorem states that generically, projecting into $\mathbb{R}^m$ will cause the dimension to drop by at most a factor of $1/(1 + \tau(X))$ provided m is sufficiently large.

Theorem 8.10 (Banach Space Projections [39])**.** *Let B be a Banach space, and let M be any subspace of the Borel measurable functions from B to $\mathbb{R}^m$ that contains the space of bounded linear functions and is contained in the space of locally Lipschitz functions. Let $X \subset B$ be a compact set with thickness exponent $\tau(X)$. Let μ be a Borel probability measure supported on X. For almost every $f \in M$, one has*

$$\dim_H(f(X)) \geqslant \min\left\{m, \frac{\dim_H(X)}{1 + \tau(X)}\right\},$$

and, for $1 < q \leqslant 2$,

$$D_q^-(f(\mu)) \geqslant \min\left\{m, \frac{D_q^-(\mu)}{1 + \tau(X)}\right\}.$$

The proof of the Banach space theorem uses only the most general information about the structure of the dual space B'. In specific situations, additional knowledge about the structure of the dual space may yield improved theorems. This does indeed happen in the Hilbert space setting.

Theorem 8.11 (Hilbert Space Projections [39])**.** *Let H be a Hilbert space, and let M be any subspace of the Borel measurable functions from H to $\mathbb{R}^m$ that contains the space of bounded linear functions and is contained in the space of locally Lipschitz functions. Let $X \subset H$ be a compact set with thickness exponent $\tau(X)$. Let μ be a Borel probability measure supported on X. For almost every $f \in M$, one has*

$$\dim_H(f(X)) \geqslant \min\left\{m, \frac{\dim_H(X)}{1 + \tau(X)/2}\right\},$$

and, for $1 < q \leqslant 2$,

$$D_q^-(f(\mu)) \geqslant \min\left\{m, \frac{D_q^-(\mu)}{1 + \tau(X)/2}\right\}.$$

The Banach and Hilbert space projection theorems are sharp in the following sense. Given $d > 0$, $1 \leqslant p \leqslant \infty$, and a positive integer m, there is a compact subset X of Hausdorff dimension d in ℓ^p such that for all bounded linear functions $\pi : \ell^p \to \mathbb{R}^m$,

$$\dim_H(\pi(X)) \leqslant \frac{d}{1 + d/q},$$

where $q = p/(p-1)$ [22]. The cases $p = \infty$ and $p = 2$ show that Theorems 8.10 and 8.11 are sharp for bounded linear functions on these particular Banach spaces.

Notice that for sets with thickness zero, the Banach space theorem is a dimension preservation result. On the other hand, suppose $\tau(X) > 0$. The Hausdorff dimension of X may be ***noncomputable*** in the sense that for every positive integer m and every subspace M of the Borel measurable functions from B to $\mathbb{R}^m$, $\dim_H(f(X)) < \dim_H(X)$ for all $f \in M$. In other words, the Hausdorff dimension of X cannot be ascertained from any finite-dimensional representation of X. It is thus natural to consider the following fundamental question. Suppose X represents the global attractor of a flow on a function space generated by an evolution equation. Under what hypotheses on the flow does one have $\tau(X) = 0$? If one assumes that the flow is sufficiently dissipative and smoothing, then X will have finite box-counting dimension. We conjecture that similar ***dynamical*** hypotheses imply that $\tau(X) = 0$. Friz and Robinson [17] obtain a result of this type. They prove that if an attractor is uniformly bounded in the Sobolev space H^s on an appropriate bounded domain in $\mathbb{R}^m$, then its thickness is at most m/s. This result implies that certain attractors of the Navier-Stokes equations have thickness exponent zero.

8.2. **Parametrization of Finite-Dimensional Sets.** As discussed previously, many infinite-dimensional dynamical systems have been shown to have compact finite-dimensional attractors. One therefore hopes to investigate the dynamics of such a system in a finite-dimensional setting. First, the attractor must be represented by finitely many coordinates. This entails projecting the attractor into a finite-dimensional space and then studying the regularity of the inverse of the projection. Hunt and Kaloshin [22] establish the following general result.

Theorem 8.12 ([22])**.** *Let $X \subset B$ be a compact subset of the Banach space B with box-counting dimension d and thickness exponent $\tau(X)$. Let $m > 2d$ be an integer, and let $\alpha \in \mathbb{R}$ satisfy*

$$0 < \alpha < \frac{m - 2d}{m(1 + \tau(X))}.$$

Then for almost every C^1 function $f : B \to \mathbb{R}^m$, there exists $C > 0$ such that for all x, $y \in X$,

$$C|f(x) - f(y)|^\alpha \geqslant |x - y|.$$

If B is a real Hilbert space, then α may be chosen such that

$$0 < \alpha < \frac{m - 2d}{m(1 + \tau(X)/2)}.$$

Substantial work on orthogonal projections preceded this result. Mañé [33] demonstrates the existence of a dense set of injective projections for a compact subset of a Banach space. Ben-Artzi, Eden, Foias, and Nicolaenko [5] study the Hölder continuity of the inverse and establish sharp bounds on the Hölder exponent in the case $X \subset \mathbb{R}^n$. Foias and Olson [16] give the first proof that the inverse is Hölder continuous when X is a subset of an infinite-dimensional space. Sauer, Yorke, and Casdagli [44] bring the notion of prevalence to bear on the parametrization problem. Theorem 8.12 has been generalized to metric spaces by Okon [38].

Friz and Robinson [18] use the result of Hunt and Kaloshin to obtain parametrizations of certain global attractors in terms of the physical domain. For example, they consider the two-dimensional Navier-Stokes equations on the two-torus $\Omega = \mathbb{T}^2$ with analytic forcing. Let $\mathcal{A}$ denote the global attractor of this system. For m comparable to the box-counting dimension of $\mathcal{A}$, the map from $\mathcal{A}$ into $\mathbb{R}^{2m}$ given by

$$E_{\boldsymbol{x}} : u \mapsto \big(u(x_1), \ldots, u(x_m)\big)$$

is one-to-one between $\mathcal{A}$ and its image for almost every $\boldsymbol{x} = (x_1, \ldots, x_m) \in \Omega^m$ (with respect to $2m$-dimensional Lebesgue measure). Therefore, observations at a finite number of points in the domain can parametrize the attractor.

9. Platonic embeddings

The fundamental work of Whitney [49] marks the genesis of embedding theory and its relation to dimension. Often, the sets of interest in dynamical systems possess intricate structure and are certainly not manifolds. The Whitney embedding theorem cannot be applied to such sets. Sauer, Yorke, and Casdagli [44] address the embedding problem for arbitrary compact sets.

Theorem 9.1 (Fractal Whitney Embedding Prevalence Theorem [44])**.** *Let $A \subset \mathbb{R}^n$ be a compact subset of box-counting dimension d, and let $m > 2d$ be an integer. For almost every C^1 map $f : \mathbb{R}^n \to \mathbb{R}^m$,*

(1) *f is injective on A, and*
(2) *if C is a compact subset of a smooth manifold contained in A, then f is an immersion on C.*

One needs to know the box-counting dimension of A in order to apply this theorem. For both philosophical and empirical reasons, one would like to have the ability to deduce that A and $f(A)$ are diffeomorphic by examining the structure of the image $f(A)$. From this point of view, *a priori* assumptions on A should be kept to a minimum, if not eliminated entirely. We reformulate the fractal Whitney embedding prevalence theorem by transferring the dimension hypothesis onto the image.

Conjecture 9.2. Let $A \subset \mathbb{R}^n$ be compact. For almost every C^1 map $f : \mathbb{R}^n \to \mathbb{R}^m$, if $\dim(f(A)) < m/2$, then f is an injective immersion on A.

In order to make this conjecture precise, the notion of dimension and the definition of injective immersion must be specified. Choosing the correct dimension characteristic requires a delicate touch because Hausdorff dimension and box-counting dimension will not work. We invoke the notion of tangent dimension.

Definition 9.3 (Generalized Tangent Space [40, 46])**.** Let $A \subset \mathbb{R}^n$ be compact and fix $x \in A$. The ***direction set*** D_xA consists of the vectors v in the unit sphere $\mathbb{S}^{n-1}$ for which there exist sequences $(y_n) \subset A$ and $(z_n) \subset A$ such that $y_n \to x$, $z_n \to x$, and $(z_n - y_n)/|z_n - y_n| \to v$. The ***tangent space at x relative to A***, denoted T_xA, is the smallest linear space containing D_xA.

Definition 9.4 (Tangent Dimension [40]). The ***tangent dimension*** of A, denoted $\dim_T A$, is given by

$$\dim_T A = \max_{x \in A} \dim(T_x A).$$

The relationship of the tangent dimension to the commonly used dimension characteristics is illuminated by the following extension theorem.

Theorem 9.5 (Manifold Extension Theorem [40]). *Let $A \subset \mathbb{R}^n$ be compact and fix $x \in A$. There exists a neighborhood N of x and a C^1 submanifold M such that $T_x A = T_x M$ and $M \supset N \cap A$. In particular,* $\dim(M) = \dim(T_x A)$.

From the extension theorem it follows that the tangent dimension bounds the box-counting dimension from above. This observation leads to a Platonic embedding theorem.

Theorem 9.6 (Platonic Whitney Embedding Theorem [40]). *Let $A \subset \mathbb{R}^n$ be compact. For almost every C^1 map $f : \mathbb{R}^n \to \mathbb{R}^m$, if $\dim_T(f(A)) < m/2$, then f is injective on A. Furthermore, f is an immersion on A in the sense that the derivative Df maps $T_x A$ injectively into $T_{f(x)} f(A)$ for each $x \in A$.*

We conjecture that this theorem remains true under the weaker assumption that $\dim_T(f(A)) < m$.

10. Absolutely continuous invariant measures

Densities have emerged as a fundamental tool for formulating unifying descriptions of phenomena that appear to be statistical in nature. For example, the introduction of the Maxwellian velocity distribution rapidly led to a unification of dilute gas theory. The field of human demography grew rapidly after the introduction of the Gompertzian age distribution. In dynamical systems, one studies the set of invariant measures associated with a transformation. The regularity of these invariant measures is of fundamental importance. In particular, the existence of an invariant measure absolutely continuous with respect to Riemannian volume (an ACIM) suggests stochastic asymptotic behavior.

Expanding dynamical systems generally admit ergodic ACIM with positive Lyapunov exponents. Tsujii [48] studies a class of dynamical systems for which the mechanism of overlap and sliding produces ACIM. Let $T : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ be given by

$$T(x, y) = (\ell x, \lambda y + f(x))$$

where $\ell \geqslant 2$ is an integer, $0 < \lambda < 1$ is real, and f is a C^2 function on $\mathbb{S}^1$. The map T is a skew product over the expanding map $\tau : x \mapsto \ell x$ and T contracts fibers uniformly. From the ergodic perspective, T is simple. There exists an ergodic probability measure μ on $\mathbb{S}^1 \times \mathbb{R}$ for which

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)} = \mu$$

for Lebesgue almost every $x \in \mathbb{S}^1 \times \mathbb{R}$. The measure μ shall be called the SRB measure for T. Tsujii studies the regularity of this SRB measure.

Observe that if $\lambda \ell < 1$, then T contracts area so μ must be totally singular with respect to Lebesgue measure. If $\lambda \ell > 1$, then the SRB measure is absolutely continuous for almost every f. Let $\mathcal{D} \subset (0, 1) \times C^2(\mathbb{S}^1, \mathbb{R})$ be the set of pairs (λ, f)

for which the SRB measure is absolutely continuous with respect to the Lebesgue measure on $\mathbb{S}^1 \times \mathbb{R}$. Let $\mathcal{D}^0 \subset \mathcal{D}$ denote the interior of $\mathcal{D}$.

Theorem 10.1 ([48]). *Let $\ell^{-1} < \lambda < 1$. There exists a finite collection of C^∞ functions $\varphi_i : \mathbb{S}^1 \to \mathbb{R}$, $i = 1, \ldots, m$, such that for any $f \in C^2(\mathbb{S}^1, \mathbb{R})$, the set*

$$\left\{ (t_1, \ldots, t_m) \in \mathbb{R}^m : \left(\lambda, f + \sum_{i=1}^{m} t_i \varphi_i \right) \notin \mathcal{D}^0 \right\}$$

has Lebesgue measure zero in $\mathbb{R}^m$. Therefore, $(\lambda, f) \in \mathcal{D}^0$ for almost every $f \in C^2(\mathbb{S}^1, \mathbb{R})$.

Tsujii has therefore shown that robust absolute continuity of the SRB measure is prevalent. The ideas behind the proof may be applied to more general skew products and partially hyperbolic dynamical systems.

11. Equivariant dynamical systems

Symmetric dynamical systems play a crucial role in the analysis of an eclectic array of phenomena. It is important to understand the extent to which the symmetries of an attractor reflect the symmetries of the underlying dynamical system. Let $\Gamma \subset \boldsymbol{O}(n)$ be a compact Lie group acting on $\mathbb{R}^n$. A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be ***Γ-equivariant*** if $f(\gamma x) = \gamma f(x)$ for every $\gamma \in \Gamma$ and $x \in \mathbb{R}^n$. For $x \in \mathbb{R}^n$, the ω-limit set $\omega(x)$ consists of the points $y \in \mathbb{R}^n$ for which there exists a sequence (t_i) such that $t_i \to \infty$ and $f^{t_i}(x) \to y$.

Suppose that A is an ω-limit set for the Γ-equivariant map f. The symmetry group of A is the subgroup

$$\Sigma_A = \{\gamma \in \Gamma : \gamma A = A\}.$$

Since A is closed, Σ_A is a closed subgroup of Γ. We consider the following questions. Which closed subgroups of Γ can be realized as the symmetry group of some ω-limit set? Given an ω-limit set A, what is the symmetry group Σ_A?

If $A = \{x\}$ is a fixed point for f, then Σ_A is the isotropy subgroup $\{\gamma \in \Gamma : \gamma x = x\}$. If A is a periodic orbit, then Σ_A contains the isotropy subgroup of the points in A and is a cyclic extension of this isotropy subgroup. For more complicated ω-limit sets, the subgroup may be quite large.

The case of a finite symmetry group Γ is well understood. Generally speaking, Σ_A can be any subgroup of Γ. Each subgroup of Γ can be realized by a structurally stable attractor [15]. The situation is dramatically different when Γ is infinite. Melbourne and Stewart [36] consider the abelian case.

If Γ is abelian, it is typically the case that the symmetry group of an ω-limit set contains the connected component of the identity Γ^0. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^k Γ-equivariant map. Let $x_0 \in \mathbb{R}^n$ and let $A = \omega(x_0)$. We perturb f by smooth cocycles.

Definition 11.1. A ***Γ-cocycle*** is a map $\phi : \mathbb{R}^n \to \Gamma^0$ satisfying

$$\phi(\gamma x) = \gamma \phi(x) \gamma^{-1}$$

for all $\gamma \in \Gamma$. The space of compactly supported C^k Γ-cocycles is denoted $\mathcal{Z}_k$.

Observe that $\mathcal{Z}_k$ is a group under pointwise multiplication and $\mathcal{Z}_k$ is abelian if Γ^0 is abelian. For each cocycle $\phi \in \mathcal{Z}_k$, define the perturbation f_ϕ by

$$f_\phi(x) = \phi(x) f(x)$$

and let A_ϕ denote the ω-limit set of x_0 under f_ϕ. The mapping f_ϕ is called the extension of f by the cocycle ϕ and is Γ-equivariant. Melbourne and Stewart establish that for a generic perturbation ϕ, $\Gamma^0 \subset \Sigma_{A_\phi}$.

Theorem 11.2 ([36]). *Let $\Gamma \subset \boldsymbol{O}(n)$ be an abelian compact Lie group, and let $A = \omega(x_0)$ be an ω-limit set for the C^k Γ-equivariant map $f : \mathbb{R}^n \to \mathbb{R}^n$. Define*

$$Z = \{\phi \in \mathcal{Z}_k : \Gamma^0 \subset \Sigma_{A_\phi}\}.$$

Then Z is a residual and prevalent subset of $\mathcal{Z}_k$.

The prevalence of Z ***implies*** that Z is residual because Z is a countable intersection of open sets. This observation is interesting given that prevalence and topological genericity are independent notions. Under mild assumptions on the dynamics, this theorem may be extended to the case in which only Γ^0 is assumed to be abelian.

References

1. *Proceedings of the International Congress of Mathematicians, Amsterdam, 1954. Vol. 2*, Erven P. Noordhoff N. V., Groningen, 1954. MR0070535 (16:1190j)
2. Robert M. Anderson and William R. Zame, *Genericity with infinitely many parameters*, Adv. Theor. Econ. **1** (2001), Art. 1, 64 pp. (electronic). MR2002579 (2004g:91038)
3. A. Araujo and P. K. Monteiro, *Generic nonexistence of equilibria in finance models*, J. Math. Econom. **20** (1991), no. 5, 489–499. MR1112342 (92g:90039)
4. M. Artin and B. Mazur, *On periodic points*, Ann. of Math. (2) **81** (1965), 82–99. MR0176482 (31:754)
5. A. Ben-Artzi, A. Eden, C. Foias, and B. Nicolaenko, *Hölder continuity for the inverse of Mañé's projection*, J. Math. Anal. Appl. **178** (1993), no. 1, 22–29. MR1231724 (94d:58091)
6. B. Birnir and H. A. Hauksson, *The basic attractor of the viscous Moore-Greitzer equation*, J. Nonlinear Sci. **11** (2001), no. 3, 169–192. MR1852939 (2002e:37132)
7. Björn Birnir, *Global attractors and basic turbulence*, Nonlinear Coherent Structures in Physics and Biology (K.M. Spatschek and F.G. Mertens, eds.), NATO ASI, vol. 329, Springer-Verlag, New York, 1994.
8. ———, *Basic attractors and basic control of nonlinear partial differential equations*, Ecmi lecture notes, Chalmers University of Technology and Göteborg University, 2001.
9. Björn Birnir and Rainer Grauer, *An explicit description of the global attractor of the damped and driven sine-Gordon equation*, Comm. Math. Phys. **162** (1994), no. 3, 539–590. MR1277476 (95d:58117)
10. Jonathan M. Borwein and Warren B. Moors, *Null sets and essentially smooth Lipschitz functions*, SIAM J. Optim. **8** (1998), no. 2, 309–323 (electronic). MR1618798 (99g:49013)
11. Jens Peter Reus Christensen, *On sets of Haar measure zero in abelian Polish groups*, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), Israel J. Math. **13** (1972), 255–260 (1973). MR0326293 (48:4637)
12. Darrell Duffie and William Zame, *The consumption-based capital asset pricing model*, Econometrica **57** (1989), no. 6, 1279–1297. MR1035113 (91c:90015)
13. Nelson Dunford and Jacob T. Schwartz, *Linear operators. Part I*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1988, General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication. MR1009162 (90g:47001a)
14. Kenneth Falconer, *Fractal geometry*, John Wiley & Sons Ltd., Chichester, 1990, Mathematical foundations and applications. MR1102677 (92j:28008)
15. Michael Field, Ian Melbourne, and Matthew Nicol, *Symmetric attractors for diffeomorphisms and flows*, Proc. London Math. Soc. (3) **72** (1996), no. 3, 657–696. MR1376773 (97d:58125)
16. C. Foias and E. Olson, *Finite fractal dimension and Hölder-Lipschitz parametrization*, Indiana Univ. Math. J. **45** (1996), no. 3, 603–616. MR1422098 (97m:58120)

17. Peter K. Friz and James C. Robinson, *Smooth attractors have zero "thickness"*, J. Math. Anal. Appl. **240** (1999), no. 1, 37–46. MR1728206 (2001f:37130)
18. ———, *Parametrising the attractor of the two-dimensional Navier-Stokes equations with a finite number of nodal values*, Phys. D **148** (2001), no. 3-4, 201–220. MR1820361 (2002a:35023)
19. Jean-Michel Ghidaglia, Martine Marion, and Roger Temam, *Generalization of the Sobolev-Lieb-Thirring inequalities and applications to the dimension of attractors*, Differential Integral Equations **1** (1988), no. 1, 1–21. MR0920485 (89d:35137)
20. Brian R. Hunt, *The prevalence of continuous nowhere differentiable functions*, Proc. Amer. Math. Soc. **122** (1994), no. 3, 711–717. MR1260170 (95d:26009)
21. Brian R. Hunt and Vadim Yu. Kaloshin, *How projections affect the dimension spectrum of fractal measures*, Nonlinearity **10** (1997), no. 5, 1031–1046. MR1473372 (98k:28010)
22. ———, *Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces*, Nonlinearity **12** (1999), no. 5, 1263–1275. MR1710097 (2001a:28009)
23. Brian R. Hunt, Judy A. Kennedy, Tien-Yien Li, and Helena E. Nusse, *SLYRB measures: natural invariant measures for chaotic systems*, Phys. D **170** (2002), no. 1, 50–71. MR1945459 (2004d:37033)
24. Brian R. Hunt, Tim Sauer, and James A. Yorke, *Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 2, 217–238. MR1161274 (93k:28018)
25. ———, *Prevalence. An addendum to: "Prevalence: a translation-invariant 'almost every' on infinite-dimensional spaces"* [Bull. Amer. Math. Soc. (N.S.) **27** (1992), no. 2, 217–238; MR1161274 (93k:28018)], Bull. Amer. Math. Soc. (N.S.) **28** (1993), no. 2, 306–307. MR1191479 (93k:28019)
26. V. Yu. Kaloshin, *Prevalence in spaces of finitely smooth mappings*, Funktsional. Anal. i Prilozhen. **31** (1997), no. 2, 27–33, 95. MR1475321 (98i:58040)
27. ———, *Some prevalent properties of smooth dynamical systems*, Tr. Mat. Inst. Steklova **213** (1997), Differ. Uravn. s Veshchestv. i Kompleks. Vrem., 123–151. MR1632241 (99h:58100)
28. Vadim Yu. Kaloshin, *An extension of the Artin-Mazur theorem*, Ann. of Math. (2) **150** (1999), no. 2, 729–741. MR1726706 (2000j:37020)
29. ———, *Generic diffeomorphisms with superexponential growth of number of periodic orbits*, Comm. Math. Phys. textbf211 (2000), no. 1, 253–271. MR1757015 (2001e:37035)
30. Vadim Yu. Kaloshin and Brian R. Hunt, *A stretched exponential bound on the rate of growth of the number of periodic points for prevalent diffeomorphisms. I*, Electron. Res. Announc. Amer. Math. Soc. **7** (2001), 17–27 (electronic). MR1826992 (2002d:37032)
31. Robert Kaufman, *On Hausdorff dimension of projections*, Mathematika **15** (1968), 153–155. MR0248779 (40:2030)
32. E. Lieb and W. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, Studies in Mathematical Physics, Princeton University Press, 1976.
33. Ricardo Mañé, *On the dimension of the compact invariant sets of certain nonlinear maps*, Dynamical systems and turbulence, Warwick 1980 (Coventry, 1979/1980), Lecture Notes in Math., vol. 898, Springer, Berlin-New York, 1981, pp. 230–242. MR0654892 (84k:58119)
34. J. M. Marstrand, *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proc. London Math. Soc. (3) **4** (1954), 257–302. MR0063439 (16:121g)
35. Pertti Mattila, *Hausdorff dimension, orthogonal projections and intersections with planes*, Ann. Acad. Sci. Fenn. Ser. A I Math. **1** (1975), no. 2, 227–244. MR0409774 (53:13526)
36. Ian Melbourne and Ian Stewart, *Symmetric ω-limit sets for smooth Γ-equivariant dynamical systems with Γ^0 abelian*, Nonlinearity **10** (1997), no. 6, 1551–1567. MR1483554 (98k:58169)
37. John Milnor, *On the concept of attractor*, Comm. Math. Phys. **99** (1985), no. 2, 177–195. MR0790735 (87i:58109a)
38. T. Okon, *Dimension estimate preserving embeddings for compacta in metric spaces*, Arch. Math. (Basel) **78** (2002), no. 1, 36–42. MR1887314 (2002m:60009)
39. William Ott, Brian Hunt, and Vadim Kaloshin, *The effect of projections on fractal sets and measures in Banach spaces*, submitted to Ergodic Theory & Dynamical Systems, 2004.
40. William Ott and James A. Yorke, *Learning about reality from observation*, SIAM J. Appl. Dyn. Syst. **2** (2003), no. 3, 297–322 (electronic). MR2031277 (2004m:37041)
41. J. C. Oxtoby and S. M. Ulam, *On the existence of a measure invariant under a transformation*, Ann. of Math. (2) **40** (1939), 560–566. MR0000097 (1:18e)

42. Yuval Peres and Wilhelm Schlag, *Smoothness of projections, Bernoulli convolutions, and the dimension of exceptions*, Duke Math. J. **102** (2000), no. 2, 193–251. MR1749437 (2001d:42013)
43. James C. Robinson, *Infinite-dimensional dynamical systems*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001, An introduction to dissipative parabolic PDEs and the theory of global attractors. MR1881888 (2003f:37001a)
44. Tim Sauer, James A. Yorke, and Martin Casdagli, *Embedology*, J. Statist. Phys. **65** (1991), no. 3-4, 579–616. MR1137425 (93c:58147)
45. Timothy D. Sauer and James A. Yorke, *Are the dimensions of a set and its image equal under typical smooth functions?*, Ergodic Theory Dynam. Systems **17** (1997), no. 4, 941–956. MR1468109 (98f:58127)
46. E. V. Shchepin and D. Repovš, *On smoothness of compacta*, J. Math. Sci. (New York) **100** (2000), no. 6, 2716–2726, Pontryagin Conference, 2, Nonsmooth Analysis and Optimization (Moscow, 1998). MR1778991 (2001f:58005)
47. Barry Simon, *Operators with singular continuous spectrum. I. General operators*, Ann. of Math. (2) **141** (1995), no. 1, 131–145. MR1314033 (96a:47038)
48. Masato Tsujii, *Fat solenoidal attractors*, Nonlinearity **14** (2001), no. 5, 1011–1027. MR1862809 (2002j:37035)
49. Hassler Whitney, *Differentiable manifolds*, Ann. of Math. (2) **37** (1936), no. 3, 645–680. MR1503303
50. Lai-Sang Young, *What are SRB measures, and which dynamical systems have them?*, J. Statist. Phys. **108** (2002), no. 5-6, 733–754, Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays. MR1933431 (2003g:37042)

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