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Polynomial identity rings, by Vesselin Drensky and Edward Formanek, Birkhäuser-Verlag, 2004, viii+200 pp., US$39.95, ISBN 3-7643-7126-9

Although the book under review is about associative rings with polynomial identity, let us start off in a somewhat broader context. Briefly, an **identity** of some entity A is an equality involving various indeterminates, valid under all substitutions of the indeterminates to elements of A. Examples of identities abound in various settings:

1. The identity $x^2 - 1 = (x+1)(x-1)$, used by the ancient Babylonians.
2. The fundamental identity $\sin^2 x + \cos^2 x = 1$ of trigonometry.
3. (Newton's Formulas [Mac]) If $e_m = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \cdots \lambda_{i_m}$ and $p_k = \lambda_1^k + \cdots + \lambda_n^k$, then for $m = 1, 2, \ldots,$
$$me_m = \sum_{k=1}^{m} (-1)^{k-1} p_k e_{m-k}.$$
4. The *Jacobi identity* $[[a,b],c]] + [[b,c],a]] + [[c,a],b]] = 0$ for Lie structures.
5. Any $n \times n$ matrix a is a root of the *Hamilton-Cayley polynomial*
$$\det(\lambda I - a) = \lambda^n + \sum_{i=1}^{n} \gamma_i(a) \lambda^{n-i}, \tag{1}$$
where $\gamma_1(a) = -\operatorname{tr}(a), \ldots, \gamma_n(a) = (-1)^n \det(a)$. Coupled with Newton's formulas, this gives us the *Hamilton-Cayley trace identity* of the form
$$a^k + \sum_{j=1}^{k} \left(\sum_{j_1 + \cdots + j_u = j} \alpha_{(j_1, \ldots, j_u)} \operatorname{tr}(a^{j_1}) \cdots \operatorname{tr}(a^{j_u}) \right) a^{k-j} = 0,$$
where $\alpha_{(j_1,\ldots,j_u)} \in \mathbb{Q}$ can be computed explicitly. (This formula was proved in 1858 for $n \le 3$ by Cayley [Ca].)

For example, over a field of characteristic $\neq 2$, any 2×2 matrix a satisfies
$$\det(a) = \frac{\operatorname{tr}(a)^2 - \operatorname{tr}(a^2)}{2} \tag{2}$$
and thus the "trace identity"
$$x^2 - \operatorname{tr}(x)x + \frac{1}{2}(\operatorname{tr}(x)^2 - \operatorname{tr}(x^2)) = 0. \tag{3}$$

(Occasionally when considering identities, one restricts the substitutions of the indeterminates; the identity $\sum_{n \in \mathbb{N}} \frac{1}{a^n} = \frac{1}{1-a}$ holds only for $|a| < 1$.)

Identities of algebraic structures

As formal algebraic structures emerged in the beginning of the twentieth century, mathematicians were able to formulate the definition of "identity" within the appropriate language, and examine how the identity affects the algebraic structure. For example, a binary operation is commutative iff the identity $xy = yx$ holds. It

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is easy to see that any group satisfying the identity $x^2 = 1$ is commutative (i.e., Abelian). In 1902, Burnside [Bu] posed his famous problem:

Must any finitely generated group of given exponent n, i.e., satisfying the identity $x^n = 1$, be finite?

(Although the answer is not difficult for $n = 3$ and was verified affirmatively for $n = 4$ by Sanov, the Burnside problem was settled negatively in the 1960's by Novikov-Adjan and Britton.)

We turn to algebras, which have both multiplication and addition. An algebra A is associative iff A satisfies the identity $(x_1x_2)x_3 = x_1(x_2x_3)$. Unless indicated otherwise, we consider only associative algebras. An identity here has the form $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$ where f and g are noncommuting polynomials or, equivalently, $f(x_1, \dots, x_n) - g(x_1, \dots, x_n) = 0$. This leads us to the subject at hand.

Definition. An *identity* of an algebra A is a polynomial $f(x_1, \dots, x_n)$ that vanishes identically for all substitutions $x_i \mapsto a_i$ to elements of A. The identity f is called a PI of A, for *polynomial identity*, if at least one of its coefficients is ± 1. An algebra satisfying a PI is called a *PI-algebra*. The PI-*degree* of a PI-algebra A is the minimal degree of a PI of A.

For example, an algebra A is commutative iff $ab = ba$, $\forall a, b \in A$, iff $x_1x_2 - x_2x_1$ is a PI of A, so PIs generalize commutativity. The requirement for some coefficient to be ± 1 circumvents the identity px, which holds in all algebras of characteristic p and thus says nothing else about the structure.

PI-theory for associative algebras was initiated by Dehn [De], who tried to describe intersection theorems for a Desarguian projective plane in terms of polynomial conditions on the underlying coordinate division algebra D. Noting that Pappus' theorem holds precisely when D is commutative, Dehn wanted to know which restrictions on D would arise from such an arbitrary intersection theorem. Although Dehn's project was only concluded much later by Amitsur [Am2], the notion of polynomial identity had taken root.

Identities fit in very well with the development of formal algebraic structures in the 1930's by Birkhoff [Bi] and Mal'cev [Mal]; indeed perhaps the most concise definition of "identity" is in the framework of universal algebra, as a universal atomic elementary sentence. We may also reverse our point of view and consider all algebras (of a certain signature) defined by a given set of identities; such a class of algebras is called a *variety*. Birkhoff and Mal'cev showed that varieties are characterized as being closed under taking subalgebras, homomorphic images, and direct products. This point of view fits in very well with the notions of algebraic geometry. For any variety $\mathcal{V}$ of associative C-algebras, the set $\mathcal{I}$ of identities of $\mathcal{V}$ is an ideal in the free (associative) algebra $C\{X\}$, where X denotes a set of noncommuting indeterminates and $\mathcal{I}$ satisfies the extra property that $\varphi(\mathcal{I}) \subseteq \mathcal{I}$ for any homomorphism $\varphi : C\{X\} \to C\{X\}$; such a set is called a *T-ideal*.

In the previous paragraph we suppressed the size of the set X. An algebra is called *affine* if it is finitely generated as an algebra. When considering affine algebras, we usually fix the generating set $a_1 \dots, a_\ell$; then X is taken either to consist of ℓ elements or to be countably infinite.

At this early stage, several basic questions emerged:

1. How can we tell if a given algebra A satisfies a PI?
2. What is the structure of an algebra A satisfying a PI?
3. What can one say about the T-ideal of identities satisfied by A?

The important connection with matrices was established in 1936 by Wagner [Wa], who showed that any matrix algebra $M_n(F)$ satisfies a PI. Wagner also noted that the polynomial $(xy - yx)^2$ evaluated on 2×2 matrices takes on only scalar values (but not always zero). Indeed, if $a = vw - wv$ for matrices v and w, then $\mathrm{tr}(a) = 0$, implying by (3) that a^2 is scalar. (This polynomial is often called *Hall's polynomial*, since it played a key role in [Ha].)

In another direction, Dubnov and Ivanov [DuI] showed in 1943 that any $\mathbb{Q}$-algebra satisfying x^m is in fact nilpotent, satisfying the identity $x_1 \cdots x_n$ for some n dependent on m. This paper was overlooked in the West; the theorem was rediscovered by Nagata and became known as the Nagata-Higman theorem (cf. [Hi]).

The early breakthroughs

Meanwhile, Jacobson's structure theory of primitive rings and the Jacobson radical appeared, and algebraists were looking for classes of associative algebras possessing a structure theory that could yield results comparable to those from commutative algebra. Kaplansky [Ka1] obliged, proving that every primitive PI-algebra is simple and finite over its center. Kaplansky's theorem remains at the heart of representation theory. In the same paper Kaplansky proved that the nil radical of a PI-algebra is locally nilpotent.

The parallel development of group varieties (cf. [Ne]) and identities of algebras led to similar conjectures, although at times the answers differed. As a test question, Kurosch had asked the analog of Burnside's problem for algebras — Is every algebraic affine algebra over a field F finite (dimensional) over F? Fortuitously Jacobson [Ja] noted that all algebraic algebras of bounded degree are PI. Utilizing the newly developed structure theory and also results from Levitzki [Lev1], Kaplansky [Ka2] showed that every algebraic affine PI-algebra is finite dimensional.

During the early 1950's the main advances were made by Levitzki and his star student, Amitsur, who after Levitzki's death in 1954 became the predominant figure in PI-theory and during the 1950's and 1960's almost singlehandedly laid out the structural foundations of PI-theory. Through an intricate analysis of the multiplication of matrix units, Amitsur and Levitzki showed that the matrix algebra $M_n(F)$ has PI-degree $2n$, satisfying the *standard identity*

$$s_{2n} = \sum_{\pi \in S_{2n}} \mathrm{sgn}(\pi) x_1 \cdots x_{2n}.$$

Levitzki [Lev2] also showed that the upper and lower nilradicals of a PI-algebra are equal. In 1955, Amitsur [Am3] showed that any PI-domain is an Ore domain, and its ring of fractions satisfies the same PI and thus is a division algebra finite over its center.

Turning to affine algebras, Amitsur, joined by Procesi [AmP], proved that any prime ideal of an affine PI-algebra is the intersection of maximal ideals. Procesi continued the study of affine PI-algebras and proved the analogs of many basic theorems from commutative algebra. However, a major question remained open: Is the radical of an affine PI-algebra nilpotent?

A more varietal approach came from Specht [Sp]. Inspired by Hilbert's basis theorem, Specht conjectured that the free algebra satisfies the ascending chain condition on T-ideals. Equivalently, Specht's conjecture asserts that any T-ideal is *finitely based*, i.e., finitely generated as a T-ideal (and thus is a consequence of a finite number of identities). Oates and Powell [OP] proved the group-theoretic version for finite groups in 1964, but the ring-theoretic version was to prove far more stubborn, even for finite-dimensional algebras. As a proposed first step, Specht showed that any identity is a consequence of identities that are sums of products of higher commutators.

While Amitsur was working out the structure theory, a completely different, combinatoric approach to Kurosch's problem was taken by Shirshov [Sh2], who showed that, for any affine PI-algebra $A = F\{a_1, \dots, a_\ell\}$ of degree d, any element is a linear combination of products $w_1^{k_1} \dots w_m^{k_m}$ of powers of words in the generators of bounded length, such that m is also bounded. Not only does this result show at once that any affine PI-algebra integral over its center C is necessarily finite as a module over C, but, furthermore, integrality need be checked at only a finite number of elements. Another advantage of Shirshov's combinatoric approach was that associativity of the algebra was not required. This is emphasized in the title of [Sh1], which ironically masked his results from the West for 18 years. On the other hand, Shirshov's work was later applied to theories of identities of alternative, Lie, and Jordan algebras.

PI-theory received another boost from structure theory after Goldie characterized orders in simple Artinian rings in 1958; in 1960, Posner verified Goldie's conditions for prime PI-algebras. In 1968, using intricate arguments from representation theory, M. Artin [Ar] used PI-theory to characterize Azumaya algebras of bounded rank. The proof was improved and generalized by Procesi, to yield

Artin-Procesi Theorem. *A C-algebra R is Azumaya of constant rank n iff A satisfies all identities of $M_n(C)$, with no homomorphic image of A satisfying s_{2n-2}.*

Developments in the 1970's

Perhaps the crowning application of PI-theory was in 1972. Amitsur earlier had given an explicit construction of $F\{X\}/I$, where I is the T-ideal of $M_n(F)$, and had showed that this algebra is a domain. Hence its algebra of fractions is a division algebra, denoted $\mathrm{UD}(n, F)$, whose elementary theory is as general as possible for a division algebra of dimension n^2 over its center. In [Am3], Amitsur confronted $\mathrm{UD}(n, F)$ with various "incompatible" examples, proving as a consequence that $\mathrm{UD}(p^3, F)$ cannot have a maximal subfield Galois over the center. Amitsur's theorem has had a profound impact on the theory division algebras, leading to a very thorough investigation of $\mathrm{UD}(n, F)$ by many researchers.

Central polynomials. Inspired by Hall's polynomial for $M_2(F)$, Kaplansky [Ka3] asked whether, for any given n, $M_n(F)$ need have nonidentities that take on only scalar values. This question attracted the attention of two brilliant young mathematicians: In 1972, Formanek [Fo1] and Razmyslov [Ra1] discovered such polynomials on $M_n(C)$, for C arbitrary commutative. Razmyslov's construction involved the so-called *Capelli polynomial* c_{n^2}, a nonidentity of $M_n(C)$ that alternates in n^2 indeterminates.

The existence of such central polynomials led to the observation that any ideal of a semiprime PI-algebra intersects the center nontrivially, thereby permitting the introduction of techniques from commutative algebra. Suppose R is a prime PI-ring, and let $S \subseteq \mathrm{Cent}(R)$ be the set of evaluations of central polynomials of R. Those prime ideals of R not containing S comprise a dense subset of the prime spectrum, for which localization at the center works beautifully. Moreover, if we embed R into some matrix algebra $M_n(F)$ and form $\hat{R}$ by adjoining the characteristic coefficients of elements of R, then $S\hat{R}$ is a common ideal of R and $\hat{R}$. But $\hat{R}$ is integral over its center, by Shirshov's theorem. This construction, called the *characteristic closure*, directly reduces much of the theory of prime affine PI-algebras to commutative theory. See [Sch] for a lovely application.

The alternating property of Razmyslov's central polynomial led to some vast simplifications in proofs of earlier theorems such as the Artin-Procesi theorem and provided new insights into Azumaya algebras; cf. [Am5]. Similar ideas were used in studying the prime spectrum of Noetherian PI-rings.

Other advances in the 1970's. 1974 was a particularly productive year for Razmyslov. In [Ra2] he proved the nilpotence of the Jacobson radical of any affine algebra satisfying a Capelli identity. In [Ra3], Razmyslov generalized the notion of polynomial identity to "trace identity" and showed that all such identities of $M_n(\mathbb{Q})$ are consequences of the Hamilton-Cayley trace identity. (This was done independently by Helling [He] and Procesi [Pr].) Finally, Razmyslov [Ra4] established that the T-ideal of $M_2(\mathbb{Q})$ is finitely based. (Drensky [Dr2] found the "best generators" — the standard polynomial s_4 and the Hall identity.)

In a completely different direction, Regev proved in his doctoral dissertation that the tensor product of PI-algebras is a PI-algebra. Perhaps more important than this result was its proof, which described identities of a given degree n in terms of left ideals of the group algebra $\mathbb{Q}S_n$ of the symmetric group (later generalized to the general linear group). This led to the asymptotic theory of codimensions, one of the most active areas of research today in PI-theory. As an early application, Krakowski and Regev [KrR] provided evidence for Specht's conjecture by showing that the identities of the Grassman algebra over $\mathbb{Q}$ are based on the single *Grassman identity* $[[x, y], z]$.

The breakthroughs of the 1980's

These three branches of the subject were to bear fruit in the 1980's. In the early 1980's, Kemer and Amitsur-Regev independently noted that, in characteristic 0, there exist identities corresponding to a *two-sided* ideal of $\mathbb{Q}S_n$. Kemer went on to prove that all affine algebras in characteristic 0 satisfy a Capelli identity (and thus have a nilpotent Jacobson radical). Braun [Br] proved the nilpotence of the Jacobson radical for affine PI-algebras over a field of any characteristic, using a different approach based on the structure theory of Azumaya algebras.

The invariant-theoretic approach of Procesi (joined by Formanek) merged with the representational methods of Regev (and then Kemer, Berele, and Drensky), leading to deep connections with Poincaré series and growth, as described in [Fo2].

But the pinnacle of the PI-theory in the 1980's was Kemer's solution of Specht's problem in characteristic 0, a deep result that required melding most of the known PI-theory with several major innovations. Kemer's solution can be outlined briefly via several steps:

1. The characteristic closure construction described above works for any representable algebra.

2. The relatively free algebra of a finite-dimensional algebra is representable.

3. Any affine PI-algebra satisfies the same identities as some finite-dimensional algebra. (The argument requires a very careful analysis of evaluations of finite-dimensional, subdirectly irreducible algebras.)

4. There is a 1:1 correspondence between sets of identities of PI-algebras over $\mathbb{Q}$ and sets of graded identities of affine superalgebras (i.e., $\mathbb{Z}/2$-graded algebras). This step relies heavily on the theory of Young diagrams. Also, the Grassman algebra plays a key role.

5. Analogously to steps 1–4, any affine PI-superalgebra satisfies the identities of a finite-dimensional superalgebra.

6. The solution to Specht's conjecture follows easily from steps 4 and 5.

On a completely different note, let us return to the Burnside problem. After the counterexamples noted above, researchers had turned to the *restricted Burnside problem*: Are there only finitely many groups (up to isomorphism) of exponent n generated by m elements? By passing to the associated Lie algebra of a nilpotent group, this problem was transformed to a very difficult question concerning certain identities (called *Engel identities*) of Lie algebras. Zelmanov [Ze1], [Ze2], modifying Shirshov's theorem and proving deep new theorems about Jordan algebras, obtained enough Engel-like identities to settle the restricted Burnside problem affirmatively.

Another significant advance of the 1980's was the development of the geometry of PI-algebras, perhaps most strikingly by Artin and Schelter [ArS].

More recent results. Ironically, to some extent, the deluge of brilliant results in the period 1987–1991 overwhelmed the mathematical community and drove some researchers from the arena. Kemer's theorem and Zelmanov's theorem each requires most of a book for a full proof. The situation is exasperated in characteristic $p > 0$: Step 4 in Kemer's proof fails, and many new mysterious T-ideals arise in characteristic p. On the positive side, Kemer [Ke4] proved the striking result that any affine PI-algebra in characteristic p satisfies all the identities of $M_n(\mathbb{Z}_p)$ for some n. Although Kemer's theory brought the Grassman algebra G to the fore, basic questions about G turned out to be rather stubborn. For example, what is the PI-degree of $M_n(G)$ for $n > 2$?

Razmyslov and Zubrilin [RaZ], [Zu1], [Zu2] discovered that the elegant arguments used earlier in conjunction with central polynomials did not require the polynomial to be central! Although this sublime observation has applications throughout the theory, shortening many proofs, the technique is formal, and its application is difficult to intuit.

Fortunately, around 2000 the PI community began to recover. Belov [Bel1] provided a counterexample to Specht's conjecture in characteristic $p > 0$ and has verified Specht's conjecture for affine PI-algebras (albeit in Russian, unpublished).

Giambruno and Zaicev [GiZ] succeeded in harnessing Kemer's methods to answer questions about asymptotic growth of the codimensions, including a positive solution to Regev's conjecture on exponential dimension.

Zelmanov recently has proved important basic results about profinite identities of algebras, and this looks like a major area for development.

The PI-literature

Exposition in PI-theory has had spurts roughly matching the breakthroughs. On page 135 in the book under review, Formanek lists twenty-five major expository works on PIs, but only one of them was published after 1993! In the last two years there has been a resurgence, including the book under review, another book by Belov and the reviewer, and a third book to appear by Giambruno and Zaicev, as well as a monograph by Iltyakov [Il].

The book under review, written by two of the foremost PI researchers over the last 30+ years, is an outgrowth of their lecture series at the CRM in Barcelona in 2003. As indicated by the title, the authors focus on the associative PI-theory. They preserve the balance between the combinatoric and structure theoretical aspects of the theory by dividing the material into two parts with minimal duplication — the Amitsur-Levitzki theorem and central polynomials for matrices — and these two key results are presented with different proofs. Both accounts are very well written, but often with pointers to the literature in place of proofs.

Drensky explains many of the most important combinatoric issues, such as the role of Shirshov's theorem in the theory, as well as modern developments such as how the exponential dimension ties in with Kemer's theory. He manages to include several lesser-known results as well, such as Kuzmin's lower bound for the Nagata-Higman Theorem, and Drensky's part should be very useful to advanced students who need a picture of the combinatoric literature.

Formanek's part, although appearing second, might be more accessible for a reader with a general background in algebra who needs PI-theory to answer related questions in algebra. Elaborating on [Fo3], Formanek has added several results, proved very elegantly, about Noetherian PI-rings. Formanek also prepares the reader for basic questions about the generic division ring, such as whether its center is rational — he calls this "the most important open question in PI-theory." It may well be the hardest open question in the theory as well and recently has been the focus of cohomological methods by Saltman and others (not covered in this book).

In conclusion, the Drensky-Formanek book is a fine introduction to the highlights of the associative PI-theory, including some recent research, given from two different expert perspectives.

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Louis Rowen
Bar-Ilan University
E-mail address: `rowen@macs.biu.ac.il`