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Abelian varieties, theta functions and the Fourier transform, by Alexander Polishchuk, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge (UK), 2003, xvi+292 pp., £48.00, ISBN 0-521-80804-9

1. Abelian and theta functions

Let V be a complex vector space, of dimension g, and Γ a lattice in V, that is, a discrete subgroup of V which spans V as a real vector space. An *abelian function* on V with respect to Γ is a meromorphic function on V which is Γ-periodic (note that by the Liouville theorem, such a function cannot be holomorphic unless it is constant). A *theta function* is a function on V which is holomorphic, but only quasi-periodic with respect to Γ; that is, there exists for each $\gamma \in \Gamma$ a holomorphic map $e_\gamma : V \to \mathbb{C}$ such that

$$\theta(z+\gamma) = e_\gamma(z)\,\theta(z) \quad \text{for all} \quad z \in V\ .$$

The theory of abelian and theta functions is a milestone of 19th century mathematics. This is a long and beautiful story, which is outside the scope of this review; a nice reference is [5]. The case $g = 1$ (elliptic functions) arose from classical problems (length of the ellipse, doubling the arc of the lemniscate); it was beautifully worked out by Euler, Legendre, Abel, Jacobi and Weierstrass, among many others. Their results are rich and deep. One of the highlights may be briefly summarized as follows: the field of elliptic functions has transcendance degree 1 over $\mathbb{C}$; in fact it is the field of rational functions on a smooth plane cubic curve (depending on Γ). Moreover any elliptic function can be written as the quotient of two theta functions.

The case of $g \geq 2$ variables was studied by Abel and Jacobi in connection with the theory of abelian integrals on curves of genus g. Riemann was the first to observe that the theory could be built independently of algebraic curves [13]. Contrary to the case $g = 1$, the existence of a nontrivial abelian function whose group of periods is Γ imposes strong conditions on the lattice Γ. These are the *Riemann bilinear relations*, which are expressed in modern language by the existence of a positive hermitian form on V whose imaginary part takes integral values on Γ. Conversely, if these conditions are satisfied, the field K_Γ of abelian functions has transcendance degree g over $\mathbb{C}$; in fact, V/Γ is an algebraic (projective) variety, and K_Γ is the field of rational functions on this variety. In this case again any abelian function can be written as the quotient of two theta functions (though this result was stated by Riemann and Weierstrass, the first written proof seems to be due to Picard and Poincaré [12]).

2. Complex tori and abelian varieties

A modern reader will immediately interpret an abelian function as a meromorphic function on the complex torus V/Γ, and a theta function as a section of a certain line bundle on this torus. This point of view, however, is relatively recent. In the case $g = 2$ it appears clearly in Picard's work (see e.g. [11]). The general

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case is treated in a paper by Scorza [14], who seems to be the first to use the term *abelian variety*. This was followed by Lefschetz's paper [6], which is very close to the modern formulation: he proves in particular that higher order theta functions embed V/Γ in projective space (Lefschetz theorem) and that every hypersurface in V/Γ is the zero locus of some theta function (Appell-Humbert theorem).

After these works the theory of theta functions and complex abelian varieties was on a firm basis. It was a major achievement of Weil [15] to recast this highly transcendental theory in a purely algebraic framework: an abelian variety (over an arbitrary field) is just a connected, projective algebraic group. Remarkably a large part of the theory extends (with some minor complications) to this more general set-up; needless to say the extension is far from trivial. The definition of the Jacobian of a curve C, for instance, requires a completely new idea: Weil constructs it by glueing open pieces of the symmetric product $C^{(g)}$. The theta functions, however, had to wait for another ingredient, the Heisenberg group.

3. The Heisenberg group

This new idea appeared in 1964 with a paper of Weil [16] and its reinterpretation by Cartier [3]. According to [3], Weil's purpose was "to throw the theta functions away;" the result was a new interpretation of theta functions in terms of group representations, which gives a powerful tool to study them and works also, with appropriate modifications, in the algebraic set-up.

Let V be a real vector space, of even dimension $2n$, with a non-degenerate alternate form E. The *Heisenberg group* $\mathcal{H}(V)$ is a central extension of V by the unit circle; it can be defined as the set $V \times \mathbb{S}^1$ with the multiplication law

$$(x, z)(x', z') = (x + x', zz' e^{\pi i E(x,x')}) \ .$$

The wonderful property of $\mathcal{H}(V)$ is that it admits a *unique* irreducible representation in a Hilbert space in which the center $\mathbb{S}^1$ acts by homotheties. Thus any two different models for this representation will be canonically isomorphic (up to a scalar). A complex structure on V such that E is the imaginary part of some hermitian positive form gives rise in a natural way to such a model, the so-called Fock space $\mathcal{F}$. If we are given moreover a lattice Γ in V on which E takes integral values, then Γ lifts as a subgroup of $\mathcal{H}(V)$; theta functions appear naturally as Γ-invariants in $\mathcal{F}$ – or rather in an enlarged space $\mathcal{F}_{-\infty}$.

Soon afterwards Mumford realized that this approach could be carried out in a purely algebraic set-up, replacing V by a finite commutative group [9]. More precisely, to any ample line bundle L on A he associates an algebraic group $G(L)$, which is a central extension of a finite commutative group by the multiplicative group $\mathbb{G}_m$. The group $G(L)$ acts on $H^0(A, L)$, and this representation is again the *unique* irreducible representation of $G(L)$ on which $\mathbb{G}_m$ acts by homotheties. There is a classical model for this representation, the Schrödinger model, which admits a natural basis; thus the vector space $H^0(A, L)$ has a *canonical* basis (up to a scalar), which plays the role of theta functions in any characteristic. Using this framework Mumford obtains a very precise description of the graded ring $\bigoplus_n H^0(A, L^{\otimes n})$, that is, of the equations of A in the embedding defined by L or its multiples.

4. The Fourier-Mukai transform

Mumford's work gives a rather complete picture of the algebraic theory of abelian varieties. Some refinements have been obtained since then along these lines, in particular by Koizumi, Kempf, Pareschi and Popa. However, perhaps the most influential progress since the seventies has been the discovery by Mukai of a remarkable equivalence between the derived categories of an abelian variety and its dual [7].

The usual framework of the Fourier transform is the following. Suppose we can associate to each locally compact abelian group G a space of complex-valued functions $C(G)$ on G, such that any group morphism $p : G \to H$ gives rise to pull-back and push-down homomorphisms $p^* : C(H) \to C(G)$ and $p_* : C(G) \to C(H)$ (take for instance $C(G) = L^1(G)$). Let $\hat{G} := \mathrm{Hom}(G, \mathbb{S}^1)$ denote the Pontryagin dual of G, p and $\hat{p}$ the projections of $G \times \hat{G}$ onto G and $\hat{G}$, and $\ell : G \times \hat{G} \to \mathbb{S}^1$ the natural pairing. Then the Fourier transform $\mathcal{F} : C(G) \to C(\hat{G})$ is the homomorphism defined by $\mathcal{F}(f) = \hat{p}_*(p^* f \cdot \ell)$.

Mukai observed that a very similar formula makes sense for an abelian variety A and its dual $\hat{A}$. The space $C(G)$ is replaced by the (bounded) *derived category* $D(A)$ (roughly, the category of bounded complexes of vector bundles on A up to quasi-isomorphism), and the pairing ℓ by the Poincaré line bundle $\mathcal{P}$ on $A \times \hat{A}$. The outcome is the *Fourier-Mukai functor* $\mathcal{S} : D(A) \to D(\hat{A})$, defined on an object E of $D(A)$ by $\mathcal{S}(E) = R\hat{p}_*(p^* E \otimes \mathcal{P})$; it turns out to be an equivalence of category.

The Fourier-Mukai transform has been used by Mukai to get information about vector bundles on A. But perhaps its main application has been outside of the framework of abelian varieties: thanks to the work of Bondal, Orlov, Bridgeland and others, the derived category is by now firmly established as an important invariant of projective varieties. Here the relevance of the Fourier-Mukai transform stems from the following theorem of Orlov: if X and $\hat{X}$ are smooth projective varieties, any fully faithful functor from $D(X)$ to $D(\hat{X})$ admitting an adjoint (in particular, any equivalence of categories) is a Fourier-Mukai functor, that is, of the form $E \mapsto R\hat{p}_*(p^* E \otimes \mathcal{P})$ for some object $\mathcal{P}$ of $D(X \times \hat{X})$ [10].

5. The book

The book under review contains three parts. Part I (analytic theory) introduces theta functions on a complex torus V/Γ. The representation theory of the Heisenberg group is skillfully used, in particular to derive the functional equation of the theta functions.

Part II contains the algebraic theory of abelian varieties; it begins with the theorem of the cube, the construction of the dual abelian variety, the biduality (proved using the biextensions introduced by Mumford and Grothendieck), and continues with the Fourier-Mukai transform and Mumford's theory of the Heisenberg group.

Part III is dedicated to Jacobians. They are constructed following Weil's idea. Fay's trisecant identity is established with the help of the Cauchy-Szegö kernel. The Torelli theorem is proved using the Fourier-Mukai transform, following a paper by Beilinson and the author [1]. A last chapter surveys some more advanced topics: Deligne's symbol, the determinant bundle, the 'strange duality' which links the moduli space of rank r vector bundles on a curve to the space of r-th order theta functions.

According to the author, the book is intended to "enhance the classical theory with more recent ideas." He has fully succeeded in carrying through this programme. The book covers most of the classical theory of theta functions and abelian varieties, and at the same time introduces the reader to such varied topics as the Maslov index, mirror symmetry, biextensions, vector bundles on curves, and Deligne's symbol. This gives the book a personal touch, which makes it hardly comparable to the other books on the subject.

As the author points out, the book is neither a textbook nor a reference book: it does not claim to be exhaustive and is too advanced for a first reading. For a textbook I would still advise a student to start with Mumford's book [8], perhaps completed by the chapter on Jacobians in [4]. For a reference book [2] is quite complete (though the restriction to the complex case may be a drawback for the readers more oriented towards arithmetical geometry).

However, I would definitely recommend this book to a reader already acquainted with abelian varieties wishing to go beyond the basics of the subject. It is stimulating and provocative and at the same time well-organized. Even the expert will learn a lot from reading it.

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