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The classification of quasithin groups I, II, by Michael Aschbacher and Stephen D. Smith, Mathematical Surveys and Monographs, vols. 111–112, American Mathematical Society, Providence, RI, 2004, 1221 pp., US$228.00, ISBN 0-8218-3410-X (Vol. 111), 0-8218-3411-8 (Vol. 112)

In 1983, Danny Gorenstein announced the completion of the Classification of the Finite Simple Groups. This announcement was somewhat premature. The Classification of the Finite Simple Groups was at last completed with the publication in 2004 of the two monographs under review here. These volumes, classifying the quasithin finite simple groups of even characteristic, are a major milestone in the history of finite group theory. It is appropriate that the great classification endeavor, whose beginning may reasonably be dated to the publication of the monumental Odd Order Paper [FT] of Feit and Thompson in 1963, ends with the publication of a work whose size dwarfs even that massive work.

1. Placing quasithin groups in the classification scheme

Let us first understand the place of quasithin groups in the overall scheme. Feit and Thompson proved that every nonabelian finite simple group has even order. This focussed attention on the prime 2 and, more specifically, on the centralizers of involutions (elements of order 2). Brauer and Fowler had proved that if H is the centralizer of an involution in a finite simple group G, then $|G| < (|H|^2)!$. Hence there are only finitely many G's for each H. The bound is totally impractical, but more significantly, any H with a center of even order is a priori a candidate to be an involution centralizer in a finite simple group G. A posteriori, we see that most of these H's do not even remotely resemble an involution centralizer in an actual simple group. The great challenge is to rule out most candidate H's simply from the hypothesis that H is an involution centralizer in a finite simple group G.

With perfect hindsight, we see that G is usually a group of Lie type defined over a finite field k. If k has odd characteristic, then involutions are semisimple elements in G and involution centralizers are so-to-speak reductive groups. We say then that a finite simple group G is of odd (or component) type if some involution centralizer H looks somewhat like a reductive group. (More precisely, if X is the largest normal odd order subgroup of H, then H/X should have a component, i.e. a subnormal subgroup L such that $L/Z(L)$ is a non-abelian simple group.) The classification of finite simple groups of odd type occupied scores of group theorists during the 1960's and 1970's. Landmarks were Bender's Strongly Embedded Theorem [Be], Gorenstein-Walter's L-Balance Theorem [GW], Glauberman's Solvable Signalizer Functor Theorem [Gl], and Aschbacher's Component Theorem [A1] and Classical Involution Theorem [A2]. The first three of these theorems were crucial in establishing that the odd order normal subgroup X must be very small, i.e. that $F^*(H)$ is indeed of reductive type. The Component Theorem then shows that H can usually be chosen so that L is the unique component of H. The possible isomorphism types for L are known inductively, and in general, the type of L determines the type of G. For example, if L is of Lie type in odd characteristic, then, using

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the Classical Involution Theorem, one can usually identify G as likewise a group of Lie type in odd characteristic. (There are of course other groups G, namely most of the alternating and sporadic groups, which also had to be identified.) But the Classical Involution Theorem rests on a prior elaborate analysis of the Small Odd Case, which culminated with the Sectional 2-Rank 4 paper of Gorenstein and Harada [GH]. The Small Odd Case is, roughly speaking, the characterization of the finite simple groups G of Lie type having Lie rank 1 or 2, which are defined over fields k of odd order, starting from the hypothesis that a Sylow 2-subgroup of G is of small "width".

All of this was completed by 1980. There remained the classification of finite simple groups of characteristic 2-type, i.e. simple groups in which the centralizer H of every involution looks very much like an involution centralizer in a finite simple group of Lie type defined over a field k of characteristic 2. Work on this case had proceeded in parallel with the odd type case, beginning with Thompson's N-Group Theorem [Th]. In the characteristic 2-type case, signalizer functor methods established easily that each involution centralizer H has a normal 2-subgroup $O_2(H)$ such that $C_H(O_2(H)) \leq O_2(H)$. Indeed this may be taken to be the defining property of a group of characteristic 2-type. In this context, Thompson had introduced a measure of width, $e(G)$, the maximum rank of an abelian subgroup of G of odd order which normalizes a non-trivial 2-subgroup of G. If it turns out that $G = G(k)$ is a Chevalley group defined over a field k with $|k| = 2^n > 2$, then usually $e(G)$ is the Lie rank of G, i.e. the rank of a maximal split torus. The landmarks in the treatment of the generic subcase ($e(G) \geq 4$) were the Gorenstein-Lyons Trichotomy Theorem [GL], the Gilman-Griess Theorem [GG], Timmesfeld's O_2 Extraspecial Theorem [Ti], and Aschbacher's Uniqueness Theorem [A5]. (Aschbacher also treated the case $e(G) = 3$ [A4].) All of this work was completed and published by 1983. But, as with the Odd Case, all of this work rests inductively on the Small Even Case or Quasithin Case: the classification of finite simple groups of characteristic 2-type with $e(G) = 1$ or 2. However, this time, only the Thin Group Case ($e(G) = 1$) had been handled in advance (by Aschbacher [A3]). The general quasithin case was undertaken by Geoffrey Mason in parallel with the work-in-progress on the generic case, but although he made substantial progress, Mason never completed work on this case.

2. Groups of characteristic 2-type

A strategy for studying a simple group G of characteristic 2-type was laid out by Thompson in the N-Group Paper [Th]. We call a subgroup N of G 2-local if N is the full normalizer in G of some non-identity 2-subgroup of G. It follows easily from the characteristic 2-type assumption that each 2-local N has a normal 2-subgroup $O_2(N)$ with $C_G(O_2(N)) \leq O_2(N)$. Fix a Sylow 2-subgroup T of G and a maximal 2-local subgroup M containing $N_G(T)$. By Bender's Strongly Embedded Theorem, if M contains $N_G(X)$ for every non-identity subgroup X of T, then G is a split BN-pair of rank 1, in particular, a known simple group. Hence we may assume that there exists $N := N_G(X)$ not contained in M. The first goal is to "push up" N, i.e. to prove that $N \leq P$, where P is a 2-local which contains both N and $N_G(T)$. A natural approach is the following: We may assume that $S := N_T(X)$ is a Sylow 2-subgroup of N. If $C(S)$ is a non-identity characteristic subgroup of S with $C(S)$ normal in N, then $N \leq N^* := N_G(C(S))$ and also

$N_T(S) \leq N_G(S) \leq N^*$. Now if $S < T$, then $S < N_T(S)$, and we have pushed up N to a 2-local N^* having a larger intersection with T. Finally, if $S = T$, then $N_G(T) \leq N^*$ and we have achieved our goal. Alas, this is not always possible. For example, if $G = PSL(2, 17)$, then $T = N_G(T)$ is a maximal subgroup of G isomorphic to a dihedral group of order 16. If X is a Klein 4-subgroup of S, then $N_G(X) \cong S_4$ is another maximal subgroup of G. It cannot be pushed up. Indeed, of course, since T is maximal in G, T is the unique maximal subgroup of G containing T. The symmetric group S_4 is a fundamental obstruction to pushing up. This seems like very bad news. Nevertheless Thompson, Glauberman, Aschbacher and others persevered and, finally, established the Global $C(G;T)$-Theorem classifying all simple groups of characteristic 2-type in which $N_G(T)$ is contained in a unique maximal 2-local subgroup of G.

Further analysis, then, focusses on a pair of distinct maximal 2-local subgroups M and N containing a common Sylow 2-normalizer $B := N_G(T)$. We can think of M and N as maximal parabolic overgroups of the Borel subgroup B. If we are trying to classify quasithin groups of characteristic 2-type, then most of our target groups are finite simple groups of Lie type of BN-rank at most 2 defined over a finite field of even order. Indeed since we have already disposed of the groups of BN-rank 1, our target groups have BN-rank 2 and M and N should be the two rank 1 parabolic overgroups of B. In Thompson's approach, the next stage is called the Weak Closure Method. Let Z denote the subgroup of $Z(T)$ generated by elements of order 2, and let V_M be the normal closure of Z in M, i.e. $V_M = \langle gZg^{-1} : g \in M \rangle$. It is easy to show that V_M is a normal subgroup of M of exponent 2. Hence V_M may be regarded as a vector space over $\mathbf{F}_2$ affording a faithful module for $M/C_M(V_M)$. Moreover N has its own V_N. If $V_M = V_N$, then $N_G(V_M)$ contains both M and N. But this is impossible, since M is a maximal 2-local subgroup of G. So something interesting is happening. Since V_M and V_N normalize each other and are abelian, we see that $[V_N, V_M, V_M] = 1 = [V_M, V_N, V_N]$. If, for example, $V_N \nleq C_M(V_M)$, then we have quadratic action of V_N on V_M; i.e. V_M is a quadratic module for $M/C_M(V_M)$. If 2 were odd, this would be strong information, and indeed, for $p > 3$, Thompson classified so-called quadratic pairs (M, V_M). However, even for $p = 2$, if $|V_N/C_{V_N}(V_M)| > 2$, this is non-trivial information. In any event, this is the beginning of a lengthy and subtle analysis, which relies eventually on the classification of certain types of modules the authors call FF-modules, FF^*-modules, and others, for known simple groups. (Note that by induction, the simple composition factors of M and N are on the list of known simple groups.)

Building on work of Tutte and Sims, Goldschmidt [Go] introduced a variant of Thompson's Weak Closure Analysis around 1980. Imitating the approach of Tits to groups with a BN-pair of rank 2, he proposed to study the bipartite coset graph whose vertex set is the set of left cosets of M and those of N, and in which (xM, yN) is an edge if and only if $xM \cap yN$ is a left coset of $M \cap N$; i.e. there exists $g \in G$ with $xM = gM$ and $yN = gN$. Indeed he lifted to the universal covering tree of this graph, which preserves the local structure. This leads to the study of stabilizers of certain substructures of the tree. It has the feel of moving down from the top of M (and N) in contrast to Thompson's approach of moving up from the bottom (V_M and V_N). Goldschmidt's approach is particularly effective in the cases when the action groups induced by M and N on the neighbors of the vertices M and N respectively are split BN-pairs of rank 1 in their natural 2-transitive actions. (In

Goldschmidt's original paper, these groups are both isomorphic to $SL(2,2)$ acting on three points.) This approach culminated in the classification of weak BN-pairs of rank two by Delgado and Stellmacher [DGS]. The emergence and effectiveness of this Amalgam Method was one reason given by Geoff Mason for abandoning his work (using Thompson's Weak Closure Methods) on quasithin groups.

3. The Aschbacher-Smith Theorem

As the 1980's progressed, the absence of a published Quasithin Theorem was an increasing embarrassment in view of the announced completion of the Classification Theorem. For a while in the late 1980's, it seemed probable that Stellmacher and his collaborators would complete a proof of the Quasithin Theorem. Indeed Stellmacher did produce new proofs of Aschbacher's Thin Group Theorem and of a substantial portion of Thompson's N-Group Theorem [St]. However, it became increasingly clear by the early 1990's that Stellmacher, Meierfrankenfeld, Stroth and their collaborators had different and more ambitious goals, which if accomplished would subsume the classification of quasithin groups in a new classification of all finite simple groups of characteristic 2-type, bypassing not only Mason's work but all of the work of Gorenstein-Lyons, Gilman-Griess, and Aschbacher on groups of characteristic 2-type. This remains a work-in-progress.

Meanwhile, a decade had passed since the announced completion of the Classification Theorem, and criticism (notably from Serre) of its unfinished state mounted. Finally in 1995, Aschbacher and Smith resolved to complete and publish a proof of the Quasithin Theorem. The product of this resolution is the two-volume set under review. In fact Aschbacher and Smith overachieved slightly, replacing the hypothesis of characteristic 2-type with the hypothesis of even characteristic, which they define as follows.

Definition. *A finite group G is of even characteristic if for every 2-local subgroup M of G which contains a Sylow 2-subgroup of G, M has a normal 2-subgroup $O_2(M)$ such that $C_M(O_2(M)) \leq O_2(M)$.*

If G were of characteristic 2-type, this hypothesis would hold for all 2-local subgroups M, not simply those containing a Sylow 2-subgroup of G. We thus have two hypotheses:

(E) G is of even characteristic

and

(QT) G is quasithin; i.e. $e(G) \leq 2$.

There is an additional hypothesis (K), which is common to every major theorem classifying simple groups of characteristic 2-type.

(K) Every simple composition factor of every proper subgroup of G is on the list of known simple groups.

Thus we call G a $QTKE$ group if G satisfies hypotheses (QT), (K) and (E), and the Main Theorem of Aschbacher-Smith (for groups of 2-rank at least 3) is a more precise version of the following statement.

Theorem QTKE. *Let G be a finite non-abelian simple $QTKE$ group of 2-rank at least 3. Then either G is a group of Lie type of characteristic 2 and Lie rank at most 2 (with the exception of $U_5(q)$, $q \neq 4$) or G is isomorphic to one of 17 other explicitly named simple groups.*

Why is Hypothesis (K) present? Almost every major theorem in finite group theory is proved using mathematical induction, i.e. by assuming that G is a minimal counterexample to the desired theorem. If the theorem has suitably inductive hypotheses, then this forces all relevant proper composition factors to be on the list of conclusions of the theorem. However, neither the hypothesis that G is quasithin nor the hypothesis that G is of even characteristic is suitably inductive. For example, it is crucial to the proof of this theorem that if M is a 2-local subgroup of G containing a Sylow 2-subgroup of G, then the non-abelian simple composition factors of M are on a known list. Now any such composition factor H is in fact strongly quasithin, in the sense that every abelian p-subgroup of H has rank at most 2 for all odd primes p. However, there is no reason to assume that H is of even characteristic. Thus the Aschbacher-Smith proof requires an a priori knowledge of all strongly quasithin finite simple groups which are not of even characteristic. The list of such groups is actually very short, being a subset of the set:

$$\{A_7, J_1, L_2(p), L_2(p^2), L_3(p), U_3(p) : p \geq 5\}.$$

The classification of all quasithin finite simple groups which are not of even characteristic is contained in the classification of all finite simple groups of odd type, which was completed by 1980, as discussed above. Thus, in the context of the proof of the Classification of the Finite Simple Groups, Aschbacher and Smith were entitled to quote this result and drop Hypothesis (K). They chose to include (K), since it makes clear exactly what is assumed and what is proved within the two monographs under review. Moreover, it is consistent with the approach of Gorenstein, Lyons and Solomon (GLS) in their monographs [GLS], which treat a large complementary subset of the Classification Theorem. Aschbacher and Smith have also accommodated the GLS approach by appending a final chapter, extending their work to classify QTK groups of even type, a fairly technical hypothesis introduced by GLS, which is related to but different from the hypothesis of even characteristic. Indeed they establish that the sporadic group J_1 is the unique simple QTK group of even type but not of even characteristic. (By contrast, for GLS, QTKE groups of 2-rank 2 (A_6, $L_3(3)$, $U_3(3)$, M_{11}, and $L_2(p)$ for p a Fermat or Mersenne prime) are not of even type.)

4. The review of these volumes

Both of these volumes are written beautifully and with great care. Part I is a particular gem, being an encyclopedic exposition of the principal methods for treating groups of even characteristic: failure of factorization analysis, pushing up, weak closure methods and amalgam methods. Stellmacher's fundamental qrc-Lemma is presented. This focusses attention on certain important classes of modules for finite simple groups, and these modules are discussed in considerable detail. Anyone wishing to gain mastery of these important tools would do well to read Part I, which could serve as a text for an advanced graduate course on local group theory.

With these tools in hand, Part II tackles the proof of the Main Theorem, employing both weak closure and amalgam methods. Preliminary analysis leads to the study of a pair L,H of overgroups of T with $O_2(L) \neq 1 \neq O_2(H)$ but with $O_2(\langle L, H\rangle) = 1$. Subject to this, H is chosen minimal, and, if possible, L is chosen to be non-solvable and to act faithfully on V_L. In general it is shown that V_L is an FF-module for $L/C_L(V_L)$. As with most major classification theorems, the proof now breaks down into a large number of cases requiring individual delicate analysis.

Almost half of Part II treats groups over $\mathbf{F}_2$, the case not handled by Mason. (A detailed and lucid outline of the proof appears at the beginning of Part II.)

The entire proof is an extraordinary tour de force. Only the bravest souls will read the second volume. This reviewer has read barely a fifth of it. However, a team of referees has covered Part II, and John Thompson has read Part I as well as substantial portions of Part II, as acknowledged in the Preface.

These volumes then represent an endpoint: the completion of the first proof of the Classification of the Finite Simple Groups. Researchers may now proceed with more confidence to quote this theorem as a tool in their investigations of other mathematical structures. However, the inquiry must not and will not die. As noted above, there is an ongoing project of GLS (in collaboration with Korchagina, Magaard, Stroth, and others) which will complement the Aschbacher-Smith classification of QTK groups of even type to produce a "second generation" proof of the Classification Theorem, resting on a bedrock of Background Results, including the Odd Order Theorem and the classification of split BN-pairs of rank 1. Also, as stated before, there is a major ongoing effort, led by Meierfrankenfeld, with the goal of producing a new proof of the classification of finite simple groups of characteristic 2-type, and perhaps more. This project utilizes the tools and theorems exposited in Part I, combined with new ideas and a somewhat different perspective. In a different vein, various researchers are investigating other mathematical objects, closely related to finite simple groups, such as locally finite groups, groups of finite Morley rank, p-compact groups, p-local groups, etc. All such researchers would be well advised to study the powerful techniques presented in Part I. The entire mathematical community owes a great debt of thanks to Michael Aschbacher and Stephen D. Smith, not simply for proving this prodigious theorem, but also for persevering to produce such a well-written presentation of their work.

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