BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 43, Number 2, Pages 255–258
S 0273-0979(06)01079-2
Article electronically published on February 15, 2006

Computational homology, by Tomasz Kaczynski, Konstantin Mischaikow, and Marion Mrozek, Appl. Math. Sci., vol. 157, Springer-Verlag, New York, 2004, xviii + 480 pp., US$54.50, ISBN 0-387-40853-3

In the beginning, Riemann defined the "connectivity" of a closed orientable surface to be $n+1$ where n is the maximal number of pairwise disjoint closed loops whose union does not disconnect the surface (hence a sphere is "simply connected": $n = 0$). Today we call n the genus of the surface. Betti in 1846 and Riemann in an unpublished note later defined "higher connectivity numbers" of a "space" X in Euclidean space R^n defined by a finite set of suitable equations and inequalities given by smooth functions. In modern terms, such a space is the intersection of the closed region defined by the inequalities with the smooth submanifold defined by the equations. The "boundary" of X is the space obtained by replacing the inequalities with equations. Adding more equations and inequalities produces a subspace of X. The "p'th connectivity number" of X is the maximal number of p-dimensional subspaces whose union does not contain the boundary of a $p+1$-dimensional subspace, plus one. In the 20th century this evolved into the p-th Betti number of a polyhedron.

There are problems with these definitions, the major ones being disregard for orientation and multiplicity. Nevertheless they proved useful in distinguishing algebraic varieties. A great advantage was the obvious invariance of the connectivity numbers under diffeomorphisms. Another was that for simple spaces these numbers could be calculated, or at least estimated, in terms of the maximal number of closed exterior p-forms that are linearly independent modulo exact forms.

In the 1880s Poincaré realized that the topology of a manifold greatly influences the long-term dynamics of vector fields on the manifold; to explore this influence he laid the foundations of algebraic and geometric topology, and of dynamical systems. A major result is his theorem that the Euler-Poincaré characteristic of a surface equals the sum of the indices of the singularities of a generic vector field on the surface. This was extended to all compact manifolds by Heinz Hopf in 1925.

In the 20th century Emma Noether made the crucial observation that the Betti numbers are not merely abstract numbers; they are the ranks of the abelian groups we call homology groups. As a result algebraic methods could be used not only for investigating spaces, but also continuous maps between spaces, because such maps determine homomorphisms between their homology groups. Solomon Lefschetz proved his wonderful fixed point theorem: The alternating sum of the traces of the homology endomorphisms induced by a generic self-map of a compact polyhedron equals the sum of the indices of the fixed points of the map. In particular, every map in a contractible polyhedron has a fixed point. Lefschetz's theorem is of great utility in dynamics and functional analysis, and is exploited in *Computational Homology.*

Poincaré discovered and was appalled by the chaotic dynamics he observed near homoclinic orbits, where the the stable and unstable manifolds of a periodic orbit meet transversely. In the 1960s Smale proved that near such an orbit there is an

2000 *Mathematics Subject Classification.* Primary 37-XX, 55-XX, 68-XX.

invariant compact subset that is chaotic in a very precise sense, implying in particular that it contains periodic orbits of arbitrarily high minimum period. Smale's theorem is the basis for most if not all mathematically rigorous tests for chaos. One of the achievements of the authors' research is a practical computational scheme for verifying this kind of chaos in specific systems. A version of it is expounded in their book.

This brings us to *Computational Homology*. The scope of the book is well stated in the preface:

> The purpose of this book is to present a computational approach to homology with the hope that such a theory will prove beneficial to the analysis and understanding of today's complex geometric challenges. Naturally this means that our intended audience includes computer scientists, engineers, experimentalists, and theoreticians in nonlinear science. As such we have tried to keep the mathematical prerequisites to an absolute minimum.
>
> ... we have insisted on a rigorous development of the subject. Thus this book can also be used as an introductory text in homology for mathematics students. It differs from the traditional introductory topology books in that a great deal of effort is spent discussing the computational aspects of the subject.
>
> ... we have tried on a variety of levels to make the book as modular as possible. On the largest scale the book is divided into three parts: Part I, which contains the core material on computational homology; Part II, which describes applications and extensions; and Part III, which contains a variety of preliminary material.

In the first ten pages the authors show how homology theory might be used in machine analysis of complex images, summarizing their viewpoint in the question:

> *Can we develop a computationally efficient algebraic tool that tells us how many connected components and enclosed regions a geometric object contains?*

They go on to say:

> For example it would be nice to be able to enter the mazelike object of Figure 1.3 into a computer and have the computer tell us whether the figure consisting of all black line segments is connected and whether there are one or more enclosed white regions in the figure. By the end of Chapter 4 the reader will be able to do this and much more.

This is followed by a discussion of how homology theory can be used to analyze dynamical systems.

As the authors emphasize, a computer program for analyzing a geometrical figure requires a finite description that can be entered into the computer. This can be done using the traditional topological method of decomposing a polyhedron into simplices. Simplicial homology groups can then be algorithmically computed. But this is computationally inefficient, as the number of simplices, equal to the number of generators for the simplicial chain groups, can be very large. More importantly, simplices are unnatural for numerical algorithms, especially when repeated subdivision of figures is needed. Cubes are much more convenient than simplices.

For these reasons in Chapter 2 the authors develop a homology theory based on the geometry and combinatorics of "cubical sets". Such a set is a union in R^d of elementary cubes, an elementary cube being a cube (of some dimension) whose vertices are in the integer lattice Z^d and whose edges have length 1. Cubical chains and boundary operators are defined in natural ways, leading to homology groups. Abstract chain complexes and their homology groups are also introduced. There is a major pedagogical advantage to this approach: It deals exclusively with concrete sets in R^n of a rather simple type, thus avoiding abstract topological spaces and triangulations of them.

Chapter 3, "Computing Homology Groups", includes a thorough introduction to the requisite matrix algebra. Computation (algebraic, not numerical) is emphasized. Short algorithms are given for elementary row and column operations, transforming matrices to row echelon form, and computing kernels and images of linear transformations. Typical theorems (with proofs included) are:

> *Algorithm* 3.99 [for finding row echelon form] *always stops. . . .*

and

> *Given an $m \times n$ matrix A on input, Algorithm 3.42 returns an $m\times(n-k)$ matrix W and an $m\times k$ matrix V such that the columns of W constitute a basis of* ker A *and the columns of V constitute a basis of* im A.

There is also a more elaborate algorithm for converting an integer matrix A to "Smith normal form" $B = Q^{-1}AR$, with integer matrices Q, Q^{-1}, R, so that the only nonzero entries in B are along the main diagonal, each dividing the next.

These algorithms are used to prove the fundamental theorem of finitely generated Abelian groups, and to construct algorithms for computing the homology groups of a free chain complex and of a cubical set. The output of these algorithms includes lists of cycles forming a basis for the homology groups.

Chapter 4 presents standard material on chain homotopies and culminates with an algorithm for replacing a chain complex by one with the same homology but fewer generators.

Chapters 5 and 6 treat the homology homomorphisms induced by a continuous map $f : X \to Y$ between cubical sets. Here the authors diverge from classical methods. The standard approach would be to approximate a continuous map by one that sends cubes to cubes. Instead, they cover the graph of the continuous map by a cubical set and view this set as the graph of a multivalued map $F : X \rightrightarrows Y$. Attention is of course restricted to a special class of such maps: $F(x)$ is an acyclic cubical subset of Y, $F(x)$ is constant over cubes in X, F is lower semicontinuous, and its graph contains the graph of f. One can find chain maps whose graphs, roughly speaking, are contained in the graph of F, and all such chain maps induce the same homology homomorphisms between the homology groups of X and Y. The homotopy axiom – that homotopic maps induce the same homology homomorphism – follows easily.

Chapter 7 presents algorithms for computing homology homomorphisms. The first step is to construct a multivalued map F that is a cubical approximation to the graph of a continuous map $f : X \to Y$ between cubical sets. This is only sketched and only for rational maps; however, it is conceptually the simplest step – all that is needed is to find for each $x \in X$ a cube of Y that contains $f(x)$. Left as exercises are the construction of algorithms for finding intersections of cubical rectangles and similar sets. Six algorithms are then presented, the last of which either computes a Smith normal form for the homology homomorphism induced by f or else stops because a finer cubic representation of the map is needed. "The details of an algorithm which can automatically perform this task are left as an exercise." For a reader who has absorbed the preceding material, this exercise should be of only moderate difficulty.

Chapter 8 describes a natural area of application for computational homology: image processing. A digital representation of a photograph can be converted to a cubical (i.e., rectangular) set by including those rectangles corresponding to dark pixels; three-dimensional images as might arise in tomography can be converted via parallel slices. Homology can be used to make coarse distinctions. As an example, the text uses cubical homology theory to investigate the well-known Cahn-Hilliard equation. This is a complicated nonlinear partial differential equation which describes alloy formation: large values of the unknown correspond to concentrations of iron, small values to concentrations of chromium. The level surfaces of a solution can be represented computationally in two ways, either as a triangulated surface or as cubical sets. The dimensions of the homology groups in the two methods differ by only two percent, indicating the robustness of homology calculations.

Chapter 10 is devoted to nonlinear dynamics. Using the representation theory of the homology homomorphism corresponding to a continuous map which was developed in Chapter 7, the authors show how to rigorously prove the existence of fixed points and chaotic dynamics, using Lefschetz's fixed point theorem and the Conley index theorem. In particular, Theorem 10.112 proves the existence of symbolic dynamics in the Hénon map.

Chapters 11, 12 and 13 contain material on homological algebra, simplicial homology, topology, and algebra which ties the book to more traditional treatments. Missing here is a theorem that the cubical and simplicial homology theories are equivalent. Such a result can be easily formulated and proved, but it is not needed for either the theory or the tools developed in the book.

An appendix describes the syntax of the computer language used in the book.

There are plenty of examples and exercises throughout the book. Many of the latter require use of computer programs (e.g., CubTop) which are referred to but not included in the book; readers are directed to the *Computational Homology Program* (CHomP) at `www.springeronline.com` for downloading.

An admirable feature of this book is that it deals with important dynamical systems such as the Cahn-Hilliard equations and the Hénon maps. Another is that while complete proofs are included, they are not necessary for applying the homological tools to dynamics.

We highly recommend *Computational Homology* to anyone needing to compute such things as Betti numbers, Lefschetz numbers, or periodic orbits in conjunction with computer simulations of dynamics. Its tools could be further developed and used to estimate Liapunov exponents and entropy. While the book is not a textbook in homology theory, it could easily be the basis for an interesting alternative to the usual courses.

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