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Functional analysis, by Peter D. Lax, Wiley-Interscience, New York, 2002, xix+ 580 pp., US$105.00, ISBN 0-471-55604-1

A functional analysis course is taught in almost all mathematics departments with a Ph.D. program. Functional analysis is primarily the study of the algebraic, analytic and geometric structures of infinite-dimensional vector spaces and operators on them. The prototypical example is spaces of functions, regarded as vector spaces, where the operators are differential and integral operators. The study of infinite dimensional spaces evolved naturally and eventually became a very rich, broad and sophisticated subject. At one time, it was difficult to find a good text book of functional analysis decades ago, but nowadays we have plenty of choices. Before having Lax's book, we saw some very successful textbooks of functional analysis; to name a few: Yosida's [10] and Conway's [2].

Lax's *Functional Analysis*, which grew out of a course taught by the author at the Courant Institute over many years, can be predicted to be another successful book. It can be easily adapted as a textbook for an introductory course, and it can be used as an excellent reference book for current and future analysts.

The instructor can easily choose topics from the first twenty five chapters to teach a one-year course of introductory functional analysis. One should not be intimidated by the number of chapters: each chapter might be deliberately kept short. While the material covered in these chapters is quite similar to other books, the author makes a very thoughtful arrangement of the order: there are many chapters on theory followed immediately by chapters titled "Applications of" or "Examples of". This reflects his intention to embed abstractions into real applications.

The book starts by recalling definitions of linear spaces and linear maps in the first two chapters. Some simple properties about convex sets are given. Further study of these properties is seen in the discussion of locally convex spaces in chapter thirteen. Some nonstandard items about the index of linear maps (e.g. the product formula for the index) are also given in the first two chapters. The discussions of various types of Hahn-Banach theorems and their extensions are given in chapter three, which includes the theorem of Agnew and Morse on a semigroup (this is seldom seen in other textbooks) and the Hahn-Banach type theorem of Bohnenblust, Sobczyk and Soukhomlinoff for complex linear spaces. The next chapter is devoted to a variety of applications of Hahn-Banach theorems and the first historical note, the tragic death of Hausdorff. It returns to normed linear spaces in chapter five. Examples of Banach spaces, which include Sobolev spaces, are given. The striking difference between a finite dimensional ball and an infinite dimensional ball is emphasized and put in one section.

Already, readers may observe the elegant structure of the book. Each section is relatively short and thus easy to digest. Exercises are offered in the middle of a section, not at the end, which surely is a natural way for people to read a book or to give lectures.

Hilbert space is introduced in chapter six. The optimal distance from a given point to a closed nonempty convex set is discussed. This naturally leads to the

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Riesz representation theorem. From this standpoint it is not easy to understand where the generalization of the Riesz representation theorem for linear functionals to bilinear functionals, the Lax-Milgram Lemma, comes from. The answer, among other things, is provided in chapter seven. The discussion of the Dirichlet problem reveals the necessity of the Lax-Milgram Lemma (for non-self-adjoint operators). The classic existence theorem of Garabedian and Schiffer is presented. It is a well-known result that a harmonic function in the distributional sense is harmonic in the classical sense (e.g. this is one of the exercises in the book of Gilbarg and Trudinger [3]). It is a pleasure to learn from Lax that this result is due to Herman Weyl. In this chapter one also learns the well-known Poincaré inequality

$$||f||_{L^2} \leq C||\nabla f||_{L^2} \quad \text{for all} \quad f \in C_0^\infty(\Omega)$$

is due to Zaremba (Lemma 1 on page 65).

Chapter eight turns to the discussion of dual spaces. The focus is on reflexive spaces and support functions of given sets. As an application the standard proof of Runge's theorem is given in the next chapter. The author also inserts another elegant application: a constructive proof of the existence of Green's function, due to the author himself. Weak topologies are discussed in chapter ten to chapter twelve with applications. Some standard examples are introduced, and the principle of uniform boundedness is given. As applications, the proof of the uniform boundedness theorem of Toeplitz and Galerkin's method to show the existence of weak solutions to certain partial differential equations are presented. Weak topologies indicate another big difference between finite dimensional spaces and infinite dimensional spaces. They have become more and more important tools in modern analysis (e.g. in Calculus of Variation; see, for example, Struwe [8]). To this end, I would like to include my favorite example, other than the standard examples, of sequences which are weakly convergent but not strongly convergent. Consider a sequence of smooth functions defined in $\mathbb{R}^n$ (for simplicity, let us assume that the dimension n is greater than or equal to five) as follows:

$$u_i(x) = (\frac{1/i}{1/i^2 + |x|^2})^{\frac{n-2}{2}}, \quad i = 1, 2, \cdots .$$

This sequence converges to zero in L^p for $p \in (2, \frac{2n}{n-2})$, but it only weakly converges to zero in $L^{\frac{2n}{n-2}}$. In fact, one can easily check that $u_i(x)$ is a solution to the following elliptic equation (and they are the only ones up to translation according to Caffarelli, Gidas and Spruck [1]):

$$-\Delta u_i = n(n-2)u_i^{\frac{n+2}{n-2}}, \quad u_i > 0, \quad \text{in} \quad \mathbb{R}^n.$$

A modification of this series is used as a sequence of test functions for the minimizing sequence of Yamabe functional energy and plays an essential role in the resolution of the famous Yamabe problem; see, for example, Lee and Park [5]. Other examples of this type can be seen in the study of minimal surfaces, Yang-Mills equations, etc.; see, for example, Sacks and Uhlenbeck [7].

Local convex topologies and the beautiful theorem of Krein-Milman are discussed in chapter thirteen. The (nowadays standard) proof of the Stone-Weierstrass density theorem by Louis de Branges is included. The elegant and useful theorem of Carathéodory is mentioned:

Every compact convex subset K in $\mathbb{R}^N$ has extreme points, and every point of K can be written as a convex combination of $N+1$ extreme points.

From this, it becomes natural to derive a beautiful generalization, the Krein-Milman theorem. Quite different from other books, fixed point theorems on convex sets are not discussed here. Instead, plenty of results on extreme points are given. Choquet's theorem on the set of extreme points is also discussed. The author even makes a bold prediction on the bright future of locally convex topologies in a note at the end of the chapter. A great variety of examples of convex sets and Choquet-type representations are discussed in the following chapter.

Bounded linear maps are discussed in chapter fifteen, which includes the open mapping principle and closed graph theorem. Plenty of applications are given in chapter sixteen. In particular, the interpolation theorem of M. Riesz about linear maps is discussed in detail, followed by its wonderful application in the regularity theory of Partial Differential Equations.

Banach algebras and spectral theory are introduced in chapter seventeen. Applications of Gelfand's theory of commutative Banach algebras are given in chapter nineteen, where one can find the following beautiful corona theorem:

Theorem. *Let $\mathcal{B}$ be the set of bounded analytic functions in the open unit disk $|Z| < 1$. If $f_1, ..., f_m$ is a collection in $\mathcal{B}$ with*

$$\sum |f_j(z)| > 1$$

for every z in $|z| < 1$, then there exist m functions g_j in $\mathcal{B}$ such that

$$\sum g_j f_j \equiv 1.$$

For a smooth nonnegative function there is a similar style theorem of C. Fefferman, which can be found in Guan [4]:

Theorem. *Let f be a $C^{3,1}$ nonnegative function on $\mathbb{R}^n$ with $||f||_{C^4(\mathbb{R}^n)} \leq A < \infty$. There are N functions $g_1, \cdots, g_N$ in $C^{1,1}(\mathbb{R}^n)$ with $||g_j||_{C^2(\mathbb{R}^n)} \leq C(A)$ such that*

$$f = \sum_{i=1}^{N} g_i^2.$$

Various examples of operators and their spectra are given in the next chapter.

Chapter twenty-one is devoted to the discussion of compact maps and the spectral theory of compact maps. Two useful examples of compact operators, integral operators and inverse Dirichlet operators are given as applications in chapter twenty-two. An additional chapter is devoted to the discussion of positive compact operators. Fredholm's theory and its applications to integral equations are given in chapter twenty-four. The whole discussion about spectra ends with the discussion of invariant subspaces in chapter twenty-five.

The rest of the book covers some interesting topics as well, for example, the Phragmén-Lindelöf principle in chapter twenty-six and variational principles in chapter twenty-eight. They may not fit in a one-year course, but are certainly excellent for extra readings.

What has been omitted from this heavy book are, as noted by the author in the preface, all of nonlinear analysis and operator algebra. On the other hand, these two are extremely active fields. Readers can easily find nice books on these topics (e.g. Nirenberg [6] and Zeidler's four-volume treatise for nonlinear analysis, Takesaki [9] for operator algebras).

The author also includes some first-hand historical notes: except for ones about Heisenberg (in chapters thirty-three, thirty-five and thirty-seven), most of them are stories about the tragic fate of many founding fathers of functional analysis in WW II, including Hausdorff, König, Banach, Tauber, Schauder and Hellinger.

The book has my strongest recommendation.

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